CFT notes

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Contents

1 Basics of conformal symmetry in two dimensions

1.1 Conformal transformations and holomorphic functions

Definition 1.1 *A **conformal transformation** between (pseudo)-Riemannian manifolds (M, g) and (N, g') is a map $f : M \to N$ such that there exists $\Lambda : M \to \mathbb{R}_{>0}$ so that for all $x \in M$, $[f^*g'](x) = \Lambda(x)g(x)$*

Let us specialize to the case of $M = N = \mathbb{R}^2 = \mathbb{C}$ with the Euclidean metric $g = g' = dx^2+dy^2$. Then for $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be a conformal transformation we must have (writing the metric components in matrix notation):

$$\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} = \Lambda g = f^*g' = \begin{bmatrix} (\frac{\partial f^0}{\partial x})^2 + (\frac{\partial f^0}{\partial y})^2 & \frac{\partial f^0}{\partial x}\frac{\partial f^1}{\partial x} + \frac{\partial f^0}{\partial y}\frac{\partial f^1}{\partial y} \\ \frac{\partial f^0}{\partial x}\frac{\partial f^1}{\partial x} + \frac{\partial f^0}{\partial y}\frac{\partial f^1}{\partial y} & (\frac{\partial f^1}{\partial x})^2 + (\frac{\partial f^1}{\partial y})^2 \end{bmatrix}$$

The diagonal elements of this matrix equation tell us that

$$(\frac{\partial f^0}{\partial x})^2 + (\frac{\partial f^0}{\partial y})^2 = \Lambda = (\frac{\partial f^1}{\partial x})^2 + (\frac{\partial f^1}{\partial y})^2 \tag{1}$$

and the off diagonal elements tell us that

$$\frac{\partial f^0}{\partial x}\frac{\partial f^1}{\partial x} + \frac{\partial f^0}{\partial y}\frac{\partial f^1}{\partial y} = 0 \tag{2}$$

If we do some algebra now, making liberal use of (1) and (2) we get the following:

$$\left[(\frac{\partial f^1}{\partial x} - \frac{\partial f^0}{\partial y})^2 + (\frac{\partial f^0}{\partial x} + \frac{\partial f^1}{\partial y})^2\right]\left[(\frac{\partial f^1}{\partial x} + \frac{\partial f^0}{\partial y})^2 + (\frac{\partial f^0}{\partial x} - \frac{\partial f^1}{\partial y})^2\right] = 0 \tag{3}$$

The first of these factors being 0 is equivalent to the Cauchy-Riemann equations for f, and the second factor being 0 is equivalent to the equations defining an anti-holomorphic function. So we have found that f being conformal is the same as f being either holomorphic or anti-holomorphic. Because of this, we move to complex coordinates $z = x+iy$, $\bar{z} = x-iy$. Associated to this coordinate transformation are these usual corresponding transformations and definitions:

$$\partial := \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \tag{4}$$

$$\bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \tag{5}$$

$$g_{\mu\nu} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \tag{6}$$

Remark 1 *In (6), the indices μ and ν run over $z, \bar{z}$*

Remark 2 *We will be interested in theories invariant under **local** conformal transformations. Local refers to invariance of our theory under any meromorphic function on $\mathbb{C}$, i.e we allow for poles. Indeed, the invertible/holomorphic transformations on the Riemann sphere are just those of the form $f(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$ (the so called **Mobius transformations**) which comprise a small subset of all possible local conformal maps from the Riemann sphere to itself.*

1.2 The Witt algebra

From basic complex analysis, we recall that any meromorphic function is equal to its Laurent series on a punctured neighborhood of 0. Therefore we have that the algebra of local conformal transformations is generated by $\{\ell_n\}_{n\in\mathbb{Z}} \cup \{\bar{\ell_n}\}_{n\in\mathbb{Z}}$ where

Definition 1.2

$$\ell_n = -z^{n+1}\partial \tag{7}$$

$$\overline{\ell_n} = -\bar{z}^{n+1}\bar{\partial} \tag{8}$$

The collection of these ℓ_n span the Lie algebra of infinitesimal conformal transformations called the **Witt algebra**. Clearly the $\bar{\ell}_n$ span an isomorphic algebra.

Remark 3 *Easy computations yield*

$$[\ell_n, \ell_m] = (m-n)\ell_{m+n} \tag{9}$$

$$[\overline{\ell_n}, \overline{\ell_m}] = (m-n)\overline{\ell_{m+n}} \tag{10}$$

$$[\ell_n, \overline{\ell_m}] = 0 \tag{11}$$

Remark 4 *The symmetry algebra of the theory is the direct sum of two copies of the Witt algebra, one for the ℓ_n's and one for the $\bar{\ell}_n$'s. Because of this splitting we will often focus our attention on the ℓ_n's (the **holomorphic part** of the theory) and identical considerations will apply to the **anti-holomorphic** $\bar{\ell}_n$'s. Furthermore we will think of z and $\bar{z}$ as independent complex coordinates, imposing the reality condition $z^* = \bar{z}$ when it is convenient.*

If we consider the singular behavior of some vector field $\sum a_n\ell_n$ as $z \to 0$ and $z \to \infty$ what we find is that the only globally well defined elements of the Witt algebra are $\{\ell_{\pm1}, \ell_0\}$. Indeed these generate the aforementioned Mobius tranformations from Remark 1.

1.3 The Virasoro algebra

When we move from a classical theory to a quantum one, we must consider projective representations of the symmetry algebra which in turn are equivalent to true representations of central extensions of that algebra. The Witt algebra has a unique central extension, as one can check by computing its second cohomology.

Definition 1.3 *The **Virasoro algebra with central charge c** is the central extension of the Witt algebra given by the relations*

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{\boldsymbol{c}}{12}(m^3 - m)\delta_{m,-n} \tag{12}$$

$$[L_n, \boldsymbol{c}] = 0 \tag{13}$$

c here is an operator in the center of the Lie algebra, and it can take specific values on representations (it will necessarily take a fixed value on an irreducible representation by Schur's lemma).

We expect that two dimensional conformal field theory should be all about representations of this algebra.

2 Fields and correlation Functions

In this section we establish some of the basics of the formalism in 2 dimensional conformal field theories.

2.1 Conformal dimension and (quasi)-primary fields

Definition 2.1 *A field $\phi(z,\bar{z})$ has **conformal weight/dimension** $(h,\bar{h})$ if under scale transformation $z \to \lambda z$, $\bar{z} \to \bar{\lambda}\bar{z}$ it transforms as*

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \lambda \bar{z}) \tag{14}$$

Equivalently, $L_0\phi = h\phi$ and $\bar{L}_0\phi = \bar{h}\phi$.

Definition 2.2 *A **quasi-primary field with conformal weight** $(h,\bar{h})$ is a field $\phi(z,\bar{z})$ such that for any global conformal transformation $z \to f(z) = \frac{az+b}{cz+d}$ $(ab - cd = 1)$*

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = (\frac{\partial f}{\partial z})^h (\frac{\partial \bar{f}}{\partial \bar{z}})^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{15}$$

Definition 2.3 *A **primary field with conformal weight** $(h,\bar{h})$ is a field ϕ transforming under the same law* (15) *for all (local in addition to global) transformations $z \to f(z)$.*

Remark 5 *From path integral methods and the transformation property* (15)*, we know that if $z \to f(z)$ is a global conformal transformation and ϕ_i are quasi primary fields of conformal weight $(h_i, \bar{h}_i)$ then*

$$\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle = [\prod_{i=1}^{n} (\frac{\partial f}{\partial z})^{h_i}\Big|_{z=z_i} (\frac{\partial \bar{f}}{\partial \bar{z}})^{\bar{h}_i}\Big|_{\bar{z}=\bar{z_i}}] \langle \phi_1(f(z_1), \bar{f}(\bar{z}_1))...\phi_n(f(z_n), \bar{f}(\bar{z}_n))\rangle \tag{16}$$

The same equation applies to primary fields with arbitrary (i.e local in addition to global) conformal transformations f.

2.2 Correlation functions of quasi-primary fields

In a quantum field theory, the objects of interest are the correlation functions $\langle \phi_1(z_1,\bar{z_1})...\phi_i(z_i,\bar{z_i})\rangle$ which are defined as either vacuum expectation values of time ordered products of operators in the operator/canonical formalism, or as path integrals (specifically functional derivatives of the source-dependent partition function of the theory) in the path integral formalism. These definitions are equivalent, but in conformal field theory it is useful to work in the operator formalism while drawing upon things like Ward identities that most naturally are seen from the path integral perspective when they are necessary.

Let's take a first look at how constraining it is for a field theory to be invariant under conformal transformations by looking at 2 and 3 point functions of quasi-primary fields. If we define $g(z,\bar{z},w,\bar{w}) = \langle \phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle$ for quasi-primary fields ϕ_i of conformal weight $(h_i,\bar{h}_i)$ then using (16) with $f(z) = z - b$ (and $\bar{f}(\bar{z}) = \bar{z} - \bar{b}$) we get

$$g(z,\bar{z},w,\bar{w}) = g(z-b, \bar{z}-\bar{b}, w-b, \bar{w}-\bar{b})$$

Specializing to $b = w$ this tells us that g only depends on the differences $z - w$ and $\bar{z} - \bar{w}$: we may write $g(z-w, \bar{z}-\bar{w})$. Now if we consider $f(z) = \lambda z$, $\bar{f}(\bar{z}) = \bar{\lambda}\bar{z}$ we get that

$$g(z-w, \bar{z}-\bar{w}) = \lambda^{h_1+h_2}\bar{\lambda}^{\bar{h}_1+\bar{h}_2} g(\lambda(z-w), \bar{\lambda}(\bar{z}-\bar{w}))$$

Picking $z - w = \bar{z} - \bar{w} = 1$ shows us that

$$g(\lambda, \bar{\lambda}) = \frac{d_{12}}{\lambda^{h_1+h_2}\bar{\lambda}^{\bar{h}_1+\bar{h}_2}}$$

Here $d_{12} := g(1,1)$ (we just use this notation to emphasize that g depends on/is defined in terms of ϕ_1 and ϕ_2). Of course λ and $\bar{\lambda}$ are arbitrary so this tells us that

$$\langle \phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle = g(z-w, \bar{z}-\bar{w}) = \frac{d_{12}}{(z-w)^{h_1+h_2}(\bar{z}-\bar{w})^{\bar{h}_1+\bar{h}_2}} \tag{17}$$

We can apply one more Mobius tranform to our coordinates, namely $f(z) = \frac{-1}{z}, \bar{f}(\bar{z}) = \frac{-1}{\bar{z}}$ to obtain the equation

$$\frac{d_{12}}{(z-w)^{h_1+h_2}(\bar{z}-\bar{w})^{\bar{h}_1+\bar{h}_2}} = z^{-2h_1}w^{-2h_2}\bar{z}^{-2\bar{h}_1}\bar{w}^{-2\bar{h}_1}\frac{d_{12}}{(\frac{-1}{z}-\frac{-1}{w})^{h_1+h_2}(\frac{-1}{\bar{z}}-\frac{-1}{\bar{w}})^{\bar{h}_1+\bar{h}_2}}$$

If $d_{12} \neq 0$ then using $\frac{-1}{z} - \frac{-1}{w} = \frac{z-w}{zw}$ it is simple to see that the above is only satisfied when $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$. So up to some constant d_{12} that vanishes unless conformal weights are equal, we have found the form of all two point functions of quasi-primary fields. We can normalize our fields by some constants to guarantee that $d_{12} = 1$ when the conformal weights agree (and is, like we just noted, 0 when they don't).
We may employ a similar process to conclude that 3 point correlation functions of quasi-primary fields ϕ_i of conformal weights $(h_i, \bar{h}_i)$ take the following form:

$$\langle \phi_1(z_1,\bar{z_1}), \phi_2(z_2,\bar{z_2}), \phi_3(z_3,\bar{z_3})\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}z_{13}^{h_1+h_3-h_2}\bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}z_{23}^{h_2+h_3-h_1}\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}} \tag{18}$$

Where we are using the notation $z_{ij} := z_i - z_j$ and $\bar{z}_{ij} := \bar{z}_i - \bar{z}_j$. N-point functions for $N > 3$ are not constrained to such a particular form, but conformal invariance does impose some conditions on them.

3 Radial quantization and the operator formalism

3.1 'Isomorphism' between the theory on the cylinder and on the Riemann sphere

Consider the cylinder $\mathbb{R} \times S^1$ with coordinates σ_0 and σ_1 (so σ_1 takes values in $[0, 2\pi)$). The map

$$f : \mathbb{R} \times S^1 \to \mathbb{C}^*, \quad (\sigma_0, \sigma_1) \xrightarrow{f} e^{\sigma_0 + i\sigma_1} \tag{19}$$

is a well defined conformal map between the cylinder and the punctured complex plane $\mathbb{C}^*$. In fact, if we adjoin points at $\sigma_0 = \pm\infty$ to the cylinder, we get the suspension of a circle (in the sense of algebraic topology) and the standard relation $SS^1 \cong S^2$ then gives an extension of f from this 'elongated' cylinder to the Riemann sphere.
If we think of σ_0 as the time direction on the cylinder, the circles centered about the origin in $\mathbb{C}$ correspond to constant time slices on the cylinder under f. We will shift perspectives and think of the cylinder as the fundamental object of study, so that the basic physical notions in our theory are derived from the geometry of the cylinder and then are translated via f to $\mathbb{C}$. For example

- equal time slices on the cylinder (σ_0 constant) correspond to circles centered about the origin in the complex plane
- integrals over all of space now become integrals over a circle centered about the origin
- time translations along the cylinder $\sigma_0 \to \sigma_0 + a$ correspond to dilations $z \to e^a z$
- The Hamiltonian, being the generator of time translations on the cylinder, becomes the generator of dilations on $\mathbb{C}$. But it is easy to see that the generator of coordinate dilations on $\mathbb{C}$ is $\ell_0 + \bar{\ell}_0$, so the Hamiltonian is (some multiple of) $L_0 + \bar{L}_0$

3.2 Stress energy tensor and currents of arbitrary conformal transformations

The second bullet point above is crucial in constructing Noether charge's from conserved currents. Usually if we have a conserved current j^μ in some $d+1$ dimensional classical field theory, we get a charge $Q = \int d^dx j^0$ whose conservation is guaranteed by using Gauss's law and $\partial_\mu j^\mu = 0$. Now the prescription above tells us that such charges will be constructed with the use of contour integrals around a circle centered about the origin:

$$Q = \oint_0 d\theta j_r(\theta) \tag{20}$$

One familiar conserved current is the stress energy tensor $T_{\mu\nu}$ that corresponds to translation symmetries. When we move to complex coordinates, $T_{\mu\nu}$ now has components $T_{zz}, T_{z\bar{z}}, T_{\bar{z}z}, T_{\bar{z}\bar{z}}$. Although the stress energy tensor obtained from the Noether method may not be manifestly symmetric, we may add to it the divergence of an appropriate antisymmetric tensor to make it symmetric; such a modification is called the **Belinfante** tensor. There is another important example of a conserved current in a conformal field theory: Dilations form a continuous family of conformal maps, and the current that we get from them (after maybe adding the divergence of an antisymmetric tensor) is $j_\mu = T_{\mu\nu}x^\nu$. Conservation of this current tells us that we have the equation $0 = \partial^\mu(T_{\mu\nu}x^\nu) = T^{\mu\nu}\delta_{\mu\nu} + [\partial^\mu T_{\mu\nu}]x^\nu = T^\mu_\mu + 0$. In this last equality we used that $\partial^\mu T_{\mu\nu} = 0$ since $T_{\mu\nu}$ is itself a conserved current. So we see that the stress energy tensor is traceless. Therefore the components $T_{z\bar{z}}, T_{\bar{z}z}$ are both 0 since they are equal to $\frac{1}{4}T^\mu_\mu$. For notational ease, in complex coordinates we define

Definition 3.1

$$T = T_{zz} \tag{21}$$

$$\bar{T} = T_{\bar{z}\bar{z}} \tag{22}$$

Since the other components vanish, T and $\bar{T}$ capture all aspects of the stress energy tensor. Furthermore, translating the conservation law $\partial^\mu T_{\mu\nu}$ into complex coordinates, and using $T_{z\bar{z}}, T_{\bar{z}z} = 0$ we get the equations $\bar{\partial}T = 0$ and $\partial\bar{T} = 0$ so we see that T only depends on z ('$T(z,\bar{z}) = T(z)$ is holomorphic') and $\bar{T}$ only depends on $\bar{z}$ ('$\bar{T}(z,\bar{z}) = \bar{T}(\bar{z})$ is anti-holomorphic'). Actually there are many more currents that we can form in a conformal field theory using the fact that arbitrary holomorphic maps are conformal. The current associated to an infinitesimal conformal transformation $x^\mu \to x^\mu + \epsilon^\mu$ (ϵ^μ a function of x) can be computed and the result is $T_{\mu\nu}\epsilon^\nu$. Once we shift to complex coordinates, combining this result with (19) gives us that the **conserved charge** associated to an infinitesimal conformal transformation $z \to z + \epsilon(z), \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ is

$$Q_{\epsilon,\bar{\epsilon}} := \frac{1}{2\pi i}\oint_0 dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \tag{23}$$

In a QFT we have a relation $\delta_\epsilon A = [Q_\epsilon, A]$ for any field A that tells us how A varies under an infinitesimal symmetry. Therefore from (23) we find the equation

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_0 dz[T(z)\epsilon(z), \phi(w,\bar{w})] + d\bar{z}[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w})] \tag{24}$$

The correlation functions that we consider in QFT are usually time ordered. The analog of this in radial quantization is the following:

Definition 3.2 *The* ***radially ordered product*** *of two fields A and B is defined*

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |w| < |z| \end{cases} \tag{25}$$

Now if we go back to (23), we may use this definition to write it in a more concise form. Let's just focus on the holomorphic part, since identical considerations apply to the anti-holomorphic part.

