Due: Thursday, October 22, in class.

Problem 1:
(a) Let $G$ be a finite group and let $X$ be a finite left $G$-set. For each $g \in G$ let $^gX = \{ x \in X \mid g.x = x \}$ be the set of points in $X$ that are fixed by $g$. Prove the Cauchy-Frobenius formula for the number of orbits:

$$|G \setminus X| = \frac{1}{|G|} \sum_{g \in G} |^gX|.$$

Hint: Consider the set $\{(g,x) \in G \times X \mid g.x = x\}$.
(b) Prove that there are $\frac{1}{12}(m^4 + 11m^2)$ ways to color the sides of a regular tetrahedron with $m$ colors, up to rotation.
(c) Find the number of ways to color the sides of a cube with $m$ colors, up to rotation.

Problem 2:
(a) Prove that there are exactly two isomorphism classes of groups of order 6.
(b) Prove that there are exactly five isomorphism classes of groups of order 12.

Hint: Any group of order 12 contains a normal Sylow subgroup.

Problems from Basic Algebra 1:
2.3: 2
2.5: 8
4.6: 8, 9, 11

Hints: For 9: Use 8 and 1.13(1). For 11: If $P$ and $Q$ are two normal subgroups of $G$ such that $P \cap Q = \{1\}$, then $PQ \cong P \times Q$. If $P$ is any $p$-group then use the homomorphism $P \rightarrow P/C(P)$ and induction to prove that $P$ is nilpotent.