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_0 dz[T(z), A(w)] = \frac{1}{2\pi i}\oint_0 dz T(z)A(w) - A(w)T(z) = \\
&\frac{1}{2\pi i}\oint_{|z|=R>w} dz T(z)A(w) - \frac{1}{2\pi i}\oint_{|z|=r<w} dz A(w)T(z) = \\
&\frac{1}{2\pi i}\oint_{|z|=R>w} dz R(T(z)A(w)) - \frac{1}{2\pi i}\oint_{|z|=r<w} dz R(A(w)T(z)) = \\
&\frac{1}{2\pi i}\oint_w dz R(A(w)T(z))
\end{aligned} \tag{26}$$

In the second equality we used Cauchy's theorem to choose the radius of the circle along which we integrate, and then the third equality is recognizing that along these radii R and r, the integrands both are equal to the radially ordered product. Applying this to the antiholomorphic part as well, we obtain

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{w,\bar{w}} dz\epsilon(z)R(T(z)\phi(w,\bar{w})) + d\bar{z}\bar{\epsilon}(\bar{z})R(\bar{T}(\bar{z})\phi(w,\bar{w})) \tag{27}$$

We can use this expression along with the transformation law (15) to obtain an expression for the singular part of $R(T(z), \phi(w,\bar{w}))$ for ϕ a primary field of conformal weight $(h,\bar{h})$ as follows. If we have some infinitesimal transformation $w \to f(w) = w + \epsilon(w), \bar{w} \to \bar{f}(w) = \bar{w} + \bar{\epsilon}(\bar{w})$ then we can compute using binomial expansion that $(\frac{\partial f}{\partial z})^h = (1 + \partial\epsilon(w))^h = 1 + h\partial\epsilon(w) + O(\epsilon^2)$ and similarly for $\bar{f}$. Taylor expanding we also get $\phi(f(w), \bar{f}(\bar{w})) = \phi(w + \epsilon(w), \bar{w} + \bar{\epsilon}(\bar{w})) = \phi(w,\bar{w}) + \partial\phi(w,\bar{w})\epsilon(w) +$

$\bar{\partial}\phi(w,\bar{w})\bar{\epsilon}(\bar{w}) + O(\epsilon^2)$. Hence going back to (15) we see that the new field after the transformation is equal to

$$\begin{aligned}\phi'(w,\bar{w}) =&(1+h\partial\epsilon(w))(1+\bar{h}\bar{\partial}\bar{\epsilon}(\bar{w}))(\phi(w,\bar{w})+\partial\phi(w,\bar{w})\epsilon(w)+\bar{\partial}\phi(w,\bar{w})\bar{\epsilon}(\bar{w})) = \\ &\phi(w,\bar{w})+\partial\phi(w,\bar{w})\epsilon(w)+\bar{\partial}\phi(w,\bar{w})\bar{\epsilon}(\bar{w})+h\phi(w,\bar{w})\partial\epsilon(w)+\bar{h}\phi(w,\bar{w})\bar{\partial}\bar{\epsilon}(\bar{w}) \end{aligned} \tag{28}$$

Therefore we compute that

$$\delta_{\epsilon,\bar{\epsilon}}\phi = \phi'(w,\bar{w})-\phi(w,\bar{w}) = \partial\phi(w,\bar{w})\epsilon(w)+\bar{\partial}\phi(w,\bar{w})\bar{\epsilon}(\bar{w})+h\phi(w,\bar{w})\partial\epsilon(w)+\bar{h}\phi(w,\bar{w})\bar{\partial}\bar{\epsilon}(\bar{w}) \tag{29}$$

Now using Cauchy's integral formula, we can write $\epsilon(w) = \frac{1}{2\pi i}\oint_w dz\frac{\epsilon(z)}{z-w}$, $\partial\epsilon(w) = \frac{1}{2\pi i}\oint_w dz\frac{\epsilon(z)}{(z-w)^2}$ and similarly for $\bar{\epsilon}(\bar{w})$ and $\bar{\partial}\bar{\epsilon}(\bar{w})$. Plugging this into (29) gives

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{w,\bar{w}} dz\epsilon(z)[\frac{h\phi(w,\bar{w})}{(z-w)^2}+\frac{\partial\phi(w,\bar{w})}{z-w}] + d\bar{z}\bar{\epsilon}(\bar{z})[\frac{\bar{h}\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2}+\frac{\bar{\partial}\phi(w,\bar{w})}{\bar{z}-\bar{w}}] \tag{30}$$

Comparing (27) and (30) and then tells us that

$$R(T(z)\phi(w,\bar{w})) = \frac{h\phi(w,\bar{w})}{(z-w)^2}+\frac{\partial\phi(w,\bar{w})}{z-w}+... \tag{31}$$

$$R(\bar{T}(\bar{z}))\phi(w,\bar{w}) = \frac{\bar{h}\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2}+\frac{\bar{\partial}\phi(w,\bar{w})}{\bar{z}-\bar{w}}+... \tag{32}$$

where the $...'s$ are standard notation for a regular/homolorphic function (no poles). In fact, (31) and (32) could have been taken as the defining property of primary fields instead of (15). Expansions of a product of local fields like this are commonly referred to as **Operator Product Expansions (OPEs)**.

Remark 6 *We could repeat this analysis with the weaker assumption that ϕ is a field with conformal dimension $(h,\bar{h})$. The result is that the $(z-w)^{-2}$ term in the $T\phi$ OPE is $\frac{h\phi(w,\bar{w})}{(z-w)^2}$, and the $(z-w)^{-1}$ term is $\frac{\partial\phi(w,\bar{w})}{z-w}$ but there can be more singular terms.*

Remark 7 *We shall stop writing R to denote radial ordering and instead have an implicit understanding that all products of operators that we write are radially ordered.*

3.3 OPE of stress energy tensor with itself

Let's expand the chiral part of the energy momentum tensor as follows:

$$T(z) = \sum_{m\in\mathbb{Z}} z^{-m-2}T_m \tag{33}$$

What are these T_m? Picking $\epsilon(z) = z^{m+1}$ in (23) we see that T_m is equal to the conserved charge associated to the local conformal transformation $z \to z+\epsilon(z)$, which is nothing other than the Virasoro algebra element L_m. Hence we have

$$T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \tag{34}$$

This realization of the energy momentum tensor along with the defining relations of the Virasoro algebra (12) allows us to compute the OPE of T with itself. The idea is to express $[L_n, L_m]$ in two ways, one of them being

$$[L_n, L_m] = (\frac{1}{2\pi i})^2\oint_0 dw\oint_w dz w^{m+1}z^{n+1}T(w)T(z) \tag{35}$$

and the other obtained in a similar fashion by writing

$$\begin{aligned}[L_n, L_m] =&(n-m)L_{n+m}+\frac{\mathbf{c}}{12}\delta_{n,-m}(n^3-n) = \\ &\frac{1}{2\pi i}\oint_0 dww^{m+1}[\frac{1}{12}\mathbf{c}(n^3-n)w^{n-2}+2(n+1)w^nT(w)+w^{n+1}\partial T(w)]\end{aligned} \tag{36}$$

Since this holds for every value of m we must have

$$\frac{1}{2\pi i}\oint_w z^{n+1}T(w)T(z) = \frac{1}{12}\mathbf{c}(n^3-n)w^{n-2}+2(n+1)w^nT(w)+w^{n+1}\partial T(w) \tag{37}$$

Finally we notice that ***for n*** $\neq -1,0,1$

$$\frac{1}{2\pi i}\oint_w dzz^{n+1}\frac{\mathbf{c}}{2(z-w)^4} = \frac{1}{12}\mathbf{c}(n^3-n)w^{n-2} \tag{38}$$

$$\frac{1}{2\pi i}\oint_w dzz^{n+1}\frac{2T(w)}{(z-w)^2} = 2(n+1)w^nT(w) \tag{39}$$

$$\frac{1}{2\pi i}\oint_w dzz^{n+1}\frac{\partial T(w)}{z-w} = w^{n+1}\partial T(w) \tag{40}$$

Using Cauchy's theorem and (37) – (40) we conclude that

$$T(w)T(z) = \frac{\mathbf{c}}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + ... \tag{41}$$

where we must allow for the $...'s$ because of the $n \neq -1, 0, 1$ condition.

Remark 8 *In accordance with remark 6, we conclude that T has conformal dimension* $(2, 0)$.

Actually (41) gives us more than just the conformal dimension of T. With some work, we can extract from it the fact that T is a quasi-primary field. Let's outline how this follows. First we use (27) with $\phi = T$, (41) and Cauchy's integral theorem to say that

$$\delta_\epsilon T(w) = \frac{1}{2\pi i}\oint_w \epsilon(z)T(z)T(w) = \frac{\mathbf{c}}{12}\partial^3\epsilon(w) + 2T(w)\partial\epsilon(w) + \epsilon(w)\partial T(w) \tag{42}$$

We won't go into the details, but (42) may be exponentiated to give

$$T'(z) = (\frac{\partial f}{\partial z})^2 T(f(z)) + \frac{\mathbf{c}}{12}S(f(z), z) \tag{43}$$

where f is any conformal transformation and S is defined to be the **Schwarzian derivative**:

Definition 3.3

$$S(w, z) := \frac{1}{(\partial_z w)^2}((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2) \tag{44}$$

Remark 9 *It is easy to check that $S(f(z), z)$ vanishes when $f(z) = \frac{az+b}{cz+d}$ is a Mobius transformation. Hence for f a global conformal transformation,* (43) *reduces to*

$$T'(z) = (\frac{\partial f}{\partial z})^2 T(f(z)) \tag{45}$$

which is the defining property of a quasi primary field of conformal dimension $(2, 0)$.

3.4 Mode expansions and commutators

We can generalize the calculation on the last page to an arbitrary field as follows.

Definition 3.4 *The **mode expansion** of a field $\phi(z, \bar{z})$ of conformal dimension $(h, \bar{h})$ is a Laurent series*

$$\phi(z, \bar{z}) = \sum_{n,m\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{46}$$

*The $\phi_{m,n}$ are called the **modes** of ϕ and they are operators with no z dependency.*

Remark 10 *We can take a contour integral of* (32) *to pop out any particular mode we want:*

$$(\frac{1}{2\pi i})^2\oint_0 dz\oint_0 d\bar{z}\phi(z, \bar{z})z^{m+h-1}\bar{z}^{n+\bar{h}-1} = \phi_{m,n} \tag{47}$$

Knowing the singular part of the OPE of two fields is equivalent to knowing the commutation relations between the modes of the two fields. We saw one direction of this biconditional above with the calculation of the TT OPE using the commutation relations of the Virasoro modes, and now we will demonstrate the other direction by using (31), the OPE of $T\phi$ where ϕ is primary of conformal dimensions $(h, \bar{h})$, to calculate $[L_m, \phi_n]$. For simplicity of notation we will assume that $\phi(w, \bar{w}) = \phi(w)$ is holomorphic so that in particular $\bar{h} = 0$ and $\phi(w) = \sum_{k\in\mathbb{Z}} w^{-k-h}\phi_k$. Then we can calculate

$$\begin{aligned}
[L_m, \phi_n] =& \frac{1}{(2\pi i)^2}\oint_0 dw\oint_w dz[z^{m+1}w^{n+h-1}T(z)\phi(w)] = \\
& \frac{1}{(2\pi i)^2}\oint_0 dw\oint_w dz[z^{m+1}w^{n+h-1}\frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}] \\
& \frac{1}{2\pi i}\oint_0 dw[w^{n+h-1}h\phi(w)(m+1)w^m + w^{n+h-1}\partial\phi(w)w^{m+1}] = \\
& \frac{1}{2\pi i}\oint_0 dw[(m+1)h\sum_{k\in\mathbb{Z}} w^{n+h-1-k-h+m}\phi_k + \sum_{k\in\mathbb{Z}}(-k-h)w^{n+h-1-k-h-1+m+1}\phi_k] = \\
& (m+1)h\phi_{n+m} + (-m-n-h)\phi_{n+m} = (m(h-1)-n)\phi_{n+m}
\end{aligned} \tag{48}$$

We can define a slightly more general mode expansion that is centered around some arbitrary point w instead of 0 as

Definition 3.5

$$\phi(z, \bar{z}) =: \sum_{n,m\in\mathbb{Z}} (z-w)^{-m-h}(\bar{z}-\bar{w})^{-n-\bar{h}}\phi_{m,n}(w, \bar{w}) \tag{49}$$

3.5 Conformal Ward identity

In the literature I have seen a couple of things referred to as the conformal Ward identity. The first is (27). The second is obtained by considering the following expression:

$$\frac{1}{2\pi i}\oint_C dz\epsilon(z)\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \tag{50}$$

where the ϕ_i are primary and the contour C is some circle large enough to contain all of the points w_i. By a contour deformation, (50) is equal to

$$\begin{aligned}
&\sum_{i=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...\oint_{w_i}dz[\frac{1}{2\pi i}\epsilon(z)T(z)\phi_i(w_i,\bar{w}_i)]...\phi_n(w_n,\bar{w}_n)\rangle = \\
&\sum_{i=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...\delta_\epsilon\phi(w_i,\bar{w}_i)...\phi_n(w_n,\bar{w}_n)\rangle = \\
&\sum_{i=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...(\epsilon(w_i)\partial_{w_i}+h\partial_{w_i}\epsilon(w_i))\phi(w_i,\bar{w}_i)...\phi_n(w_n,\bar{w}_n)\rangle = \qquad (51)\\
&\sum_{i=1}^{n}\langle\phi_1(w_1,\bar{w}_1)...\oint_{w_i}dz\epsilon(z)\Big[(\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i})\phi_i(w_i,\bar{w}_i)\Big]...\phi_n(w_n,\bar{w}_n)\rangle = \\
&\oint_C dz\epsilon(z)\sum_{i=1}^{n}\Big[\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\Big]\langle\phi_1(w_1,\bar{w}_1)...\phi_i(w_i,\bar{w}_i)...\phi_n(w_n,\bar{w}_n)\rangle \qquad (52)
\end{aligned}$$

The first three lines are just using the various expressions that we have come up with for $\delta_\epsilon\phi(w,\bar{w})$ on page 5. The third to the fourth line is a simple application of Cauchy's integral theorem, in fact the same application that got us the $T\phi$ OPE in the first place. The fourth to the fifth line is a contour deformation to change an integral about w_i to an integral about C (allowed to do this because the only poles of the thing being integrated in the *ith* term is at w_i). Comparing (50) and (52) and realizing that they must hold for all $\epsilon(z)$, we get

$$\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \sum_{i=1}^{n}\Big[\frac{h_i}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\Big]\langle\phi_1(w_1,\bar{w}_1)...\phi_i(w_i,\bar{w}_i)...\phi_n(w_n,\bar{w}_n)\rangle \tag{53}$$

Equation (53) is the second thing that I have seen called the conformal Ward identity. The last thing that I have seen called the conformal Ward identity is derived by starting with with the equality of (50) and (51), noticing that (50) is $\delta_\epsilon\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle$, and then using the fact that this quantity must vanish for ϵ corresponding to an infinitesimal global conformal symmetry $\epsilon(z)=\beta+2\alpha z-\gamma z^2$ (this is the first order approximation of the infinitesimal Mobius transformation $\frac{(1+\alpha)z+\beta}{\gamma z+1-\alpha}$). Accordingly, taking ϵ constant, linear and quadratic in (51) yields

$$0=\delta_\epsilon\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \sum_{i=1}^{n}\partial_{w_i}\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \tag{54}$$

$$0=\delta_\epsilon\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \sum_{i=1}^{n}(w_i\partial_{w_i}+h_i)\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \tag{55}$$

$$0=\delta_\epsilon\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \sum_{i=1}^{n}(w_i^2\partial_{w_i}+2w_ih_i)\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle \tag{56}$$

These are the third set of equations that I have seen referred to as the conformal Ward identity. In some sense they are all consequences of (27) so I suppose that this is the 'master' Ward identity.

4 Conformal families and Verma Modules

4.1 The Hilbert space

We are currently in a strange situation where we are mainly working in the canonical/operator formalism to make most of our calculations and definitions, and yet the calculation of the correlation functions and identities that they satisfy (e.g the conformal Ward identity) follow from our path integral intution. It is time to remedy this by going into some detail about the Hilbert space of our theory. First, an axiom. We assume that our Hilbert space has a vacuum vector $|0\rangle$ such that if ϕ is a primary field of conformal dimension $(h,\bar{h})$, then $\phi_{m,n}|0\rangle=0$ if $-h<m$ or $-\bar{h}<n$. In addition, we require that $L_n|0\rangle=0, \bar{L}_n|0\rangle=0$ for $n\geq-1$. Because of this axiom we may write

Definition 4.1 *The **primary state** $|h,\bar{h}\rangle$ associated to a primary field ϕ of conformal weight $(h,\bar{h})$ is*

$$\lim_{z,\bar{z}\to0}\phi(z,\bar{z})|0\rangle = \lim_{z,\bar{z}\to0}\sum_{m,n\in\mathbb{Z}}z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n}|0\rangle = \phi_{-h,-\bar{h}}|0\rangle \tag{57}$$

Remark 11 *The last equality in the above definition follows because for* $(m,n) \neq (-h,-\bar{h})$*, either*

$$\phi_{m,n}|0\rangle = 0$$

by our axiom or

$$\lim_{z,\bar{z}\to 0} z^{-m-h}\bar{z}^{-n-\bar{h}} = 0$$

because $-m-h, -n-\bar{h} > 0$.

Let's check that these states deserve the label $|h,\bar{h}\rangle$ by acting on them with L_0. Using equation (48) with $m=0$ and $n=-h$, we get the following:

$$L_0|h,\bar{h}\rangle = L_0\phi_{-h,-\bar{h}}|0\rangle = [L_0,\phi_{-h,-\bar{h}}]|0\rangle = -(-h)\phi_{-h,-\bar{h}}|0\rangle = h|h,\bar{h}\rangle \tag{58}$$

and similarly for the action of $\bar{L}_0$. The second equality in (58) follows from axioms since $L_0|0\rangle = 0$. Now let's recall equation (12), the defining commutation relation of the Virasoro algebra. It tells us that

$$[L_0, L_{-m}] = mL_{-m} \tag{59}$$

Similarly, (48) tells us that

$$[L_0, \phi_{-m}] = m\phi_{-m} \tag{60}$$

Taking m to be positive in either (59) or (60) tells us then that the action on $|h,\bar{h}\rangle$ with L_{-m} or ϕ_{-m} increases the L_0 eigenvalue (which, in anticipation of the state operator correspondence, we will refer to as the conformal dimension of the state) by m:

$$\begin{aligned} &L_0L_{-m}|h,-\bar{h}\rangle = L_0L_{-m}\phi_{-h,h}|0\rangle = [L_0,L_{-m}]\phi_{-h,-\bar{h}}|0\rangle + L_{-m}L_0\phi_{-h,-\bar{h}}|0\rangle = \\ &mL_{-m}\phi_{-h,-\bar{h}}|0\rangle + L_{-m}h\phi_{-h,-\bar{h}} = (m+h)L_{-m}|h,-\bar{h}\rangle \end{aligned} \tag{61}$$

$$\begin{aligned} &L_0\phi_{-m}|h,-\bar{h}\rangle = L_0\phi_{-m}\phi_{-h,h}|0\rangle = [L_0,\phi_{-m}]\phi_{-h,-\bar{h}}|0\rangle + \phi_{-m}L_0\phi_{-h,-\bar{h}}|0\rangle = \\ &m\phi_{-m}\phi_{-h,-\bar{h}}|0\rangle + \phi_{-m}h\phi_{-h,-\bar{h}} = (m+h)\phi_{-m}|h,-\bar{h}\rangle \end{aligned} \tag{62}$$

Definition 4.2 *We can repeatedly apply negative Virasoro (or primary field) modes to* $|h,\bar{h}\rangle$ *to obtain* ***descendent states***

$$L_K|h,\bar{h}\rangle := L_{-k_1}...L_{-k_n}|h,\bar{h}\rangle \tag{63}$$

By the same calculation we did in (61) *and some induction we obtain that the conformal dimension of a descendent state is* $h+k_1+...+k_n$. *We make the assumption that* $0 \leq k_1 \leq ... \leq k_n$.

We will take the collection of primary states and their descendents to be a basis for our Hilbert space. This explains the $k_1 \leq ... \leq k_n$ condition that we have imposed on descendent states; a primary state acted on by negative Virasoro modes in an arbitrary order can always be written as a linear combination of these ordered descendent states by making use of the Virasoro algebra. If a primary state is acted on by a positive Virasoro mode, it is annhilated. In this manner we ensure that each primary state is the lowest weight state of the subrepresentation that it generates.

Definition 4.3 $V_{(h,\bar{h})} = span\{L_{-k_1}...L_{-k_n}|h,h\rangle | 0 \leq k_1 \leq ... \leq k_n\}$ *is called the* ***Verma module*** *associated to* $|h,\bar{h}\rangle$

Finally we make note that we just have an vector space so far; we haven't defined an inner product on it yet, so it is not a Hilbert space. We delay this construction to the next section after we talk about descendent fields.

4.2 Normal ordering, descendent fields and state operator correspondence

We now talk about normal ordering which will be useful in section 5 when we deal with explicit examples of conformal field theories, and also connects to the notion of descendent fields that we will define later in this subsection. Normal ordering is usually first seen in the context of free fields in QFT where it manifests as the prescription of putting annhilation operators on the right. This normal ordering will be more general in the sense that it applies to arbitrary field theories, not just free ones, but it will be less general in the sense that it only defines the normal ordering for fields inserted at the same point. Let's give a definition (we assume holomorphic fields for simplicity):

Definition 4.4 *We define the normal ordering of two fields* $\phi_1(z)$ *and* $\phi_2(w)$ *as*

$$:\phi_1\phi_2:(w) = \mathcal{N}(\phi_1\phi_2)(w) = (\phi_1\phi_2)(w) \equiv \frac{1}{2\pi i}\oint_w dz \frac{\phi_1(z)\phi_2(w)}{z-w} \tag{64}$$

If we write the OPE of ϕ_1 *and* ϕ_2 *as*

$$\phi_1(z)\phi_2(w) = \sum_{n=-\infty}^{N} (z-w)^{-n}\{\phi_1\phi_2\}_n(w) \tag{65}$$

then plugging this into the definition of normal ordering gives us

$$\mathcal{N}(\phi_1\phi_2)(w) = \{\phi_1\phi_2\}_0(w) \tag{66}$$

Remark 12 *By mode expanding ϕ_1 and ϕ_2 in (64) we can obtain*

$$\mathcal{N}(\phi_1\phi_2)_m(w) = \sum_{n\leq -h_1} (\phi_1)_n(w)(\phi_2)_{m-n}(w) + \sum_{n>h_1} (\phi_2)_{m-n}(w)(\phi_1)_n(w) \tag{67}$$

Using (64) and (65) we can calculate

$$(\partial^k\phi_1\phi_2)(w) = \frac{1}{2\pi i}\oint_w dz \frac{\partial_z^k \phi_1(z)\phi_2(w)}{z-w} = \frac{1}{2\pi i}\oint_w dz \frac{\partial_z^k \sum_{n=-\infty}^{N}(z-w)^{-n}\{\phi_1\phi_2\}_n(w)}{z-w} =$$
$$\frac{1}{2\pi i}\oint_w dz \frac{k!\{\phi_1\phi_2\}_{-k}(w)}{z-w} = k!\{\phi_1\phi_2\}_{-k}(w) \tag{68}$$

Therefore we can write the OPE of ϕ_1 and ϕ_2 as

$$\phi_1(z)\phi_2(w) = \sum_{n=1}^{N}(z-w)^{-n}\{\phi_1\phi_2\}_n(w) + \sum_{n=0}^{\infty}\frac{(z-w)^n}{n!}(\partial^n\phi_1\phi_2)(w) \tag{69}$$

Definition 4.5 *We define*

$$\phi^{(-n)}(w) = (L_{-n}\phi)(w) = \{T\phi\}_{-n+2}(w) \tag{70}$$

Remark 13 *This last equality in Definition 4.5 is true, but it is not a definition - it needs to be derived.*

We can iterate the construction of $\phi^{(-n)}$ to get **descendent fields**

$$\phi^{(-k_1,\ldots,-k_n)} \equiv (\phi^{(-k_2,\ldots,-k_n)})^{(-k_1)} \tag{71}$$

Let's show that the descendent fields deserve their names. We will show by induction that

$$\phi^{(-k_1,\ldots,-k_n)}(0)\,|0\rangle = L_{-k_1}...L_{-k_n}\,|h\rangle \tag{72}$$

Assuming that it is true for the descendent field $\phi^{(-k_1,\ldots,-k_{n-1})}$, we get that

$$\phi^{(-k_1,\ldots,-k_n)}(0)\,|0\rangle = \{T\phi^{(-k_2,\ldots,-k_n)}\}_{-k_1+2}(0)\,|0\rangle = \frac{1}{2\pi i}\oint dz z^{1-k_1}T(z)\phi^{(-k_2,\ldots,-k_n)}(0)\,|0\rangle =$$
$$\frac{1}{2\pi i}\oint dz z^{1-k_1}T(z)L_{-k_2}...L_{k_n}\,|h\rangle = L_{-k_1}...L_{-k_n}\,|h\rangle \tag{73}$$

So we have proved (72) which tells us that the descendent fields create the descendent states.
In section 2, it may have seemed strangely restrictive that we calculated correlation functions for quasi primary fields only. Now we will demonstrate how a correlator involving descendant fields can be expressed in terms of correlators of primaries. For example, say that ϕ_i $(1 \leq i \leq n)$ are primaries of weight h_i (assume holomorphic for simplicity) and we want to know the correlator $\langle\phi_1^{(-k)}(w_1)\phi_2(w_2)...\phi_n(w_n)\rangle$. We can calculate

$$\langle\phi_1^{(-k)}(w_1)\phi_2(w_2)...\phi_n(w_n)\rangle = \frac{1}{2\pi i}\oint_{w_1} dz(z-w_1)^{1-k}\langle T(z)\phi_1(w_1)\phi_2(w_2)...\phi_n(w_n)\rangle$$
$$\frac{-1}{2\pi i}\sum_{i=2}^{n}\oint_{w_i} dz(z-w_1)^{1-k}\langle\phi_1(w_1)...T(z)\phi_i(w_i)...\phi_n(w_n)\rangle$$
$$\frac{-1}{2\pi i}\sum_{i=2}^{n} dz(z-w_1)^{1-k}\langle\phi_1(w_1)...[\frac{h_i\phi_i(w_i)}{(z-w_i)^2} + \frac{\partial_{w_i}\phi_i(w_i)}{z-w_i}]...\phi_n(w_n)\rangle$$
$$\frac{-1}{2\pi i}\sum_{i=2}^{n}\oint_{w_i} dz(z-w_1)^{1-k}[\frac{h_i}{(z-w_i)^2} + \frac{\partial_{w_i}}{z-w_i}]\langle\phi_1(w_1)...\phi_n(w_n)\rangle$$
$$-\sum_{i=2}^{n}\Big[(1-k)h_i(w_i-w_1)^{-k} + (w_i-w_1)^{1-k}\partial_{w_i}\Big]\langle\phi_1(w_1)...\phi_n(w_n)\rangle \tag{74}$$

The first line is inserting (70) into the correlator. The second line is viewing the contour tightly wound around w_1 instead as a contour going the other way around wound around the complement of a neighborhood of w_1. The third line is inserting the $T\phi_i$ OPE into the equation, and then we just do residue calculus to get the final result. For simplicty of notation we define

Definition 4.6 *Let $\mathcal{L}_{-k}$ be the differential operator appearing in* (74) *:*

$$\mathcal{L}_{-k} = \sum_{i=2}^{n}(k-1)h_i(w_i-w_1)^{-k} - (w_i-w_1)^{1-k}\partial_{w_i} \tag{75}$$

Taking $k=1$ in (75), we see that the first term vanishes and the second term is $-\sum_{i=2}^{n}\partial_{w_i}$. By translation invariance of correlators ((16) with $f=z-b$) we know that $\sum_{i=1}^{n}\partial_{w_i}$ annhilates any correlator of n fields inserted at the w_i, and hence when acting on correlators we have that the action of $\mathcal{L}_{-1}$ is the same as the action of ∂_{w_1}. We can also iterate the procedure (74) to derive more correlators of more complicated descendents such as

$$\langle\phi_1^{(-k_1,\dots,-k_p)}(w_1)\dots\phi_n(w_n)\rangle=\mathcal{L}_{-k_1}\dots\mathcal{L}_{-k_p}\langle\phi_1(w_1)\dots\phi_n(w_n)\rangle \tag{76}$$

Definition 4.7 *The map $L_{-k_1}\dots L_{-k_n}|h_\phi\rangle\to\phi^{(-k_1,\dots,-k_n)}$ is called the **state operator correspondence** and is an injective linear map when extended to the vector spaces that these elements span.*

We define

Definition 4.8 *The Hermitean conjugate of a quasi-primary field $\phi(z,\bar{z})$ is*

$$\phi^\dagger=\bar{z}^{-2h}z^{-2\bar{h}}\phi(\frac{1}{\bar{z}},\frac{1}{z}) \tag{77}$$

The factor $\bar{z}^{-2h}z^{-2\bar{h}}$ is present to ensure that the following quantity is well defined. One can check using the mode expansion of a field that this condition is equivalent to imposing $\phi^\dagger_{m,n}=\phi_{-m,-n}$.

Definition 4.9 *The inner product of two vectors $L_{-k_1}\dots L_{-k_n}|h_1,\bar{h}_1\rangle$ and $L_{-s_1}\dots L_{-s_r}|h_2,\bar{h}_2\rangle$ is defined by*

$$\langle h_1,\bar{h}_1|\,L_{k_n}\dots L_{k_1}L_{-s_1}\dots L_{-s_r}\,|h_2,\bar{h}_2\rangle \tag{78}$$

which is evaluated by

- *using the Virasoro algebra repeatedly to put the positive modes on the right and negative modes on the left*
- *imposing $L_n|h\rangle=0$ for $n>0$, $\langle h|L_m=0$ for $m<0$, and $L_0|h\rangle=h|h\rangle$*
- *using*

$$\langle h_1,\bar{h}_1||h_2,\bar{h}_2\rangle=\lim_{z,\bar{z},w,\bar{w}\to 0}\langle 0|\,w^{-2\bar{h}_1}\bar{w}^{-2h_1}\phi_1(\frac{1}{\bar{w}},\frac{1}{w})\phi_2(z,\bar{z})\,|0\rangle \tag{79}$$

Plugging the two point function (17) in the last bullet point shows that the limit is well defined.

Remark 14 *We are calling the above an inner product, but all that we really know about it right now is that it is a Hermitean bilinear form. We need states to have non-negative norms with respect to this form in order for the theory to be unitary.*

4.3 Kac determinant and restrictions imposed by unitarity

Now we turn our attention to the Verma modules $V_{\mathbf{c},h}$ and see what restrictions are imposed on h and $\mathbf{c}$ in order for the theory to be unitary. The primary state in a Verma module is labeled $|h\rangle$, and we described how to take inner products between all of its descendants $L_K|h\rangle$ in Definition 4.8. Recall that the conformal dimension (a.k.a L_0 eigenvalue) of a descendent state $L_{-k_1}\dots.L_{-k_n}|h\rangle$ is $h+k_1+\dots+k_n$.

Definition 4.10 *The **level** of the descendent state $L_{-k_1}\dots.L_{-k_n}|h\rangle$ is defined to be $k_1+\dots+k_n$.*

Remark 15 *A level N descendent of $|h\rangle$ has conformal dimension $h+N$.*

If $N\neq M$, a level N and level M descendent of $|h\rangle$ will be orthogonal. This follows from an inductive argument, passing positive modes in the inner product formula (78) to the right until we get a sum of terms which all vanish:

- Base case:

$$\begin{aligned}&\langle h|\,L_{k_n}\dots L_{k_1}L_{-1}\,|h\rangle=\langle h|\,L_{k_n}\dots L_{k_2}\Big(L_{-1}L_{k_1}+[L_{k_1},L_{-1}]\Big)\,|h\rangle=\\&\langle h|\,L_{k_n}\dots L_{k_2}\Big((k_1+1)L_{k_1-1}+((-1)^3-(-1))\frac{\mathbf{c}}{12}\delta_{k_1,1}\Big)\,|h\rangle=\\&(k_1+1)\,\langle h|\,L_{k_n}\dots L_{k_2}L_{k_1-1}\,|h\rangle\end{aligned} \tag{80}$$

  The second equality follows since $L_{k_1}|h\rangle=0$ and using (12), the definition of the Virasoro algebra. This last expression vanishes because if $k_1>1$ then L_{k_1-1} is a positive mode, and if $k_1=1$ then since by assumption $k_1+\dots+k_n\neq 1$ (we are proving the claim for inner products of vectors at different levels) we must have $n>1$ so the first term vanishes because it is $\langle h|\,L_{k_n}\dots L_{k_2}(k_1+1)h\,|h\rangle$.

- Assume that the claim has been proved for all descendents of level N and below, and say that we have states $L_{-j_1}\dots L_{-j_d}|h\rangle$ with $j_1+\dots+j_d=N+1$ and $L_{-k_1}\dots L_{-k_n}|h\rangle$ with $k_1+\dots+k_n=M\neq N+1$

- We calculate

$$\langle h| L_{k_n}...L_{k_1} L_{-j_1}...L_{-j_d} |h\rangle =$$
$$\langle h| L_{k_n}...L_{k_2}[(k_1+j_1)L_{k_1-j_1} + (k_1^3 - k_1)\frac{\mathbf{c}}{12}\delta_{k_1,j_1}]L_{-j_2}...L_{j_d} |h\rangle$$
$$(k_1+j_1)\langle h| L_{k_n}...L_{k_2} L_{k_1-j_1} L_{-j_2}...L_{j_d} |h\rangle + (k_1^3 - k_1)\frac{\mathbf{c}}{12}\delta_{k_1,j_1}\langle h| L_{k_n}...L_{k_2} L_{-j_2}...L_{j_d} |h\rangle \tag{81}$$

If $k_1 = j_1$, then the first and second terms are proportional to $\langle h| L_{k_n}...L_{k_2} L_{-j_2}...L_{-j_d} |h\rangle$ which vanishes by the induction hypothesis. If $k_1 \neq j_1$ then the second term automatically vanishes, and the first term vanishes by the induction hypothesis since we can lump $L_{k_1-j_1}$ with the positive or negative modes depending on the sign of $k_1 - j_1$.

Therefore if we make a basis for the (infinite dimensional) Verma module $V_{\mathbf{c},h}$ consisting of vectors with well defined levels, the 'Gram' matrix - the matrix of all inner products between basis elements - will be block diagonal with blocks corresponding to the levels. We put quotes around Gram, because returning to remark 14 we just have a Hermitean bilinear form and not necessarily an inner product. In order for our theory to be unitary, it is necessary that this form be a true inner product, i.e it cannot admit negative norm states. Technically 0 norm states are not allowed either, but if the Hermitean bilinear form is positive semi-definite there is the obvious construction of a new vector space with a positive definite bilinear form by quotienting by the span of all zero norm vectors. From basic linear algebra, it is easy to see that the 'Gram' matrix of some Hermitean bilinear form on a vector space tells us information about whether the Hermitean bilinear form is an inner product or not; if the eigenvalues of the Gram matrix are all positive, it is an inner product, if the eigenvalues are all non-negative it is positive semi-definite, and if there are negative eigenvalues then there are negative norm states and unitarity is spoiled. So we would like to know something about the determinant of the Gram matrix (we will stop putting quotes around it now), since this would tell us about the product of all the eigenvalues . Actually, this discussion applies to finite dimensional matrices which our matrix is not, but each of the blocks comprising it are so we will examine the determinants of these level blocks.

Definition 4.11 *There is a formula that tells us everything we want to know about the determinant of the level N block $M^{(N)}$ of the Gram matrix, which is termed the **Kac determinant**.*

$$\det(M^{(N)}) = \alpha_N \prod_{r,s\geq 1, rs\leq N} [h - h_{r,s}(\mathbf{c})]^{p(N-rs)} \tag{82}$$

where $h_{r,s}(\mathbf{c})$ satisfies the following defining set of equations:

$$c = 1 - \frac{6}{m(m+1)} \tag{83}$$
$$h_{r,s}(\mathbf{c}) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \tag{84}$$

p is the partition counting function of an integer, and α_N is a positive constant whose particular value doesn't matter to us.

We won't go into the details, but here are some results that can be derived from (82) – (84) and/or other more basic considerations.

- $V_{(\mathbf{c},h)}$ has negative norm states if c and h are not both non-negative.
- If $c \geq 1$ and $(h \geq 0)$ then $V_{(\mathbf{c},h)}$ is unitary.
- If $c < 1$ and $c \neq 1 - \frac{6}{m(m+1)}$ for any integer m then $V_{(\mathbf{c},h)}$ contains negative norm states.
- If $c = 1 - \frac{6}{m(m+1)} < 1$ for an integer m but $h \neq h_{r,s}(\mathbf{c})$ for some integers r, s such that $1 \leq r < m$ and $1 \leq s < r$ then $V_{(\mathbf{c},h)}$ contains negative norm states.
- The last remaining possibility is that $c = 1 - \frac{6}{m(m+1)} < 1$ for an integer m and $h = h_{r,s}(\mathbf{c})$ for some integers r, s such that $1 \leq r < m$ and $1 \leq s < r$. It is indeed true in this case that $V_{(\mathbf{c},h)}$ will not contain negative norm states, although this does not follow from (82) – (84) or elementary considerations. Of course having $h = h_{r,s}(\mathbf{c})$ implies the existence of zero norm states by the formula for the Kac determinant (82), so $V_{(\mathbf{c},h)}$ will not be unitary in this case, but like we mentioned above we can mod out by the span of the zero norm states to obtain a unitary representation. We call this quotientied representation space $M_{\mathbf{c},h}$.

4.4 Fusion rules

As was mentioned twice in the last section, it is a possibility that we obtain states with zero norm in a Verma module $V_{(\mathbf{c},h)}$. In fact we know that this happens exactly when the Kac determinant (82) vanishes. Looking more closely at (82), we notice that it tells us not only which values of h

will yield a representation with zero norm states, but also which level these states first appear at since $p(N-rs)=0$ for $rs>N$. Hence $V_{(\mathbf{c},h_{r,s}(\mathbf{c}))}$ contains its first zero norm state at level $N=rs$. Furthermore, since $p(0)=1$ there is exactly 1 zero eigenvalue at this level so there is 1 zero norm state of level rs.

Definition 4.12 *If $h=h_{r,s}(\boldsymbol{c})$ then the unique (up to scaling) vector $|\chi\rangle$ at level rs with zero norm in $V_{\boldsymbol{c},h}$ is called the* ***singular vector*** *or* ***null vector****.*

The existence of null vectors in a Verma module has significant consequences because it can be shown that the null vector is itself a primary state and hence generates a sub representation inside of the Verma module: for $n>0$ and $|\chi\rangle$ a null vector we have

$$L_n\,|\chi\rangle = 0 \tag{85}$$

Of course a null vector, being a state at level rs, is itself a (linear combination of) descendent states of the primary state $|h\rangle$ that generates the whole Verma module $V_{(\mathbf{c},h)}$. Let's see this in action with a simple example. Writing the most general possible level 2 state $(L_{-2}+aL_{-1}^2)\,|h\rangle$ and using the Virasoro algebra to write

$$\begin{aligned} L_1(L_{-2}+aL_{-1}^2)\,|h\rangle &= (3+2a+2ha)L_{-1}\,|h\rangle \\ L_2(L_{-2}+aL_{-1}^2)\,|h\rangle &= (\frac{\mathbf{c}}{2}+4h+6ha)\,|h\rangle \\ L_n(L_{-2}+aL_{-1}^2)\,|h\rangle &= 0 \text{ for n} \geq 3 \end{aligned} \tag{86}$$

we find that the condition for $(L_{-2}+aL_{-1}^2)\,|h\rangle$ to be a primary state amounts to $3+2a+2ha=0$ and $\frac{\mathbf{c}}{2}+4h+6ha=0$. The solution of these equations, assuming $\mathbf{c}$ is just some fixed number is

$$h = \frac{5-\mathbf{c}\pm\sqrt{(\mathbf{c}-1)(\mathbf{c}-25)}}{16} \tag{87}$$

$$a = \frac{-3}{2(2h+1)} \tag{88}$$

If we compare (87) with (83) + (84) we find that these values of h correspond to $h=h_{1,2}(\mathbf{c})$ and $h=h_{2,1}(\mathbf{c})$ which is a good consistency check: we have found explicit null vectors at level 2 promised by the vanishing of the level 2 Kac determinant. Now that we have explicit forms for the singular vectors, we can use the state operator correspondence (Definition 4.7) and the fact that $|\chi\rangle$ being primary means that it is orthogonal to the whole Verma module $V_{\mathbf{c},h}$ to get

$$0 = \langle\chi(z)X\rangle = \langle[\phi^{(-2)}(z) - \frac{3}{2(2h+1)}\phi^{(-1,-1)}(z)]X\rangle = [\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2]\langle\phi(z)X\rangle \tag{89}$$

where X is a string of quasi-primary fields. This is the simplest example of a **BPZ equation**. BPZ equations are differential equations obeyed by correlators involving a degenerate field, and they all arise in this same manner. Taking X to be a single quasi-primary field so that $\langle\phi(z)X\rangle$ is a 2 point function does not yield anything interesting since the formula (17) automatically satisfies (89). However if we take $X=\phi_1(z_1)\phi_2(z_2)$ for quasi primary (holomorphic, for simplicity) ϕ_1 and ϕ_2 with conformal weight h_1 and h_2, we get a non-trivial equation for the 3 point correlation function (18). When all is said and done, the solution can be written as follows. We define

$$h(\alpha) := \frac{\mathbf{c}-1}{24} + \frac{1}{4}\alpha^2 \tag{90}$$

Then say $h_{2,1}(\mathbf{c})=h(\alpha_{2,1})$, $h_{1,2}(\mathbf{c})=h(\alpha_{1,2})$, $h_1=h(\alpha_1)$ and $h_2=h(\alpha_2)$. Requiring (89) to hold (if $\langle\phi(z)\phi_1(z_1)\phi_2(z_2)\rangle$ does not vanish) we must have $\alpha_2=\alpha_1\pm\alpha_{1,2}$ or $\alpha_2=\alpha_1\pm\alpha_{2,1}$ (depending on whether we chose $h=h_{1,2}$ or $h=h_{2,1}$ for our Verma module). Let's see what this requirement can tell us about OPE's of primary fields. Say that

$$\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1) = \sum_{p,\{k,\bar{k}\}} C_{\phi\phi_1}^{\phi_p\{k,\bar{k}\}}(z-z_1)^{h_p-h-h_1+K}(\bar{z}-\bar{z}_1)^{\bar{h}_p-\bar{h}-\bar{h}_1+\bar{K}}\phi_p^{\{k,\bar{k}\}}(z_1,\bar{z}_1) \tag{91}$$

is the OPE of ϕ with ϕ_1. At first glance (91) seems like an arbitrary expression, but it is simply the most general possible sum of fields that could occur in the OPE of these fields multiplied by factors that make them scale correctly. $\{k,\bar{k}\}$ is a label where k and $\bar{k}$ are strings of integers that dictate which descendant of the primary field ϕ_p we are talking about, and K and $\bar{K}$ denote the level of that descendant. Now we can look at the 3 point function $\langle\phi(z)\phi_1(z_1)\phi_2(z_2)\rangle$ using the RHS and the LHS of (91). Looking at the LHS and remembering our discussion that we just had after (90), we get that

$$\langle\phi(z)\phi_1(z_1)\phi_2(z_2)\rangle \sim \delta_{\alpha_2,\alpha_1\pm\alpha_{1,2/2,1}} \tag{92}$$

Looking that the RHS and recalling that the two point functions $\langle\phi_p^{\{k,\bar{k}\}}(z_2)\phi_2(z)\rangle$ of ϕ_2 with descendants of primaries $\phi_p^{\{k,\bar{k}\}}$ can be expressed as linear differential operators acting on $\langle\phi_p(z_1)\phi_2(z_2)\rangle$, we conclude that $\phi_p^{\{k,\bar{k}\}}$ can only occur in the OPE of ϕ with ϕ_1 if α_p $(h(\alpha_p)=h_p)$ satisfies

$\alpha_p = \alpha_2 = \alpha_1 \pm \alpha_{1,2/2,1}$. So if we write $\phi_{(\alpha)}$ for a primary field of conformal weight $h(\alpha)$ then we get the following

$$\phi_{2,1} \times \phi_1 = \phi_{(\alpha_1+\alpha_{2,1})} + \phi_{(\alpha_1-\alpha_{2,1})} \tag{93}$$
$$\phi_{1,2} \times \phi_1 = \phi_{(\alpha_1+\alpha_{1,2})} + \phi_{(\alpha_1-\alpha_{1,2})} \tag{94}$$

Some explanation is in order of the notation in (93) and (94). The products on the LHS are known as **fusion products** and the RHS denotes which conformal families are allowed to appear in the OPE of the two LHS fields.
The process employed here to come up with the fusion rules of $\phi_{1,2}$ and $\phi_{2,1}$ with other primary fields may be replicated for other values of $h = h_{r,s}(\mathbf{c})$. This particular value of h will yield a null vector $|\chi\rangle$ and a corresponding null field $\chi(z)$ which is a descendant of the primary field $\phi_{r,s}$. Imposing that the correlator of $\chi(z)$ with any other string of fields vanishes gives us differential equations for $n-$point functions that can be solved to give constraints on which fields may appear in the OPE of $\phi_{r,s}$ with other fields. The result is that

$$\phi_{r,s} \times \phi_{(\alpha)} = \sum_{\substack{k=1-r \\ k+r=1 mod 2}}^{k=r-1} \sum_{\substack{\ell=1-s \\ \ell+s=1 mod 2}}^{\ell=s-1} \phi_{(\alpha+k\alpha_{2,1}+\ell\alpha_{1,2})} \tag{95}$$

By taking $\phi_{(\alpha)}$ to be in the conformal family of a primary field ϕ_{r_2,s_2} associated to $h_{r,s}(\mathbf{c})$, relabeling r, s as r_1, s_1 and using commutativity of the OPE along with (95) we get

$$\phi_{r_1,s_1} \times \phi_{r_2,s_2} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 mod 2}}^{k=r_1+r_2-1} \sum_{\substack{\ell=1+|s_1-s_2| \\ \ell+s_1+s_2=1 mod 2}}^{\ell=s_1+s_2-1} \phi_{k,\ell} \tag{96}$$

which is fewer conformal families than we get by blindly applying (95). For example, (95) in the case of $(r_1, s_1) = (1, 2)$ and $(r_2, s_2) = (2, 1)$ gives us

$$\phi_{1,2} \times \phi_{2,1} = \phi_{2,0} + \phi_{2,2} \tag{97}$$

but (96) gives us

$$\phi_{1,2} \times \phi_{2,1} = \phi_{2,2} \tag{98}$$

There is no contradiction between these equations because the RHS of these fusion equations simply denote which conformal families are ALLOWED to appear in the OPE, not which ones are required to.

4.5 Unitary Minimal Models - The Ising Model

Since we are thinking about unitary CFTs, we know from section 4.3 that we must have Verma modules $V_{\mathbf{c},h}$ with $\mathbf{c} = 1 - \frac{6}{m(m+1)}$ $(m \in \mathbb{N})$, $h = h_{r,s}(\mathbf{c})$. We would like to have some examples of unitary CFTs where there are only a finite number of conformal families. Looking at (96) this hope seems far off because if we have one conformal family, we automatically get conformal families with primaries of conformal dimension $h_{r,s}(\mathbf{c})$ with r, s arbitrarily large by using the fusion rules. The way around this is to have more of the truncation phenomenon like (95) – (98) occur where differential equations arising from null vectors of the theory give rise to restrictions in the form of fusion rules. Let's take a look at why this might happen. From equation (84) we can calculate with a bit of algebra

$$h_{r,s} = h_{m-r,m+1-s} \tag{99}$$
$$h_{m+r,m+1-s} = h_{r,s} + rs = h_{m-r,m+1+s} \tag{100}$$
$$h_{r,s} + (m-r)(m+1-s) = h_{r,2m+2-s} = h_{2m-r,s} \tag{101}$$

These equations tell us something interesting - the null vectors themselves are the cyclic vectors of degenerate Verma modules i.e there exist descendants of the null vectors that are null themselves. The result of this is that when we have a Verma module $V_{\mathbf{c},h_{r,s}(\mathbf{c})}$ there is automatically an infinite number of null vectors. For example, from equation (100) we obtain that there are null vectors at levels krs for all integers k, and there are more from equation (101). In chapter 8 of diFrancesco it is shown that the analysis of the differential equations that arise from the vanishing of correlators involving the fields corresponding to these vectors yield a finite number of conformal families that close under fusion. Specifically

Definition 4.13 *The* ***unitary minimal model*** $\mathcal{M}(m+1, m)$ *is a conformal field theory with fields* $\phi_{r,s}$ *of conformal weight* $h_{r,s}(\boldsymbol{c})$ *for* $1 \le r < m$ *and* $1 \le s < r$. *It has fusion rules*

$$\phi_{r_1,s_1} \times \phi_{r_2,s_2} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 mod 2}}^{r_1+r_2-1} \sum_{\substack{\ell=1+|s_1-s_2| \\ \ell+s_1+s_2=1 mod 2}}^{s_1+s_2-1} \phi_{k,\ell} \tag{102}$$

Equation (102) may make it seem like we don't get the promised closure under fusion for the specified r and s values, but this issue is dealt with by employing (99) – (101).

Remark 16 *The notation $\mathcal{M}(m+1,m)$ has to do with the fact that there are non-unitary minimal models indexed by coprime integers p and p' (choosing $p=m+1$ and $p'=m$ is the only way that these are unitary).*

Remark 17 *We haven't showed that such a conformal field theory actually exists, all of this analysis was just at the level of representations of the Virasoro algebra. These theories do exist, though, and they may be realized through the coset construction that we will review later in these notes.*

As a simple example, lets look at $\mathcal{M}(4,3)$ which is termed the **Ising Model**. From the restrictions $1 \leq r < 3$ and $1 \leq s < r$ we get that the primary fields of this model will be

$$\phi_{1,1} =: 1 \tag{103}$$
$$\phi_{2,1} =: \epsilon \tag{104}$$
$$\phi_{2,2} =: \sigma \tag{105}$$

From the formula $c = 1 - \frac{6}{m(m+1)}$ we calculate $c = 1 - \frac{6}{12} = \frac{1}{2}$, and from the formula $h_{r,s} = \frac{[(m+1)r-ms]^2-1}{4m(m+1)}$ we can calculate that the conformal dimensions of the fields (103) – (105): for 1 we get 0, for ϵ we get $\frac{1}{16}$ and for σ we get $\frac{1}{2}$. Applying (102) to $\mathcal{M}(4,3)$ we see that

$$1 \times 1 = 1 \tag{106}$$
$$1 \times \sigma = \sigma \tag{107}$$
$$1 \times \epsilon = \epsilon \tag{108}$$
$$\epsilon \times \epsilon = 1 \tag{109}$$
$$\epsilon \times \sigma = \sigma \tag{110}$$
$$\sigma \times \sigma = 1 + \epsilon \tag{111}$$

Notice that the full symmetry algebra of a CFT is the direct sum of two commuting copies of the Virasoro algebra, and since the analysis in this section was for the representation theory of a single copy of the Virasoro algebra we expect that our CFT Hilbert space will decompose into a sum of tensor products of these representations.

5 Examples of CFTs

Now we go through some basic examples of CFT's. These CFT's and the ones that we will consider in the WZW model section arise from a Lagrangian, but this need not be the case. For example, when we do the coset construction it will not be the case that there is a readily avaliable Lagrangian description at hand.

5.1 Free boson

Consider the theory of a massless free boson on the plane with action $S = \frac{1}{4\pi\ell_s^2}\int d^2\sigma \partial_\alpha X \partial^\alpha X$. When we move to complex coordinates this action becomes

$$S = \frac{1}{2\pi\ell_s^2}\int d^2z \partial_z X \partial_{\bar{z}} X \tag{112}$$

At a classical level, we see that the equations of motion are $\bar{\partial}\partial X = 0$ and hence X is a sum of a holomorphic function and and anti holomorphic function. From standard QFT procedures, in the canonical formalism it is simple to obtain the following results:

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\ell_s^2}{2}[\ln(z-w) + \ln(\bar{z}-\bar{w})] \tag{113}$$
$$T(z) = -\frac{1}{\ell_s^2} : \partial X \partial X : (z) \tag{114}$$
$$\bar{T}(\bar{z}) = -\frac{1}{\ell_s^2} : \bar{\partial} X \bar{\partial} X : (\bar{z}) \tag{115}$$

We can take derivatives with respect to z and w of (113) to obtain

$$\langle \partial X(z) \partial X(w)\rangle = -\frac{\ell_s^2}{2}\frac{1}{(z-w)^2} \tag{116}$$

(116) is in the correct form (17) for a correlation function of a quasi-primary field of conformal weight $(1,0)$, so we start to suspect that ∂X might be quasi-primary or primary. We can verify this

suspicion by find its OPE with $T(z)$ and seeing that it is of the form (31):

$$
\begin{aligned}
T(z)\partial X(w) &= -\frac{1}{\ell_s^2} : \partial X \partial X : (z) \partial X(w) = \frac{-2}{\ell_s^2} \langle \partial X(z) \partial X(w) \rangle \partial X(z) \\
&= \frac{\partial X(z)}{(z-w)^2} = \frac{\partial X(w) + \partial\partial X(w)(z-w) + O(z-w)^2}{(z-w)^2} = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial \partial X(w)}{z-w} + ... \qquad (117)
\end{aligned}
$$

On the first line we used the fact that Wick's theorem works on a time ordered product of a normal ordered field and another field by contracting the fields inside the normal ordering with the field outside the normal ordering, but not between themselves. On the second line we just Taylor expanded. So we have verified that ∂X is a primary field of conformal weight $(1, 0)$. A similar analysis goes through for $\bar{\partial} X$ to show that it is primary of conformal weight $(0, 1)$. Another way to make this conclusion starting from (116) would be as follows. First note that ∂X has classical dimension $(1, 0)$ which is clear from dimensional analysis of the action (112). We are working with a free theory so these are also the conformal dimensions of ∂X in the quantum theory. We could therefore mode expand $j(z) := i\partial X(z)$ as

$$
j(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j_n \qquad (118)
$$

Using equation (67) which expresses the modes of a normal ordered product $\mathcal{N}(AB)$ in terms of the modes of the fields A and B we obtain

$$
L_m = \frac{1}{\ell_s^2} \mathcal{N}(jj)_m = \frac{1}{\ell_s^2} \sum_{n\leq -1} j_k j_{m-k} + \frac{1}{\ell_s^2} \sum_{n>1} j_{m-k} j_k \qquad (119)
$$

Now using (116) we can calculate

$$
[j_m, j_n] = \frac{\ell_s^2}{2} m \delta_{m,-n} \qquad (120)
$$

This calculation goes through in the same way that we calculated the mode algebra from the singular part of an OPE before. Using (119) and (120) we obtain after some calculation

$$
[L_m, j_n] = -n j_{m+n} \qquad (121)
$$

which is (48) with $h = 1$, in agreement with the fact that j (and hence ∂X) is primary of weight $(1, 0)$. Using (119) we can also verify that the modes of T in this theory satisfy the Virasoro algebra with central charge $\mathbf{c} = 1$. We can define a continuous family of primary fields in the free boson theory as follows

Definition 5.1 *The **vertex operator** $V_\alpha(z, \bar{z})$ is defined as*

$$
V_\alpha(z, \bar{z}) = \mathcal{N}(e^{i\alpha X(z,\bar{z})}) = \sum_{i=0^\infty} \frac{(i\alpha)^n}{n!} : X^n : (z, \bar{z}) \qquad (122)
$$

After doing a couple of Wick contractions and Taylor expansions we can obtain

$$
\begin{aligned}
T(z) V_\alpha(w, \bar{w}) &= \frac{\alpha^2 \ell_s^2 V_\alpha(w, \bar{w})}{4(z-w)^2} + \frac{i\alpha \partial V_\alpha(w, \bar{w})}{z-w} + ... \\
\bar{T}(\bar{z}) V_\alpha(w, \bar{w}) &= \frac{\alpha^2 \ell_s^2 V_\alpha(w, \bar{w})}{4(\bar{z}-\bar{w})^2} + \frac{i\alpha \bar{\partial} V_\alpha(w, \bar{w})}{\bar{z}-\bar{w}} + ... \qquad (123)
\end{aligned}
$$

So we see that V_α is a primary field of conformal weight $(\frac{\alpha^2 \ell_s^2}{4}, \frac{\alpha^2 \ell_s^2}{4})$. It is customary to label the states that these vertex operators create as

$$
|\alpha\rangle = \lim_{z,\bar{z}\to 0} V_\alpha(z, \bar{z}) |0\rangle \qquad (124)
$$

instead of $|\frac{\alpha^2 \ell_s^2}{4}\rangle$. The latter notation would of course be more consistent with how we labeled the state created by a primary field of conformal weight h by $|h\rangle$.
Notice that the action (112) is invariant under translations $X \to X + \epsilon$ since it only depends on derivatives of X. From Noether we get that the associated conserved current of this continuous family of symmetries is (up to some normalization constant) simply the current $j(z) = i\partial X(z)$ defined above.

5.2 Free fermion

Here we will be brief and summarize some relevant formulae for the theory of a free fermion. These can be derived analogously from an action as in section 5.1, but we will just state the field content and the relevant OPE's. For the theory of a free fermion we have Grassman fields $\psi(z)$ and $\bar{\psi}(\bar{z})$ that satisfy

$$
\psi(z)\psi(w) = \frac{-1}{z-w} \qquad (125)
$$

$$
\bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) = \frac{-1}{\bar{z}-\bar{w}} \qquad (126)
$$

The holomorphic component of the stress energy tensor is given by

$$T(z) = \frac{1}{2}\mathcal{N}(\psi\partial\psi)(z) \tag{127}$$

and using Wick's theorem (the version for Grassman fields) we get

$$T(z)\psi(w) = \frac{\psi(w)}{2(z-w)^2} + \frac{\partial_w\psi(w)}{z-w} + ... \tag{128}$$

which tells us that ψ is primary and has holomorphic conformal weight $h = \frac{1}{2}$. The corresponding results hold for $\bar{T}$ and $\bar{\psi}$. ψ does not have to be single valued on the plane; it can satisfy either $\psi(e^{2\pi i}z) = \psi(z)$ (called the **Neveu-Schwarz [NS] sector**) or $\psi(e^{2\pi i}z) = -\psi(z)$ (called the **Ramond [R] sector**). Which sector we are in affects whether the mode expansion of ψ goes over integer or half integers.

6 Extended symmetry, duality and modular invariance

This section will develop some formal aspects of CFTs that will be useful when we come to WZW models. This section follows the CFT notes of Fuchs, as well as those of Luis Alvarez Gaume.

6.1 Chiral symmetry algebra

It is often the case that the full symmetry algebra of a conformal field theory is actually larger than just the direct sum of two Virasoro algebras. On general grounds we expect that this maximal symmetry algebra $\mathcal{W}_{total}$ has the following properties:

1. $\mathcal{W}_{total}$ is a Lie algebra.
2. $\mathcal{W}_{total}$ splits into the sum of a a holomorphic and anti holomorphic Lie subalgebras,

$$\mathcal{W}_{total} = \mathcal{W} \oplus \bar{\mathcal{W}}' \tag{129}$$

   each of which contain a copy of the Virasoro algebra. We will focus our attention on $\mathcal{W}$ which we call the **chiral symmetry algebra** of our theory.
3. There is a countable basis for $\mathcal{W}$

$$\mathcal{W} = span(\{W^i_n | i \in \mathbb{Z}_{\geq 0} \wedge n \in \mathbb{Z}\} \cup \{C_\ell \| \ell \in \mathbb{N}\}) \tag{130}$$

   where the C_ℓ are central, and the subscripts provide a $\mathbb{Z}$-grading for $\mathcal{W}$ (the subscripts on the W's; the central terms are all grade 0). This grading condition tells us that many of the structure constants of the algebra must vanish.
4. The Virasoro modes are a part of the basis from 3.:

$$L_n = W^0_n \tag{131}$$

   and we have the relation for $m = 0, \pm 1$

$$[L_m, W^i_n] = ((\Delta_i - 1)m - n)W^i_{m+n} \tag{132}$$

   where Δ_i is a positive integer. Compare to (48). Notice that this relation respects the grading condition from 3.

We take the above list of properties as axioms. If we take $m = 0$ in (132) we get the relation

$$[L_0, W^i_n] = -nW^i_n \tag{133}$$

The $\mathbb{Z}$ grading on $\mathcal{W}$ gives us a decomposition

$$\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^0 \oplus \mathcal{W}^- \tag{134}$$

Where $\mathcal{W}^+$ ($\mathcal{W}^-$) contains all of the positive (negative) sub-scripted W^i_n's and, $\mathcal{W}^0$ contains all of the central terms and the W^i_0's. If we let $\mathcal{W}_0$ be the maximal abelian subalgebra of $\mathcal{W}^0$ containing L_0 (generically smaller than all of $\mathcal{W}^0$) then we can find subalgebras $\mathcal{W}_\pm \subset W^0 \oplus W^\pm$ such that we get a triangular decomposition of $\mathcal{W}$:

$$\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_- \tag{135}$$
$$[\mathcal{W}_\pm, \mathcal{W}_0 \oplus \mathcal{W}_\pm] \subset W_\pm \tag{136}$$
$$[\mathcal{W}, \mathcal{W}_-] \subset \mathcal{W}_0 \tag{137}$$

6.2 State space (Highest weight reps and Verma modules)

Denote the state space of our theory by $\mathcal{H}$. Obviously $\mathcal{H}$ carries a representation of $\mathcal{W}_{total}$ (and hence of $\mathcal{W}$) for this is what it means for $\mathcal{W}_{total}$ to be a symmetry algebra of the theory. We continue the tradition of not using any special notation for the image of an element of $\mathcal{W}$ under the representation homomorphism. We also assume that $\mathcal{H}$ has an inner product $(\ ,\ )$ and that $\mathcal{W}$ is a $*$ algebra such that the representation of $\mathcal{W}$ on $\mathcal{H}$ is unitary: for all $W \in \mathcal{W}$, $(v', Wv) = (W^* v', v)$. In fact, we assume that $*$ takes the form $(W^i_n)^* = W^{\pi(i)}_{-n}$ for some involution $\pi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Lastly we assume that $(L_n)^* = L_{-n}$ (so that $\pi(0) = 0$) and L_0 is diagonalizable with eigenvalues bounded below. Let's examine the consequences of this last assumption. Firstly, (133) tells us that if $v \in \mathcal{H}$ is an eigenvector of L_0 (say $L_0 v = hv$) then

$$L_0(W^i_n v) = (h - n) W^i_n v \tag{138}$$

and hence for n sufficiently large we must have $W^i_n v = 0$ because otherwise we would get arbitrarily large (in magnitude) negative L_0 eigenvalues. Also, for any positive n, there exists $N > 0$ such that $(W^i_n)^N v = 0$ for the same reason. A particular instance of this that we will need later is that L_1^N annhilates v.
The triangular decomposition (135) – (137) of $\mathcal{W}$ allows us to define special types of representations of $\mathcal{W}$.

Definition 6.1 *A* ***highest weight module*** *for $\mathcal{W}$ is a representation space V for $\mathcal{W}$ such that*

- *$V \subset \mathfrak{U}(\mathcal{W}_-)v$ where $\mathfrak{U}(\mathcal{W}_-)$ is the universal enveloping algebra of $\mathcal{W}_-$ (a.k.a the tensor algebra on $\mathcal{W}_-$ modulo the ideal generated by elements of the form $a \otimes b - b \otimes a - [a, b]$), and $\mathfrak{U}(\mathcal{W}_-)v$ means the set of all symbols Wv for $W \in \mathfrak{U}(\mathcal{W}_-)$ (so, as a vector space, isomorphic to $\mathfrak{U}(\mathcal{W}_-)$).*
- *All of the elements in $W' \in \mathcal{W}_-$ act in the obvious way on elements $Wv \in \mathfrak{U}(\mathcal{W}_-)v$:*

$$W'(Wv) := (W' \otimes W)v \tag{139}$$

  For this to make sense, it is clear that the subspace V of $\mathfrak{U}(\mathcal{W}_-)$ needs to be an ideal of the universal enveloping algebra.
- *$\mathcal{W}_+ v = 0$*
- *v is a simultaneous eigenvector for $\mathcal{W}_0$: there exists $\lambda_v : \mathcal{W}_0 \to \mathbb{C}$ such that for all $W_0 \in \mathcal{W}_0$*

$$W_0 v = \lambda(W_0) v \tag{140}$$

Because v is an eigenvector of $L_0 \in W_0$ and we know that the only operators that lower the eigenvalue of a given eigenvector are in $\mathcal{W}_+$. But $\mathcal{W}_+$ annhilates the highest weight module, and therefore we do have that the set of L_0 eigenvalues of a highest weight representation is bounded below. We did not make mention of any bilinear product on our highest weight module. This is because once we specify the value (v, v) (normally just taken to be 1 for simplicity), all of the other products on a highest weight rep follow immediately from the $*$ structure on $\mathcal{W}$ along with the condition that $\mathcal{W}_+$ annhilates the v. It is possible that this product as defined does not constitute an inner product. It could give rise to either null vectors (vectors w of zero norm $(w, w) = 0$) or vectors with negative norm. If the highest weight rep only contains null vectors and not negative norm vectors, we can quotient out by the submodule formed by these states to obtain a representation with a true inner product, i.e a unitary representation. Notice that this whole process of declaring what the adjoints of the operators are by using the $*$ structure of $\mathcal{W}$, and then after-the-fact deciding whether the induced norm is an inner product is backwards compared to what we normally do. We normally have a given inner product on a vector space and use it to define the adjoint operators.

Definition 6.2 *If in definition 6.1 we take V to be the entire space $\mathfrak{U}(W_-)v$, we obtain a* ***Verma module****. Notice then that as a vector space, all Verma modules for $\mathcal{W}$ are the same object and that it is really only the function $\lambda_v : \mathcal{W}_0 \to \mathbb{C}$ that distinguishes different Verma modules from each other.*

Remark 18 *These Verma modules are more general than the Verma modules that we considered in earlier sections which corresponded to the case of $\mathcal{W}$ being the Virasoro algebra.*

In fact any highest weight rep can be obtained as a quotient of a Verma module, so the Verma modules are in some sense the most fundamental highest weight reps. The state space $\mathcal{H}$ of our theory decomposes as a sum of unitary irreducible highest weight reps.

$$\mathcal{H} = \bigoplus_A \mathcal{H}_A \tag{141}$$

This decomposition was an assumption that we were implicitly working with for the first 5 sections - the state space was a sum of highest weight reps with generating vectors given by the states created by primary fields. In addition, one of these so called **sectors** $\mathcal{H}_A$ is distinguished; we denote this sector by $\mathcal{H}_0 = \mathfrak{U}(\mathcal{W}_-)v_0$ and call it the **vacuum sector**. In earlier sections we used the notation

$v_0 = |0\rangle$. $\mathcal{H}_0$ satisfies $\lambda_0 = 0$, i.e it is annhilated by the entire zero mode algebra $\mathcal{W}^0$ (there was a one step argument to get from $\lambda_0 = 0$ to $\mathcal{W}^0$ annhilates v_0 since the domain of λ_0 is $\mathcal{W}_0$, but it is an easy argument: the rest of $\mathcal{W}^0$ belongs to $\mathcal{W}_+$ which already annhilates v_0). In addition, v_0 is annhilated by $L_n = W_n^0$ for $n \geq 1$, and will also generally be annhilated by any operators W_n^i so long as this annhilation is compatible with the bracket structure of $\mathcal{W}$. For a generic sector $\mathcal{H}_A = \mathfrak{U}(\mathcal{W}_-)v_A$ we define $\Delta_A = \lambda_A(L_0)$ to be the conformal weight of v_A.
When $\mathcal{W}$ was just the Virasoro algebra, we were able to use the Kac determinant formula to put constraints on h and c necessary for unitarity of the highest weight representations. In principle we could do a similar analysis of $\mathcal{W}$ by systematically computing the the norms of states. For example, we can show that the conformal weights Δ_A must be non-negative for unitarity to be achieved. Earlier we noted that for any vector $v \in \mathcal{H}$ there is some N such that $L_1^N v = 0$. Let N_A be the smallest such integer for which $L_1^N v_A = 0$. Then $L_1^{N_A-1} v_A \neq 0$ but $L_1(L_1^{N_A-1}v_A) = 0$. The conformal dimension of the descendant $L^{N_A-1}v_A$ is $\Delta_{L_1^{N_A-1}v_A} = \Delta_A - (N-1)$, and hence if $\Delta_A < 0$, we have $\Delta_{L_1^{N_A-1}v_A} < 0$. This is impossible however:

$$\begin{aligned}0 \leq ||L_{-1}L_1^{N_A-1}v_A|| =& (L_{-1}L_1^{N_A-1}v_A, L_{-1}L_1^{N_A-1}v_A) = (L_1L_{-1}L_1^{N_A-1}v_A, L_1^{N_A-1}v_A)\\ & (\cancel{L_{-1}L_1^{N_A}v_A} + [L_1, L_{-1}]L_1^{N_A-1}v, L_1^{N_A-1}v_A) = (2L_0L_1^{N_A-1}v, L_1^{N_A-1}v) = \\ & 2\Delta_{L_1^{N_A-1}v}||L_1^{N_A-1}v||^2 \end{aligned} \tag{142}$$

So we have demonstrated that indeed $\Delta_A \geq 0$.

Definition 6.3 *Any state that is annhilated by* L_1 *is called* ***quasi-primary****.* $L_1^{N_A-1}v_A$ *for example is quasi-primary.*

Earlier in the notes we did not have a notion of quasi-primary states, but we did have a notion of quasi primary fields. Every field, including quasi-primary fields, is a (linear combination of) descendant fields of primaries. Therefore we could have defined quasi-primary states as the descendant states corresponding to these descendant fields, and we would have seen then that the action of L_1 on them vanished.
The state-operator correspondence from previous sections applies equally well to these conformal field theories. However, there are particularly interesting fields that we can get from our extended chiral symmetry algebra.

Definition 6.4 *We define the generating field*

$$W^i(z) = \sum_{n\in\mathbb{Z}} z^{-n-\Delta_i} W_n^i \tag{143}$$

As an example, when we take $i = 0$ we recover the energy momentum tensor T:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n = \sum_{n\in\mathbb{Z}} z^{-n-\Delta_0}W_n^0 = W^0(z) \tag{144}$$

We actually have a way to generalize the notion of primary and fields now. The notion of primary from the previous sections will now be known instead as **Virasoro-primary** so that we can define

Definition 6.5 *A* $\mathcal{W}$***-primary field*** $\phi(z,\bar{z})$ *is one that creates a state* $v_\phi = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})v_0$ *such that* v_ϕ *is the cyclic vector of one of the sectors* $\mathcal{H}_A$*. Therefore* $\mathcal{W}_+ v_\phi = 0$*.*

This explains why quasi-primary fields are sometimes called $\mathfrak{sl}(2)$-primaries, since they are primaries for the $\mathfrak{sl}(2)$ algebra generated by $\{L_0, L_{\pm 1}\}$.

6.3 Conformal blocks

Let's recall (91), the OPE between two primary fields (reindexed to be useful below):

$$\phi_n(z,\bar{z})\phi_m(0,0) = \sum_{p,\{k,\bar{k}\}} C_{nm}^{p\{k,\bar{k}\}} z^{h_p-h_n-h_m+K}\bar{z}^{\bar{h}_p-\bar{h}_n-\bar{h}_m+\bar{K}}\phi_p^{\{k,\bar{k}\}}(0,0) \tag{145}$$

If we use the LHS and RHS to compute the 3 point function $\langle\phi_n(z_1,\bar{z}_1)\phi_m(z_2,\bar{z}_2)\phi_p(z_3,\bar{z}_3)\rangle$, we will obtain that $C_{nm}^{p,\{0,0\}} = C_{nmp}$, i.e the non-descendant structure constants of the OPE's are precisely the same constants that appear in the 3 point functions (18). Furthermore, the coefficients $C_{nm}^{p\{k,\bar{k}\}}$ for the descendants are completely determined by C_{nmp}, $\mathbf{c}$ and the conformal dimensions of ϕ_n, ϕ_m and ϕ_p. Therefore knowing the 3 point function coefficients of primary field is equivalent to knowing the OPE's of all fields in the theory (the OPE's of descendants follow from those of the primaries), and in turn this is equivalent to knowing all of the correlators of the theory (we can use the OPE's to reduce any correlator to 2-point functions). So we would like to know these 3 point function coefficients. In fact, it turns out to be useful to study the 4 point functions of primary fields $\langle\phi_k(z_1,\bar{z}_2)\phi_l(z_2,\bar{z}_2)\phi_n(z_3,\bar{z}_3)\phi_m(z_4,\bar{z}_4)\rangle$ in order to derive restrictive equations that these C_{ijk}'s must satisfy. The first thing that we do is use Mobius transformations to fix 3 out of 4 of the points in our function to be $1, \infty$ and 0: we get

Definition 6.6

$$G(z,\bar{z}) := \langle \phi_k(\infty,\infty)\phi_l(1,1)\phi_n(z,\bar{z}), \phi_m(0,0)\rangle = \langle k|\, \phi_l(1,1)\phi_n(z,\bar{z})\,|m\rangle \tag{146}$$

Using (145) + (146) we obtain

$$G(z,\bar{z}) = \sum_{p,\{k,\bar{k}\}} C_{nm}^{p\{k,\bar{k}\}} z^{h_p-h_n-h_m+K}\bar{z}^{\bar{h}_p-\bar{h}_n-\bar{h}_m+\bar{K}} \langle k|\,\phi_l(1,1)\phi_p^{\{k,\bar{k}\}}(0,0)\,|0\rangle =$$

$$\sum_p C_{pnm}C_{klp}\mathcal{F}_{nm}^{lk}(p|z)\overline{\mathcal{F}_{nm}^{lk}(p|z)} \tag{147}$$

This last line requires some explanation. As mentioned above, each $C_{nm}^{p\{k,\bar{k}\}}$ is determined by C_{nmp}, **c** and the conformal dimensions of ϕ_n, ϕ_m and ϕ_p. We write

$$C_{nm}^{p\{k,\bar{k}\}} = C_{nmp}\beta_{nm}^{p\{k\}}\beta_{nm}^{p\{\bar{k}\}} \tag{148}$$

Inserting this expression into the first line of (147), we then perform the sum over all descendant states for each p, divide by $C_{klp} = \langle k|\,\phi_l(1,1)\,|p\rangle$ and get

Definition 6.7 *A **chiral conformal block** is*

$$\mathcal{F}_{nm}^{lk}(p|z) := \sum_{\{k\}} \beta_{nm}^{p\{k\}} \frac{\langle k|\,\phi_l(1,1)L_{\{k\}}\,|p\rangle}{\langle k|\,\phi_l(1,1)\,|p\rangle} z^{h_p-h_n-h_m+K} \tag{149}$$

where $L_{\{k\}}\,|p\rangle$ is the descendant state associated to the descendant field $\phi_p^{\{k\}}$

Remark 19 *I am unsure why everyone writes the denominator like this instead of as C_{klp}. Perhaps it has something to do with the fact that the blocks are 'universal' data of CFTs and can be calculated without reference to the three point structure constants, and this is somehow more apparent when the denominator is written as is.*

Remark 20 *Conformal blocks must obey the same differential equations that the correlation functions do. For example, they must obey the Ward identities (54)-(56), and if one of the fields are degenerate, they must obey the corresponding BPZ equation. In fact, the conformal blocks form a basis for the space of solutions to these equations.*

The corresponding definition of course holds for the anti-chiral conformal block $\overline{\mathcal{F}}$. The conformal blocks are not functions on spacetime, but rather sections of some vector bundle over spacetime. However $G(z,\bar{z})$ is a genuine single valued function, and so we already have a restriction imposed on the 3 point structure constants - they need to make the expression (147) single valued. There are other restrictions that get imposed on both the three point structure constants and the conformal blocks; the so called **crossing symmetry** equations that follow from associativity and commutativity of the OPE which are quadratic and linear equations for the three point constants, and the fact that the conformal blocks need to carry a representation of the braid group which give rises to the so-called **polynomial equations** for the fusing and braiding matrices that represent the action of the braid group.

Fusion rules, modular transformations and the Verlinde formula

(Equations in this section are not numbered yet because I wrote it later and haven't had the time to number these equations and shift back all the numbers in the subsequent sections).
We studied a simple version of the fusion ring when we talked about unitary minimal models. These were equations in the formal symbols Φ_i associated to primary fields ϕ_i that told us which other conformal families were allowed to appear in the OPE between ϕ_i and ϕ_j. For a generic **rational CFT** (finitely many sectors and hence finitely many $\mathcal{W}-$primaries), the fusion rules can take the following general form:

$$\Phi_A \star \Phi_B = \sum_C N_{AB}^C \Phi_C$$

The symbols N_{AB}^C are zero if the $\mathcal{W}-$family of Φ_C does not appear in the OPE of Φ_A with Φ_B, and is a positive integer if Φ_C does appear. Which positive integer does appear can be figured out from the OPE as follows. Ward identities for OPE's gives us formulas for the coefficients $C_{AB}^{C\{k,\hat{k}\}}$ of descendant fields in terms of the coefficients of their ancestors. We pick the field in the conformal family of ϕ_C appearing in the $\phi_A\phi_B$ OPE of lowest grade (i.e a descendant field $W_{-n_1}^{i_1}...W_{-n_k}^{i_k}\phi_C$ with $n_1+...+n_k$ minimized), and subtract off the contributions to the OPE $\phi_A\phi_B$ of $W_{-n_1}^{i_1}...W_{-n_k}^{i_k}\phi_C$ and all of the contributions from its descendants that follow from Ward identities. After this subtraction process, if there are still contributions from the ϕ_C family, then we repeat this process, finding the next lowest grade descendant appearing and subtracting off its contributions. The number of times we need to do this to remove the ϕ_C family completely from the OPE is N_{AB}^C. The ring (called the fusion ring) of these symbols Φ_i is a commutative (from commutativity of the OPE) ring with identity. The identity is the symbol Φ_0 associated to the $\mathcal{W}$ family of the identity field (which includes the stress energy tensor). It is an assumption that the $N_{AB}^0 = \delta_{A^+,B}$ where $+$ is an involution of the set of primary field labels called **conjugation**. So the $\mathcal{W}-$family of the identity field only appears in the OPE between ϕ_A and ϕ_{A^+}. Technically we do not need our theory to be rational for the fusion ring to make sense, i.e there can be infinitely many sectors (and therefore infinitely many $\mathcal{W}$-primaries). Rather, we just need only finitely many families to appear in the OPE between any two given fields. Such a CFT is termed **quasi rational**. We can take the formal symbols Φ_A to be basis elements of an algebra over $\mathbb{C}$ instead of over $\mathbb{Z}$, and the resulting finite dimensional commutative algebra is called the **fusion algebra**. Its finite dimensional representations are fully reducible, while its irreducible representations are all one dimensional so any finite dimensional representation of the fusion algebra splits into a direct sum of one dimensional representations. In particular, the adjoint representation of the fusion algebra (given by $\Phi_A \to N_A$ where $(N_A)_B^C := N_{AB}^C$ is an integer matrix) is finite dimensional and hence reducible to a direct sum of one dimensional representations. This means that the matrices N_A can all be simultaneously diagonalized by a unitary matrix S. After identifying a couple of other key properties of the matrix S, it follows that

$$N_{AB}^C = \sum_D \frac{S_{AD} S_{BD} (S^{-1})_{CD}}{S_{0D}}$$

This is not the Verlinde formula yet; the Verlinde formula is the above equation once we identify S with the matrix that implements the effect of modular transformations on characters these CFTs (modular transformations and characters are discussed just below).
Earlier in these notes, we talked about the equivalence between a CFT on a cylinder and a CFT on the plane. An interesting requirement to impose on a theory on the plane is consistency conditions on correlation functions of higher genus Riemann surfaces. **Modular invariance** is the study of the dependence of the 0 point torus correlation function (i.e the torus partition function) on the parameter $\tau := \frac{\omega_1}{\omega_2}$. Here ω_1 and ω_2 are complex numbers such that the torus is obtained as the quotient $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$. The complex structure of tori obtained in this way is invariant under the action of the **modular group** $PSL_2(\mathbb{Z})$ on τ (acting by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \tau = \frac{a\tau+b}{c\tau+d}$), and hence we expect that the partition function $Z(\tau)$ will be invariant under the modular group as well.

7 Affine Lie algebras

7.1 Simple Lie algebras

In this section we go over the basics of finite dimensional simple Lie algebra theory. This follows chapter 13 of diFrancesco and chapter 1 of Fuchs and Schwiegert's Affine Lie algebras and quantum groups. Let $\mathfrak{g}$ be such an algebra. Then

Definition 7.1 *A **Cartan-Weyl** basis for* $\mathfrak{g}$ *consists of*

- *elements* $\{H^1, ..., H^r\}$ *of a maximal abelian subalgebra. These subalgebras are not unique, but unique up to automorphisms of* $\mathfrak{g}$*. The span of these elements*

$$\mathfrak{g}_0 := span(\{H^i\}_{i=1}^r) \tag{150}$$

  *is termed a **Cartan subalgebra** of* $\mathfrak{g}$*, and* r *is called the **rank** of* $\mathfrak{g}$*.*

- *Simultaneous eigenvectors of* $\mathfrak{g}_0$ *(called **step operators**) in the adjoint representation*

$$\{E^\alpha | ad_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha\} \tag{151}$$

  These α *- called **roots** - are non-degenerate and can be thought of as elements of the dual space of the Cartan subalgebra* $\mathfrak{g}_0^*$*. We denote the set of roots by* Φ*.*

Since $\mathfrak{g}$ is simple it has a canonical non-degenerate inner product defined as follows.

Definition 7.2 *The **Killing form** is an inner product on simple Lie algebras*

$$k(X,Y) := \frac{1}{2g} Tr(ad_x \circ ad_y) \tag{152}$$

where g *is a constant that we will specify later.*

The Killing form satisfies the identity

$$K([X,Y],Z) = K(X,[Y,Z]) \tag{153}$$

The Killing form is still non-degenerate once restricted to $\mathfrak{g}_0$. Hence it induces an isomorphism between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, and using this isomorphism we get an inner product $(\, ,)$ on $\mathfrak{g}_0^*$. By definition the elements of $\mathfrak{g}_0$ have trivial brackets amongst themselves, so all of the interesting brackets will involve step operators. We already know the bracket between elements of $\mathfrak{g}_0$ and E^α, so all that is left is to examine the bracket structure of the step operators. Using the Jacobi identity we can compute that

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta] \tag{154}$$

Using this, after some work we can compute

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \alpha+\beta \in \Phi \\ k(E^\alpha, E^{-\alpha}) H^\alpha & \alpha = -\beta \\ 0 & \alpha+\beta \notin \Phi \end{cases} \tag{155}$$

In the first of the above cases, $N_{\alpha,\beta}$ is an unspecificed constant and the fact that the commutator is proportional to $E^{\alpha+\beta}$ follows from non-degeneracy of the roots. In the second case, we have defined $H^\alpha = \sum (k^{-1})_{ij} \alpha^i H^j \in \mathfrak{g}_0$ (which is the image of α under the isomorphism provided by the killing form). Here we are using the components $(k^{-1})_{ij}$ of k^{-1}, the inner product induced on g_0^*. We can work in a basis of $\mathfrak{g}_0$ such that $k|_{\mathfrak{g}_0}$ is the Euclidean metric, and hence so is its inverse. In this basis, we get the simpler

$$H^\alpha = \sum \alpha^i H^i \tag{156}$$

Of course the step operators were only defined up to a constant, so (155) reflects our choice of normalization conventions. In these conventions it follows that $k(E^\alpha, E^{-\alpha}) = \frac{2}{\sum \alpha^i \alpha^i} = \frac{2}{(\alpha,\alpha)} =: \frac{2}{|\alpha|^2}$. Say that R is a representation of $\mathfrak{g}$ with a basis of simultaneous eigenvectors of $\mathfrak{g}_0$, $|\lambda\rangle$:

$$H^i |\lambda\rangle = \lambda^i |\lambda\rangle \tag{157}$$

These $\lambda \in \mathfrak{g}_0^*$ are called **weights**. Clearly roots are weights of the adjoint representation. It is simple to compute that

$$H^i(E^\alpha |\lambda\rangle) = (\lambda^i + \alpha^i) E^\alpha |\lambda\rangle \tag{158}$$

If R is finite dimensional, the set of roots is finite, and thus some power of E^α must eventually kill any eigenvector $|\lambda\rangle$. We know that $E^{-\alpha}$ is also a step operator if E^α is, and hence there exist smallest minimal non-negative integers p_λ and q_λ such that

$$(E^\alpha)^{p_\lambda+1} |\lambda\rangle = 0 \tag{159}$$

$$(E^{-\alpha})^{q_\lambda+1} |\lambda\rangle = 0 \tag{160}$$

Now notice that the operators $\{E^\alpha, E^{-\alpha}, \frac{1}{|\alpha|^2} H^\alpha\}$ form an algebra isomorphic to $\mathfrak{sl}(2)$ with $\frac{1}{|\alpha|^2} H^\alpha \to J^3$, $E^{\pm\alpha} \to J^\pm$.

$$[E^\alpha, E^{-\alpha}] = 2\frac{1}{|\alpha|^2} H^\alpha \tag{161}$$

$$[\frac{1}{|\alpha|^2} H^\alpha, E^\alpha] = E^\alpha \tag{162}$$

$$[\frac{1}{|\alpha|^2} H^\alpha, E^{-\alpha}] = -E^{-\alpha} \tag{163}$$

Hence we know the representation theory of this algebra. We can look at the module

$$span\{(E^{-\alpha})^{q_\lambda} |\lambda\rangle ..., E^{-\alpha} |\lambda\rangle, |\lambda\rangle, E^\alpha |\lambda\rangle, ..., (E^\alpha)^{p_\lambda} |\lambda\rangle\}$$

which we identify as a highest weight $\mathfrak{sl}(2)$ representation. We know that such a representation has dimension $2j+1$, and clearly here we have that the lowest (highest) $J^3 \sim \frac{1}{|\alpha|^2} H^\alpha$ eigenvalue will be given by acting on $(E^{-\alpha})^{q_\lambda} |\lambda\rangle$ ($(E^{-\alpha})^{p_\lambda} |\lambda\rangle$) with $\frac{1}{|\alpha|^2} H^\alpha$ to get

$$j = \frac{(\alpha, \lambda)}{|\alpha|^2} + p_\lambda \tag{164}$$

$$-j = \frac{(\alpha, \lambda)}{|\alpha|^2} - q_\lambda \tag{165}$$

$$\tag{166}$$

and together these imply

$$2\frac{(\alpha, \lambda)}{|\alpha^2|} = q_\lambda - p_\lambda \in \mathbb{Z} \tag{167}$$

This relation holds for any weight λ, and since roots are examples of weights it holds in particular for roots.

Generally the set Φ of roots will span $\mathfrak{g}_0^*$ but $\dim g_0 - r \geq r$ so that there will be some linear dependencies between the elements of Φ. We can come up with a basis for $\mathfrak{g}_0^*$ consisting of a subset of Φ termed the **simple roots**. To construct the simple roots, we cut $\mathfrak{g}_0$ by a hyperplane that doesn't contain any of the roots, call the roots lying on one half the **positive roots** Φ^+ and the roots lying on the other half the **negative roots** Φ^-. From the positive roots, we define the simple roots to be those that cannot be written as a sum of two positive roots. It can be shown that

there are r of these simple roots, labeled $\{\alpha_1, ..., \alpha_r\}$, and that they are linearly independent. Note that subscripts on a root indicate different simple roots, while superscripts on a root indicate the components of some generic root. In addition, any positive root has non-negative integer coefficients in the basis of simple roots, and any negative root has non-positive integer coefficients in the basis of simple roots. Since every positive root is a $\mathbb{Z}_{\geq 0}$ sum of simple roots, we can define the **highest root** $\theta = \sum_i a_i \alpha_i$ to be the positive root with the greatest sum of coefficients $\sum a_i$ (θ is unique). These a_i are called the **marks**. In fact, all of the roots can be obtained by subtracting off an appropriate non-negative integer sum of simple roots from θ.
Going back to (167), we define the **simple coroots**

$$\alpha_j^\vee := \frac{2\alpha_j}{|\alpha_j|^2} \tag{168}$$

so that any of the inner products $(\alpha_i, \alpha_j^\vee)$ are integers. Since we have just normalized the roots in order to define the coroots, the coroots obviously also form a basis for $\mathfrak{g}_0^*$. If we expand θ in this basis $\theta = \sum_i a_i^\vee \alpha_i^\vee$ then we call the coefficients $a_i^\vee$ the **comarks**. From the comarks we obtain

Definition 7.3 *The* ***dual Coxeter number*** *is*

$$g := 1 + \sum_i a_i^\vee \tag{169}$$

This dual Coxeter number is the g that appears in the normalization of the Killing form (152). Of course we have made a large number of arbitrary choices to get to this point, so it is not obvious that the dual Coxeter number thus defined does not depend on these choices, but this is in fact the case; g is an invariant of $\mathfrak{g}$. We can form the matrix of all inner products between simple roots and simple coroots $A_{ij} := (\alpha_i, \alpha_j^\vee)$ called the **Cartan matrix**. From our previous observations, the Cartan matrix is an integer matrix. From the definition of coroots, it is clear that the diagonal elements A_{ii} of the Cartan matrix are all equal to 2. All of the off diagonal elements of the Cartan matrix are equal to $0, -1, -2$ or -3. Clearly $A_{ij} = 0$ implies $A_{ji} = 0$ but in general it is not true that $A_{ij} = A_{ji}$. However we note that for $i \neq j$ $A_{ij}A_{ji} = \frac{4(\alpha_i,\alpha_j)^2}{|\alpha_i|^2|\alpha_j|^2} < 4$ by Cauchy Schwarz (α_i and α_j linearly independent so we get strict inequality), so $A_{ij}, A_{ji} \neq 0$ tells us that at least one of A_{ij} and A_{ji} must be -1 ($-2 \cdot -2$, $-3 \cdot -2$ and $-3 \cdot -3$ are all bigger than 3). The simple root lengths $|\alpha_i^2|$ turn out to take at most two values, and we call those with the greater of these values the **long roots** and those with the smaller the **short roots**. Clearly non-equality of A_{ij} and A_{ji} results from α_i and α_j have different lengths. In the case there is just one root length we call $\mathfrak{g}$ **simply laced**.
From our simple roots we can define a particular Cartan-Weyl basis known as the **Chevalley basis**. Actually, we have already encountered the necessary definitions when we analyzed the commutation relations (161) – (163). We define for each simple root α_i three operators

$$e^i := E^{\alpha_i} \tag{170}$$
$$f^i = E^{-\alpha_i} \tag{171}$$
$$h^i = \frac{2H^{\alpha_i}}{|\alpha_i|^2} \tag{172}$$

The point of the Chevalley basis is that it is simple to write down commutation relations between these elements in terms of the Cartan matrix:

$$[h^i, h^j] = 0 \tag{173}$$
$$[h^i, e^j] = A_{ij} e^j \tag{174}$$
$$[h^i, f^j] = -A_{ji} f^j \tag{175}$$
$$[e^i, f^i] = \delta_{ij} h^i \tag{176}$$

The h^i do constitute a basis for $\mathfrak{g}_0$ but the e^i and f^i do not account for all possible step operators, so this is not a full basis yet. In order to get the rest of the step operators, we need to employ the **Serre relations**

$$[ad(e^i)]^{1-A_{ji}} e^j = 0 \tag{177}$$
$$[ad(f^i)]^{1-A_{ij}} f^j = 0 \tag{178}$$
$$\tag{179}$$

This is nothing but equations (159) and (160) for the adjoint representation in the Chevalley basis, but now we explicitly know p_λ and q_λ. The rest of the step operators are then simply of the form $[ad(e^i)]^s e^j$ for $0 \leq s \leq -A_{ji}$ and $[ad(f^i)]^r f^j$ for $0 \leq s \leq -A_{ij}$. The upshot of the Chevalley basis and the Serre relations is that the entire simple Lie algebra $\mathfrak{g}$ can be reconstructed from its Cartan matrix.
Now we discuss yet another basis of $\mathfrak{g}_0^*$ (the third one!) called the basis of **fundamental weights**. These are the elements ω_i of $\mathfrak{g}_0^*$ such that

$$(\omega_i, \alpha_j^\vee) = \delta_{ij} \tag{180}$$

i.e the basis vectors of the dual lattice of the simple coroot lattice. Since we already showed that the weights λ of any representation have integer inner products with any simple coroot, the coefficients of any such λ in the basis of fundamental weights will be integral:

$$\lambda = \sum_i \lambda_i \omega_i =: (\lambda_1, ..., \lambda_r) \in \mathbb{Z}^r \tag{181}$$

These λ_i are known as the **Dynkin labels** of λ. What are the Dynkin labels of the simple roots? If we write $\alpha_i = \sum_{k=1}^r c_k \omega_k$, then taking inner products with $\alpha_j^\vee$ yields

$$A_{ij} = (\alpha_i, \alpha_j^\vee) = \sum_k c_k \delta_{jk} = c_j \implies \alpha_i = \sum_{k=1}^r A_{ik}\omega_k = (A_{i1}, ..., A_{ir}) \tag{182}$$

and so the Dynkin labels of α_i are the *ith* row of the Cartan matrix. For any weight λ we have

$$h^i |\lambda\rangle = \frac{2H^{\alpha_i}}{|\alpha_i|^2} |\lambda\rangle = \frac{2}{|\alpha_i|^2} \sum_{k=1}^r \alpha_i^k H^k |\lambda\rangle = \frac{2}{|\alpha_i|^2} \sum_{k=1}^r \alpha_i^k \lambda^k = (\alpha_i^\vee, \lambda) = \lambda_i \tag{183}$$

where the last equality follows from expanding λ in the fundamental weight basis and using the duality between fundamental weights and simple coroots. If we take all possible inner products $F_{ij} := (\omega_i, \omega_j)$ between fundamental weights, we get the **quadratic form matrix**. This matrix is nothing other than a change of basis matrix from the basis of fundamental weights to the basis of simple coroots:

$$\omega_i = \sum F_{ij} \alpha_b^\vee \tag{184}$$

Multiplying (182) through by $\frac{2}{|\alpha_i|^2}$ we get

$$\alpha_i^\vee = \sum \frac{2}{|\alpha_i|^2} A_{ij} \omega_j \tag{185}$$

The change of basis matrices between two bases are inverses of each other by basic linear algebra, so we get

$$F_{ij} = (A^{-1})_{ij} \frac{|\alpha_j|^2}{2} \tag{186}$$

Clearly the quadratic form matrix allows us to take inner products in the fundamental weight basis by multiplying F with Dynkin labels.
If we return now to the fact that $\{E^\alpha, E^{-\alpha}, \frac{1}{|\alpha|^2} H^\alpha\}$ forms a $\mathfrak{sl}(2)$ algebra and

$$span\{(E^{-\alpha})^{q_\lambda} |\lambda\rangle ..., E^{-\alpha} |\lambda\rangle, |\lambda\rangle, E^\alpha |\lambda\rangle, ..., (E^\alpha)^{p_\lambda} |\lambda\rangle\}$$

is a highest weight representation space for it, taking $|\lambda\rangle = |\beta\rangle := E^\beta$ (working in the adjoint representation), from the representation theory of $\mathfrak{sl}(2)$ we get that $|\beta\rangle$ has some integer eigenvalue m for $J^3 \sim \frac{1}{|\alpha|^2} H^\alpha$

$$mE^\beta = \frac{1}{|\alpha|^2}[H^\alpha, E^\beta] = \frac{1}{|\alpha|^2} \sum \alpha^i [H^i, E^\beta] = \frac{1}{|\alpha|^2} \sum \alpha^i \beta^i E^\beta = \frac{1}{2}(\alpha^\vee, \beta) E^\beta \tag{187}$$

and so we see that $2m = (\alpha^\vee, \beta)$. Furthermore, so long as $m \neq 0$ we know that the spectrum of J^3 includes $-m$ since it contains m (the spectrum is $\{-2j-1, -2j+1, ..., 2j-1, 2j+1\}$). Looking at the set of eigenvectors listed above, we see that the eigenvector with J^3 eigenvalue $-m$ has to be of the form $|\beta + \ell\alpha\rangle$ for some integer $q_\beta \leq \ell \leq p_\beta$. Computing the action of J^3 on this vector returns

$$-mE^{\beta+\ell\alpha} = \frac{1}{2}(\alpha^\vee, \beta + \ell\alpha) E^{\beta+\ell\alpha} = \frac{1}{2}(2m + 2\ell) E^{\beta+\ell\alpha} \tag{188}$$

and hence $-2m = 2m + 2\ell$. Therefore we have $\beta + \ell\alpha = \beta - 2m\alpha = \beta - (\alpha^\vee, \beta)\alpha$, and this is another root. Hence if we define

Definition 7.4 *A* ***Weyl reflection*** $s_\alpha : \mathfrak{g}_0^* \to \mathfrak{g}_0^*$ *is defined by*

$$s_\alpha \beta = \beta - (\alpha^\vee, \beta)\alpha \tag{189}$$

Here α is a root and β is any element of $\mathfrak{g}_0$, not necessarily a root as above. The group W of all Weyl reflections is the ***Weyl group*** *and it is generated by the Weyl reflections of simple roots $W = \langle s_{\alpha_1}, ..., s_{\alpha_r} \rangle$.*

then we see that the Weyl group permutes Φ, the set of all roots. As mentioned in the definition, the Weyl group is generated by Weyl reflections of simple roots, so if we know the relations among these simple reflections we know the entire group. The relations between these simple reflections is easy to write down, and so we get the following presentation for W :

$$W = \langle s_{\alpha_1}, ..., s_{\alpha_r} | s_{\alpha_i}^2 = 1, (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = 1 \rangle \tag{190}$$

where $m_{ij} := 2$ if $A_{ij} = 0$ and $m_{ij} := \frac{4\pi}{4\pi - (\alpha_i^\vee, \alpha_j^\vee)} = \frac{\pi}{\pi - \theta_{ij}}$. θ_{ij} is the angle between α_i and α_j, defined using the inner product induced by the inverse Killing form on $\mathfrak{g}_0^*$. Notice that acting with a simple reflection on a simple root yields the nice formula

$$s_{\alpha_i}\alpha_j = \alpha_j - A_{ji}\alpha_i \tag{191}$$

which follows immediately from the definition of the simple reflection and the definition of the Cartan matrix. The union of the orbits of the simple roots under the Weyl group is the set of all roots. A quick computation shows that the inner product on $\mathfrak{g}_0^*$ is invariant under the action of the Weyl group.
We can define the integer span of any of our bases (simple roots, simple coroots, or fundamental weights) to be the **root lattice** Q, the **coroot lattice** $Q^\vee$ and the **fundamental weight lattice** P. The root lattice is a sublattice of the fundamental weight lattice, $Q \leq P$, while the P and $Q^\vee$ are by definition dual lattices. We give a special name to a particular element of the fundamental weight lattice:

Definition 7.5 *The* ***Weyl vector*** *is the element of* $\mathfrak{g}_0^*$

$$\rho = \sum_{i=1}^{r} \omega_i \tag{192}$$

We can make use of the Weyl group to prove a different characterization of the Weyl vector, namely half the sum of the positive roots $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. We just need to show that $\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ has the correct coefficients in the fundamental weight basis, i.e we need to show that $(\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, \alpha_i^\vee) = 1$. We can do this by making use of the invariance of the inner product under the Weyl group. Namely,

$$\begin{aligned}(\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, \alpha_i^\vee) = (s_{\alpha_i}\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, s_{\alpha_i}\alpha_i^\vee) = (\frac{1}{2}[-\alpha_i + \sum_{\alpha\in\Phi^+}\alpha] - \frac{1}{2}\alpha_i, -\alpha_i) = \\ (\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, -\alpha_i^\vee) + (-\alpha_i, -\alpha_i^\vee) = -(\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, \alpha_i^\vee) + 2 \implies (\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, \alpha_i^\vee) = 1\end{aligned} \tag{193}$$

In this computation we used the fact that $s_{\alpha_i}\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \frac{1}{2}[-\alpha_i + \sum_{\alpha\in\Phi^+}\alpha] - \frac{1}{2}\alpha_i$ which follows because s_{α_i} permutes all of the positive roots besides α_i amongst themselves, while it sends α_i to $-\alpha_i$; these are both consequences of (191) and the fact that positive roots are $\mathbb{Z}_{\geq 0}$ sums of simple roots. We can use the Weyl group elements to partition $\mathfrak{g}_0^*$ into a complete fan whose cones are given by

Definition 7.6 *The* ***Weyl chamber*** C_w *for* $w \in W$ *is the cone*

$$C_w := \{\lambda \in \mathfrak{g}_0^* | (w\lambda, \alpha_i) \geq 0, i = 1, ..., r\} \tag{194}$$

Remark 21 *Are there any interesting properties of the toric variety associated to the fan of Weyl chambers...*

We also define the **shifted Weyl reflection** associated to $w \in W$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho \tag{195}$$

where ρ is of course the Weyl vector and $\lambda \in \mathfrak{g}_0^*$. The shifted Weyl reflections still give a group action on $\mathfrak{g}_0^*$.
Now let's talk about the finite dimensional representation theory of simple Lie algebras. This boils down to two steps: finding the irreducible highest weight representations (**irreps**) and expressing tensor products of irreps as direct sums of irreps. We don't go into the details of the derivation, but here is how the classification works. We define an **integral dominant weight** $\Lambda \in P$ to be a $\mathbb{Z}_{\geq 0}$ linear combination of the fundamental weights

$$\Lambda = \sum_{i=1}^{r} \Lambda_i\omega_i, \quad \Lambda_i \in \mathbb{Z}_{\geq 0} \tag{196}$$

There is a one to one correspondence between dominant integral weights and irreps. Namely, for each integral dominant weight Λ we get an irrep R_Λ with the following features:

1. The **weight set** Ω_Λ or R_Λ is given by the following inductive algorithm:
   - Define $K_1 = \{\Lambda\}$
   - Add K_n to Ω_Λ
   - Define
   $$K_{n+1} := \bigcup_{\lambda = \sum \lambda_i\omega_i \in K_n} \{\lambda - k\alpha_i | 1 \leq k \leq \lambda_i\}$$
   - Repeat the last two steps until they can't be done anymore, i.e until K_n contains only $\lambda = \sum \lambda_i\omega_i$ such that each λ_i is non-positive

2. The multiplicities of each weight $\lambda \in \Omega_\Lambda$ (how many simultaneous eigenvectors in R_Λ have the same weight λ) is given by the **Freudenthal recursion formula**

$$[|\Lambda+\rho|^2 - |\lambda+\rho|^2] mult_\Lambda(\lambda) = 2 \sum_{\alpha\in\Phi^+, k\geq 1} (\lambda + k\alpha, \alpha) mult_\Lambda(\lambda + k\alpha) \tag{197}$$

Here ρ is the previously defined Weyl vector. Using this formula, we can start at the unique highest weight Λ ($mult_\Lambda(\Lambda) = 1$) of R_Λ and obtain the multiplicities of all of the other weights systematically.

Next we talk about a distinguished element of the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{U}(\mathfrak{g})$, called the **quadratic Casimir operator**. In a basis $\{T^a\}_{a=1}^{\dim \mathfrak{g}}$ of the Lie algebra (not one specially chosen to make the Killing form components Euclidean) it is given by

$$\mathcal{Q} := \sum_{a,b} k_{ab} T^a T^b \tag{198}$$

$\mathcal{Q}$ acts as a constant on any irrep, and that constant is

$$\mathcal{Q} \sim (\Lambda, \Lambda + 2\rho) \text{ in } R_\Lambda \tag{199}$$

The highest weight of the adjoint representation is the highest root θ discussed previously. If we compute (199) for $\Lambda = \theta$ we get that

$$\mathcal{Q} \sim 2g \text{ in } R_\theta \tag{200}$$

where g is the previously defined dual coxeter number.
We can form an inner product on the image of $\mathfrak{g}$ under a representation homomorphism R_Λ (we are using R_Λ to denote both the representation space and the homomorphism from $\mathfrak{g}$ into $\mathfrak{gl}(R_\Lambda)$):

$$k_\Lambda(T^a, T^b) := Tr_{R_\Lambda}(R_\Lambda(T^a) R_\Lambda(T^b)) \tag{201}$$

k_Λ is proportional to the killing form $k = k_\theta$

$$(k_\Lambda)_{ab} = |\theta|^2 x_\Lambda k_{ab} \tag{202}$$

and this proportionality constant x_Λ is known as the **index** of R_Λ. We actually still have some freedom in normalization of the roots of $\mathfrak{g}$ left over (so far we have only normalized them relative to each other so that the $k(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2}$) and we can use this freedom to take $|\theta|^2 = 2$. With this convention it is easy to compute from (199) + (202) that

$$x_\Lambda = \frac{\dim R_\Lambda \cdot (\Lambda, \Lambda + 2\rho)}{2 \dim g} \tag{203}$$

Given a representation R_Λ, we define

Definition 7.7 *The **character of** R_Λ is*

$$\chi_\Lambda := \sum_{\lambda\in\Omega_\Lambda} mult_\Lambda(\lambda) e^\lambda \tag{204}$$

where e^λ is a formal symbol that satisfies $e^{\lambda_1} e^{\lambda_2} = e^{\lambda_1+\lambda_2}$ and can be evaluated on weights $e^\lambda(\lambda') := exp((\lambda, \lambda'))$.

There is a formula due to Weyl for these characters. We define for any $w \in W$ the **length** of w, denoted $\ell(w)$, to be the minimum number of simple reflections needed to be multiplied together to obtain w. Then we define $\epsilon(w) := (-1)^{\ell(w)}$. Using these ϵ, we can write

$$\chi_\Lambda = \frac{\sum_{w\in W} \epsilon(w) e^{w(\Lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e^{w(\rho)}} \tag{205}$$

which is called the **Weyl character formula**. It can be used to derive a formula for the dimension of an irrep, but we won't write it here. The **Littlewood-Richardson** coefficients of a tensor product are the multiplicity with which other irreducible representations appear in the direct sum decomposition:

$$R_\Lambda \otimes R_\mu = \sum_\nu \mathcal{N}^\nu_{\Lambda,\mu} R_\nu \tag{206}$$

where ν ranges over dominant integral weights. There are a number of schemes used to calculate these coefficients, but we only highlight one that generalizes nicely to affine Lie algebras. This is the **character method**. Equation (204) for the character of an irrep can easily be extended to the character of any arbitrary representation of $\mathfrak{g}$. Once we do this, it is easy to show that $\chi_{R_\Lambda \oplus R_{\Lambda'}} = \chi_\Lambda + \chi_{\Lambda'}$ and $\chi_{R_\Lambda \otimes R_{\Lambda'}} = \chi_\Lambda \cdot \chi_{\Lambda'}$ which tells us that taking character is a ring homomorphism from the ring of representations of $\mathfrak{g}$ to the character ring of $\mathfrak{g}$. Using (206) along with these rules for how characters combine we obtain

$$\chi_\Lambda \chi_\mu = \sum_\nu \mathcal{N}_{\Lambda,\mu}^{\nu} \chi_\nu \tag{207}$$

Using (204) for χ_μ and (205) for χ_Λ and χ_ν, after clearing denominators we get

$$\sum_{w\in W} \epsilon(w) e^{w(\Lambda+\rho)} \sum_{\mu'\in\Omega_\mu} mult_\mu(\mu') e^{\mu'} = \sum_{\nu,w\in W} \mathcal{N}_{\Lambda,\mu}^{\nu} \epsilon(w) e^{w(\nu+\rho)} \tag{208}$$

Let's isolate the terms on the RHS and LHS with formal exponentials $e^{\bar{\nu}+\rho}$ for some fixed dominant weight $\bar{\nu}$. Note that $\bar{\nu}+\rho$ is dominant, being that it is the sum of two dominant weights. For some generic exponential appearing on the RHS, $e^{w(\nu+\rho)}$, the only w that can make $\nu+\rho$ dominant is $w=1$ since the W orbit of each weight intersects the fundamental chamber C_1 exactly once, and hence only $e^{\nu+\rho}$ can contribute. But in order for $e^{\nu+\rho}$ to be equal to $e^{\bar{\nu}+\rho}$ we need $\nu=\bar{\nu}$, and hence the only term on the RHS with formal exponential $e^{\bar{\nu}+\rho}$ is $\nu=\bar{\nu}$, $w=1$. This term is $N_{\Lambda,\mu}^{\bar{\nu}}\epsilon(1)e^{\bar{\nu}+\rho} = N_{\Lambda,\mu}^{\bar{\nu}} e^{\bar{\nu}+\rho}$. Now we examine the LHS. First we rewrite the inner sum as $\sum_{\mu'\in\Omega_\mu} mult_\mu(\mu') e^{w(\mu')}$ which holds because the Weyl groups permutes Ω_μ, and multiplicities are invariant under the action of the Weyl group. So we have

$$\sum_{w\in W,\mu'\in\Omega_\mu} \epsilon(w) mult_\mu(\mu') e^{w(\Lambda+\mu'+\rho)}$$

and we would like to know when the weight in the formal exponential is equal to $\bar{\nu}+\rho$.. This happens when $w\cdot(\Lambda+\mu) = w(\Lambda+\mu'+\rho)-\rho = \bar{\nu}$. Hence equating the contributions from the RHS and the LHS to $e^{\bar{\nu}+\rho}$ we get

$$\sum_{\mu'\in\Omega_\mu} \sum_{\substack{w\in W \\ w\cdot(\Lambda+\mu')=\bar{\nu}}} \epsilon(w) mult_\mu(\mu') = N_{\Lambda,\mu}^{\bar{\nu}} \tag{209}$$

Finally, we can write

$$mult_\mu(\mu') = mult_\mu(\rho+\bar{\nu}-w(\Lambda+\rho)) = mult_\mu(w^{-1}(\bar{\nu}+\rho)-\rho\Lambda) = mult_\mu(w^{-1}\cdot\bar{\nu}-\Lambda) \tag{210}$$

where we have made use of the condition under the sum in (209) in the first equality, invariance of multiplicities under multiplication by w^{-1} in the second equality, and the defintion of the shifted Weyl reflection in the last equality. $\epsilon(w)=\epsilon(w^{-1})$, and hence using (210) we can write the sum (209) in the simpler form

$$\mathcal{N}_{\Lambda\mu}^{\bar{\nu}} = \sum_{w\in W} \epsilon(w) mult_\mu(w\cdot\bar{\nu}-\Lambda) \tag{211}$$

Lastly for this section, we note that we should talk about embeddings of Lie algebras and branching rules, but we leave that discussion for later when we talk about it in the affine case.

7.2 Affine Lie algebras

Affine Lie algebras are infinite dimensional Lie algebras built from a simple Lie algebra $\mathfrak{g}$. They can be described starting from a Cartan matrix A_{ij} and relaxing the condition that $\det(A)>0$ to the easier condition that the determinant of each of the minors of the Cartan matrix is greater than 0, but we won't take this route here. Instead, we present a constructive approach to them. We begin with the so-called **loop algebra of** $\mathfrak{g}$, defined as

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] \tag{212}$$

i.e the set of all Laurent polynomials with coefficients in $\mathfrak{g}$. We denote a basis element $T^a\otimes z^n$ of $\mathcal{L}(\mathfrak{g})$ by T_n^a, and we define commutators on this algebra by

$$[T_n^a, T_m^b] = \sum_{c=1}^{\dim\mathfrak{g}} i f_c^{ab} T_{n+m}^c \tag{213}$$

We can centrally extend this loop algebra by an element $\hat{k}$ so that the modified commutation relations now read (in a basis such that $k(T^a,T^b)=\delta_{ab}$)

$$[T_n^a, T_m^b] = \sum_{c=1}^{\dim g} i f_c^{ab} T_{n+m}^c + \hat{k}n\delta_{n,-m} \tag{214}$$

$$[T_n^a, k] = 0 \tag{215}$$

For a Cartan Weyl basis $\{H^1,...,H^r\}\cup\{E^\alpha|\alpha\in\Phi\}$ of $\mathfrak{g}$ such that the Killing form restricted to $\mathfrak{g}_0$ is $\delta^{ij}=k(H^i,H^j)$, and $k(E^\alpha,E^{-\alpha})=\frac{2}{|\alpha|^2}$ these relations become

$$[H_n^i, H_m^j] = \hat{k}n\delta^{ij}\delta_{n,-m} \tag{216}$$

$$[H_n^i, E_m^\alpha] = \alpha^i E_{n+m}^\alpha \tag{217}$$

$$[E_n^\alpha, E_m^\beta] = \begin{cases} \frac{2}{|\alpha|^2}(H_{n+m}^\alpha + kn\delta_{n,-m}) & \alpha=-\beta \\ N_{\alpha,\beta}E_{n+m}^{\alpha+\beta} & \alpha+\beta\in\Phi \\ 0 & else \end{cases} \tag{218}$$

This central extension of $\mathcal{L}(\mathfrak{g})$ is not quite an affine Lie algebra yet: we must adjoin an element D to it which acts by

$$[D, T^a_n] = nT^a_n \tag{219}$$
$$[\hat{k}, D] = 0 \tag{220}$$

Once we have adjoined D, we have

Definition 7.8 *An* ***(untwisted) affine Lie algebra*** $\hat{\mathfrak{g}}$ *is the vector space* $\mathcal{L}(g) \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}D$ *supplemented with the commutation relations* (214), (215), (219) *and* (220).

The theory of these untwisted affine Lie algebras closely mirrors the theory of simple Lie algebras. Firstly, we note that $\hat{\mathfrak{g}}$ contains $\mathfrak{g}$ as the **horizontal subalgebra**, $span(\{T^a_0\})$. The isomorphism is clear using (214). As with $\mathfrak{g}$, we have a maximal abelian algebra for $\hat{\mathfrak{g}}$:

Definition 7.9 *The* ***Cartan subalgebra*** $\hat{\mathfrak{g}}_0$ *of* $\hat{\mathfrak{g}}$ *is spanned by* $\mathfrak{g}_0 \cup \{\hat{k}, D\}$.

We can again put a Killing form K on $\hat{\mathfrak{g}}$. We require that the invariance property (153) still holds, and up to normalization this requires

$$K(T^a_n, T^b_m) = \delta^{ab}\delta_{n,-m} \tag{221}$$
$$K(D, \hat{k}) = 1 \tag{222}$$
$$K(T^a_n, \hat{k}) = K(\hat{k}, \hat{k}) = K(T^a_n, D) = K(D, D) = 0 \tag{223}$$

This new Killing form is non-degenerate when restricted to $\hat{\mathfrak{g}}_0$, so we again get an inner product on $\hat{\mathfrak{g}}_0$, a canonical isomorphism between $\hat{\mathfrak{g}}_0$ and $\hat{\mathfrak{g}}_0^*$, and an induced inner product on $\hat{\mathfrak{g}}_0^*$. We can label elements of the weight space $\hat{\mathfrak{g}}_0^*$ (called **affine weights**) by their values on the Cartan subalgebra $\hat{\mathfrak{g}}$ like usual:

$$\hat{\lambda} = (\hat{\lambda}(H^1_0), ..., \hat{\lambda}(H^r_0), \hat{\lambda}(\hat{k}), \hat{\lambda}(D)) \in \hat{\mathfrak{g}}_0^* \tag{224}$$

The first r components of this vector form a standard weight λ in $\mathfrak{g}_0^*$, the dual of the Cartan subalgebra of the horizontal subalgebra of $\hat{\mathfrak{g}}$, so we can write $\hat{\lambda} = (\lambda, k_\lambda = \hat{\lambda}(\hat{k}), n_\lambda = \hat{\lambda}(D))$. In terms of these components, the induced inner product on $\mathfrak{g}_0^*$ takes the explicit form

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda \tag{225}$$

where (λ, μ) is the induced inner product on the weight space $\mathfrak{g}_0^*$. **Affine roots**, i.e affine weights of the adjoint representation of $\hat{\mathfrak{g}}$ on itself, take the form

$$\hat{\beta} = (\beta, 0, n) \in \hat{\Phi} \tag{226}$$

for $\beta \in \mathfrak{g}_0^*$ and $n \in \mathbb{Z}$. This holds because $\hat{k}$ has 0 eigenvalues in the adjoint representation ((215) and (220)), while D gives the $\mathbb{Z}$ gradation of $\mathcal{L}(g)$, and commutes with itself and $\hat{k}$. Applying the formula (225) then tells us that we can just remove hats when taking inner products of affine roots. There are two types of affine roots; the **real roots** are the roots associated to the vectors E^α_n, which are $\hat{\alpha} = (\alpha, 0, n)$, and the **imaginary roots** are the roots associated the vectors H^i_n which are $(0, 0, n)$. We define $\delta := (0, 0, 1)$ and $\alpha := (\alpha, 0, 0)$ so that the roots associated to H^i_n can be written more concisely as $n\delta$ and the roots associated the E^α_n can be written more concisely as $\alpha + n\delta$. Using (225) again we can calculate that the imaginary roots have 0 norm $(n\delta, m\delta) = 0$. Unlike the case of $\mathfrak{g}$ simple, there are degeneracies amongst the affine roots. Namely, $n\delta$ is the affine root of each H^i_n for $i = 1, ..., r$ and is therefore $r-$fold degenerate, while the roots $\alpha + n\delta$ are non-degenerate (which follows from the non-degeneracy of the roots of $\mathfrak{g}$). Clearly a basis for the set of all affine roots can be given by the simple roots $\alpha_i = (\alpha_i, 0, 0)$ plus δ, since the simple roots form a basis for the root space of $\mathfrak{g}$, but we choose instead to take as a basis the simple roots plus $\alpha_0 = -\theta + \delta$ where θ is the highest root i.e the highest weight of the adjoint representation of $\mathfrak{g}$. Therefore the preferred basis, which we shall call the **affine simple roots** $\hat{\Phi}_s$, for $\hat{\mathfrak{g}}_0^*$ is $\{\alpha_0, ..., \alpha_r\}$. The **affine positive roots** $\hat{\Phi}^+$ are the $\mathbb{Z}_{\geq 0}$ linear span of the affine simple roots intersected with $\hat{\Phi}$ - this was also the case for $\mathfrak{g}_0^*$, but the construction here is reversed (for $\mathfrak{g}_0^*$ we started with positive roots and then found the simple roots as a subset of them). We can show that the set of all affine positive roots is

$$\hat{\Phi}^+ = \{\alpha + n\delta | n > 0, \alpha \in \Phi \cup \{0\}\} \cup \Phi^+ \tag{227}$$

It is clear that $\Phi^+ \subset \hat{\Phi}^+$, and for the other elements we can write $\alpha + n\delta = \alpha + n(\alpha_0 + \theta) = n\alpha_0 + n\theta + \alpha$. θ has the property that $\theta + \alpha$ is always dominant for any $\alpha \in \Phi \cup \{0\}$ (usually this is phrased as $\theta - \alpha$ is always dominant, but of course $\alpha \in \Phi \iff -\alpha \in \Phi$), so this expression is indeed a positive root. For the other inclusion, the only other types of roots that exist are of the form

- $\alpha + n\delta = \alpha + n(\alpha_0 + \theta)$ for $n < 0, \alpha \in \Phi \cup \{0\}$. $n\theta + \alpha$ is a linear combination of the α_i for $i > 0$, so the coefficient n on α_0 in the expansion of $n\delta$ is negative and hence this is not an affine positive root (the coefficients on the other α_i's are negative too, just easier to see it for α_0).

- α for $\alpha \in \Phi/\Phi^+$

From this characterization of the set of affine positive roots, we see that that there is no highest root so the adjoint representation is not a highest weight rep.
As mentioned at the beginning of this section, we could have started with Cartan matrices to get to affine Lie algebras (and more general algebras as well called **Kac Moody algebras** which would be those whose Cartan matrix had no restriction on determinant), but we did not take this path. We instead describe the Cartan matrix $\hat{A}$ that results from our constructed $\hat{\mathfrak{g}}$. First we define

Definition 7.10 *The **affine simple coroots** are*

$$\alpha_i^\vee := \frac{2}{|\alpha_i|^2}\alpha_i \tag{228}$$

just as in the simple $\mathfrak{g}$ case, except now we have $i = 0, 1..., r$.

Obviously these affine simple coroots will be the same as the simple coroots for $i > 0$ will be the same as those of $\mathfrak{g}$ We just need to calculate the norm of $\alpha_0 = -\theta + \delta$:

$$|\alpha_0|^2 = (\theta,\theta) - 2(\theta,\delta) + \cancel{(\delta,\delta)} = 2 - 2(\theta,\delta) = 2 \tag{229}$$

This last equality follows because δ is orthogonal to all of the ($\mathfrak{g}$) simple roots $\alpha_i = (\alpha_i, 0, 0)$, and θ is a sum of such roots. Hence we see that $\alpha_0^\vee = \alpha_0$. Given these coroots, we define

Definition 7.11 *The **Cartan matrix** associated to $\hat{\mathfrak{g}}$ is the matrix of inner products*

$$\hat{A}_{ij} := (\alpha_i, \alpha_j^\vee) \tag{230}$$

So the Cartan matrix $\hat{A}$ for $\hat{\mathfrak{g}}$ is an $(r+1)\times(r+1)$ matrix whose minor obtained by deleting the $0th$ row and column gives the Cartain matrix A for $\mathfrak{g}$. So the only new entries in the Cartan matrix $\hat{A}$ are in the $0th$ row and column. We can calculate them.

$$\hat{A}_{00} = (\alpha_0, \alpha_0^\vee) = 2 \tag{231}$$

$$\hat{A}_{0,j} = (\alpha_0, \alpha_j^\vee) = -(\theta, \alpha_j^\vee) + \cancel{(\delta, \alpha_j^\vee)} = -\sum_{i=1}^{r} a_i(\alpha_i, \alpha_j^\vee) = -\sum_{i=1}^{r} a_i A_{ij} \tag{232}$$

$$\hat{A}_{j,0} = (\alpha_j, \alpha_0^\vee) = (\alpha_j, \alpha_0) = -(\alpha_j, \theta) = -\sum_{i=1}^{r} a_i^\vee A_{ji} \tag{233}$$

where the a_i and $a_i^\vee$ are the marks and comarks from the last section (the coefficients of θ in the simple root and simple coroot basis respectively). We set a convention where the **zeroeth mark** a_0 and the **zeroeth comark** $a_0^\vee$ are both equal to 1. These definitions allow us to calculate $\sum_{i=0}^{r} a_i \hat{A}_{ij} = 0$, and hence the generalized Cartan matrix has a zero eigenvalue. This is in sharp contrast to the Cartan matrix of $\mathfrak{g}$ which had positive determinant. The given definition for the zeroeth mark and comark also allow us to write

$$\delta = \sum_{i=0}^{r} a_i\alpha_i = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee \tag{234}$$

$$g = 1 + \sum_{i=1}^{r} a_i^\vee = \sum_{i=0}^{r} a_i^\vee \tag{235}$$

so we get nice expressions for δ and the dual Coxeter number.
Again mirroring simple Lie algebra theory, we can define the dual vectors to the simple coroots called the **affine fundamental weights**:

$$(\hat{\omega}_i, \alpha_j^\vee) = \delta_{ij} \tag{236}$$

We can calculate that $\hat{\omega}_0 = (0,1,0) = (0, \alpha_0^\vee, 0), \hat{\omega}_1 = (\omega_1, a_1^\vee, 0), ..., \hat{\omega}_r = (\omega_r, a_r^\vee, 0)$. $\hat{\omega}_0$ is the **basic fundamental weight**. If we define $\omega_i = (\omega_i, 0, 0)$ then using the basic fundamental weight we can write $\hat{\omega}_i = \omega_i + a_i^\vee \omega_0$ for $i = 1, ..., r$. The inner product between the basic fundamental weight and any other affine fundamental weight vanishes because of (225), and the inner product between $\hat{\omega}_i$ and $\hat{\omega}_j$ for $i, j > 0$ is just the inner product between ω_i and ω_j, i.e elements of the quadratic form matrix:

$$(\hat{\omega}_i, \hat{\omega}_j) = (\omega_i, \omega_j) = F_{ij} \tag{237}$$

Now a point of clarification: $\hat{\mathfrak{g}}_0$ (and hence $\hat{\mathfrak{g}}_0^*$) is an $r+2$ dimensional space, while there are only $r+1$ affine simple roots, affine simple coroots, and affine fundamental weights, so each of these linearly independent sets needs one more element to form a whole basis for $\hat{\mathfrak{g}}_0^*$. For example, we can add $\hat{\omega}_0$ to the affine simple roots or affine simple coroots, and we can add δ to the affine fundamental weights to get actual bases for $\hat{\mathfrak{g}}_0^*$. In this latter basis, we can expand any weight $\hat{\lambda} \in \hat{\mathfrak{g}}_0^*$ to get $\hat{\lambda} = r\delta + \sum_{i=0}^{r} \lambda_i \hat{\omega}_i$ where the λ_i are again known as the Dynkin labels and r is just some real coefficient. If we evaluate $\hat{\lambda}$ on $\hat{k}$ we get

Definition 7.12 *The* ***level*** *of an affine weight* $\hat{\lambda}$ *is*

$$k := \lambda(\hat{k}) \tag{238}$$

We can evaluate the level by looking at $\hat{\lambda} = r\delta + \sum_{i=0}^{r} \lambda_i \hat{\omega}_i$ and summing up the contributions to the second slot $\hat{\lambda} = (\lambda, \hat{\lambda}(\hat{k}), \hat{\lambda}(D))$. $r\delta = (0, 0, r)$ contributes nothing, and each term $\lambda_i \hat{\omega}_i = (\lambda_i \omega_i, \lambda_i \alpha_i^\vee)$ contributes $\lambda_i \alpha_i^\vee$ yielding

$$k = \sum_{i=0}^{r} \lambda_i \alpha_i^\vee = \lambda_0 + (\lambda, \theta) \tag{239}$$

Sometimes it is convenient to forget about the D eigenvalue of a given weight and write things like $\lambda = [\lambda_0, ..., \lambda_r]$ suggesting that a weight is characterized by its Dynkin labels, but this is obviously incomplete information about the weight λ. We can define the Weyl vector $\hat{\rho}$ in this way as $[1, ..., 1]$. I believe that when we do this, we are implicitly assuming that $\hat{\rho}(D) = 0$ but maybe that is not the case (I will modify this statement if I find out differently later). We define **dominant integral affine weights** as we did for dominant integral weights; those weights that possess all non-negative integer Dynkin labels.
Continuing on in parallel with how we developed the theory of simple Lie algebras, we can define the **affine Weyl group** $\hat{W}$ to be the group of all Weyl reflections by *real* roots. That is, we have for any real root $\hat{\alpha} = (\alpha, 0, m)$ and arbitrary weight $\hat{\lambda} = (\lambda, k, n)$

$$s_{\hat{\alpha}}\lambda := \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha} \tag{240}$$

and

$$\hat{W} := \{s_{\hat{\alpha}} | \hat{\alpha} \in \hat{\Phi}\} \tag{241}$$

We also define the shifted affine Weyl transformation analogously to the finite case:

$$w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho} \tag{242}$$

We can calculate the effect of an affine Weyl reflection on a generic weight $\hat{\lambda}$ using (225) :

$$\begin{aligned} &s_{\hat{\alpha}}\lambda = (\lambda, k, n) - \frac{2}{|\alpha|^2}[(\lambda, \alpha) + km + 0 \cdot n](\alpha, 0, m) = \\ &(\lambda - \frac{2}{|\alpha|^2}((\lambda, \alpha) + km)\alpha, k, n - \frac{2}{|\alpha|^2}((\lambda, \alpha) + km)m) = \\ &(s_\alpha(\lambda + km\alpha^\vee), k, n - ((\lambda, \alpha) + km)\frac{2m}{|\alpha|^2}) \end{aligned} \tag{243}$$

The first line is just applying (225), the second line is grouping together the terms, and the last line is noticing that

$$\begin{aligned} &s_\alpha(\lambda + km\alpha^\vee) = \lambda + km\alpha^\vee - (\lambda + km\alpha^\vee, \alpha^\vee)\alpha = \\ &\lambda + km\alpha^\vee - (\lambda, \alpha^\vee)\alpha - (km\alpha^\vee, \alpha^\vee)\alpha = \\ &\lambda - \frac{2}{|\alpha|^2}(\lambda, \alpha)\alpha + km\alpha^\vee - (km\alpha, \alpha^\vee)\alpha^\vee = \\ &\lambda - \frac{2}{|\alpha|^2}(\lambda, \alpha)\alpha + km\alpha^\vee - 2km\alpha^\vee = \\ &\lambda - \frac{2}{|\alpha|^2}(\lambda, \alpha)\alpha - \frac{2}{|\alpha^2|}km\alpha \end{aligned}$$

Because δ is orthogonal to every root, affine Weyl transformations do not affect it: $s_{\hat{\alpha}}\delta = \delta$. Affine Weyl transformations, like their finite counterparts, preserve the induced inner product on $\hat{\mathfrak{g}}_0^*$. Also like their finite counterparts, affine Weyl orbits of affine weights of some given representation have constant multiplicity. If we define for any coroot $\alpha^\vee$ the translation operator $t_{\alpha^\vee} = s_\alpha s_{\alpha+\delta}$ after some algebra, and using (242) we get that $s_{\hat{\alpha}} = s_\alpha(t_{\alpha^v})^m$. We can also compute that $t_{\alpha^\vee}t_{\beta^\vee} = t_{\alpha^\vee+\beta^\vee}$ and hence the set of these translations is isomorphic to the coroot lattice of $\mathfrak{g}$, $Q^\vee$. We can write an explicit for for the action of $t_{\alpha^\vee}$:

$$t_{\alpha^\vee}(\lambda, k, n) = (\lambda + k\alpha^\vee, k, n + \frac{1}{2k}(|\lambda|^2 - |\lambda + k\alpha^\vee|^2)) \tag{244}$$

When $k = 0$ we first expand out the inner products in the third slot of this formula and cancel the denominator k. For any $s_{\hat{\alpha}} = s_\alpha(t_{\alpha^v})^m = s_\alpha t_{m\alpha^\vee}$ we can compute the effect of conjugation by $s_{\hat{\alpha}}$ on $t_{\beta^\vee}$, remembering that $s_{\hat{\alpha}} = (s_{\hat{\alpha}})^{-1}$ is order 2:

$$s_{\hat{\alpha}} t_{\beta^\vee} s_{\hat{\alpha}}^{-1} = s_\alpha t_{m\alpha^\vee+\beta^\vee} s_\alpha t_{m\alpha^\vee} = t_{s_\alpha(m\alpha^\vee+\beta^\vee)+m\alpha^\vee} \tag{245}$$

and hence we see that the set of $t_{\beta^\vee}$'s form a normal subgroup of $\hat{W}$. Furthermore this normal subgroup intersects $W \leq \hat{W}$ trivially, so by the decomposition $s_{\hat{\alpha}} = s_\alpha(t_{\alpha^v})^m$ we get

$$\hat{W} \cong Q^\vee \rtimes W \tag{246}$$

Remark 22 *In all of the sources that I have seen, they claim that for any* $\sigma \in \hat{W}$, $\sigma t_{\beta^\vee}\sigma^{-1} = t_{\sigma(\beta^\vee)}$ *but I don't understand this. If we take* $\sigma = s_{\alpha_0}$ *to be the element of the affine Weyl group corresponding to the simple root* $\alpha_0 = (-\theta, 0, 1)$*, then using the formula (242) with* $\lambda = \beta^\vee = (\beta^\vee, 0, 0)$ *we get* $s_{\alpha_0}(\beta^\vee) = (s_{\alpha_0}(\beta^\vee), 0, -(\beta^\vee, \theta^\vee))$ *which has a non-zero last component and is hence not a coroot anymore. It doesn't even make sense to write* $t_{s_{\alpha_0}(\beta^\vee)}$*. What am I missing here? I think the argument that I gave above is better - the key difference is that I only act on coroots with the finite reflection* s_α *instead of the affine reflection* $s_{\hat{\alpha}}$*. I just did the calculation* $s_\alpha t_{m\alpha^\vee+\beta^\vee} s_\alpha = t_{s_\alpha(m\alpha^\vee+\beta^\vee)}$ *by hand by acting on an arbitrary* $\hat{\lambda} = (\lambda, k, n)$*.*

Once again following the development of the theory of $\mathfrak{g}$, we can define **affine Weyl chambers**

$$\hat{C}_w = \{\hat{\lambda} | (w\hat{\lambda}, \alpha_i) \geq 0, i = 0, ..., r\} \tag{247}$$

and make note that the affine Weyl orbits intersect the affine fundamental chamber $\hat{C}_1$ exactly once. Now we discuss the representation theory of affine Lie algebras. We will again focus our attention on the highest weight representations, but now can define a subclass of these highest weight representations that are of special importance: the **integrable highest weight representations**. In short, to get an integrable highest weight representation we take the Verma module of some dominant integral affine weight $\hat{\lambda} = [\lambda_1, ..., \lambda_r]$ at level k (notice we don't specify $\hat{\lambda}(D)$: this is because it can always be set to 0 once we focus on a specific rep by shifting the definition of the generators of $\hat{\mathfrak{g}}$), and quotient out by the null vectors (the vectors that have zero norm if we impose $(T^a_n)^\dagger = T^a_{-n}$. Equivalently, vectors that generate a maximal submodule of $V_{\hat{\lambda}}$ so that the quotient is irreducible). The condition $\lambda_0 \in \mathbb{Z}_{\geq 0}$ along with equation (239) tells us that for $\lambda = (\lambda_1, ..., \lambda_r)$ we must have

$$(\lambda, \theta) \leq k \in \mathbb{Z}_{\geq 0} \tag{248}$$

The inequality above is obvious, and the condition $k \in \mathbb{Z}_{\geq 0}$ follows from the fact that $k = \lambda_0 + (\lambda, \theta) = \lambda_0 + (\lambda, \theta^\vee)$; $\lambda_0 \in \mathbb{Z}_{\geq 0}$ and $\theta = \theta^\vee$ has integer inner product with λ since λ is a weight of some rep of $\mathfrak{g}$. Expanding θ in the simple coroot basis of $\mathfrak{g}$ and λ in the fundamental weight basis of $\mathfrak{g}$, (248) reads

$$\sum_{i=1}^{r} \lambda_i a_i^\vee \leq k \tag{249}$$

Since each comark $a_i^\vee$ is positive, and the λ_i are positive integers, this gives us only finitely many integrable highest weight representations at a given level $k \in \mathbb{Z}_{\geq 0}$. So, to sum up, to form a highest weight integrable representation

- Pick a level $k \in \mathbb{Z}_{\geq 0}$
- Choose a dominant integral weight $\hat{\lambda} = (\lambda, k, 0)$ such that $\lambda_i \in \mathbb{Z}_{\geq 0}$ for each $i = 1, ..., r$ and $(\lambda, \theta) \leq k$. There are only finitely many such λ.
- Form the Verma module $V_{\hat{\lambda}} = \mathfrak{U}(\hat{\mathfrak{g}}_-)v_{\hat{\lambda}}$ where $\hat{\mathfrak{g}}_-$ is the subalgebra of $\hat{\mathfrak{g}}$ generated by all of the step operators corresponding to negative roots.
- Impose $\hat{\mathfrak{g}}_+ v_{\hat{\lambda}} = 0$ ($\hat{\mathfrak{g}}_+$ is the subalgebra of $\hat{\mathfrak{g}}$ generated by the step operators corresponding to positive roots) and $\hat{\mathfrak{g}}_0 = span\{H^1_0, ..., H^r_0, K, D\}$ acts by $H^i_0 = \lambda_i$, $\hat{k} = k$ and $D = 0$. Equivalently we could have started with the Verma module $V'_{\hat{\lambda}} = \mathfrak{U}(\hat{\mathfrak{g}})v_{\hat{\lambda}}$ and quotientied out by these relations.
- Finally, quotient the resulting object out by the null vectors: these are the vectors $(E_0^{-\alpha_i})^{\lambda_i} v_{\hat{\lambda}}$ for $i = 1, ..., r$ and $(E^\theta_{-1})^{\lambda_0+1} v_{\hat{\lambda}}$ and their descendants.

To find the weights and multiplicities of these integrable highest weight representations, we follow the same procedure that we did for $\mathfrak{g}$: we subtract positive integer sums of simple roots from the dominant weight $\hat{\lambda}$ in an algorithmic way - we won't review exactly which sums get subtracted here, because it is exactly the same idea as for simple $\mathfrak{g}$, except that the algorithm does not terminate. The multiplicities follow from the same recursion formula (197) with ρ replaced with the affine Weyl vector $\hat{\rho}$, and a factor that accounts for the degeneracy of imaginary roots:

$$[|\hat{\lambda} + \hat{\rho}|^2 - |\hat{\lambda} + \hat{\rho}|^2] mult_{\hat{\lambda}}(\hat{\lambda}') = \sum_{\hat{\alpha}>0} mult_{\hat{\lambda}}(\hat{\alpha}) \sum_{p=1}^{\infty} mult_{\hat{\lambda}}(\hat{\lambda}' + p\hat{\alpha})(\hat{\lambda}' + p\hat{\alpha}, \hat{\alpha}) \tag{250}$$

Now we discuss the character theory of affine Lie algebras. We start with the same definition:

Definition 7.13 *The* ***(affine) character*** *of a representation* R *of* $\hat{\mathfrak{g}}$ *is*

$$\chi_R = \sum_{\hat{\mu} \in \hat{\Omega}_R} mult_R(\hat{\mu}) e^{\hat{\mu}} \tag{251}$$

However this definition comes with an added subtlety. The weight set $\hat{\Omega}_R$ of a representation of an affine Lie algebra is infinite dimensional, so we have convergence issues to worry about when it comes to evaluating such a sum acting on another weight. However, it can be shown that these characters converge for reasonable representations such as the irreducible integrable highest weight representations that we just defined. Just as with the finite dimensional simple Lie algebras $\mathfrak{g}$, we get a nice formula for the characters of the irreducible integrable highest weight representations, this time called the **Weyl-Kac character formula**. It looks identical to the last character formula:

$$\chi_{\hat{\lambda}} = \frac{\sum_{w\in\hat{W}} \epsilon(w)e^{w(\hat{\lambda}+\hat{\rho})}}{\sum_{w\in\hat{W}} \epsilon(w)e^{w(\hat{\rho})}} \tag{252}$$

Using the decomposition $\hat{W} = Q^\vee \rtimes W$ of the affine Weyl group into a semidirect product, we can manipulate (252) further. First we define the **generalized Theta functions**

$$\Theta_{\hat{\lambda}} := e^{\frac{-|\hat{\lambda}|^2\delta}{2k}} \sum_{\alpha^\vee\in Q^\vee} e^{t_{\alpha^\vee}\hat{\lambda}} \tag{253}$$

After some manipulations (doing some algebra on the generalized Theta function, and using the semidirect product structure of $\hat{W}$ to rewrite (252)) we get that the Weyl-Kac formula transforms into

$$\chi_{\hat{\lambda}} = e^{s_{\hat{\lambda}}} \frac{\sum_{w\in W} \epsilon(w)\Theta_{w(\hat{\lambda}+\hat{\rho})}}{\sum_{w\in W} \epsilon(w)\Theta_{w(\hat{\rho})}} \tag{254}$$

where

$$s_{\hat{\lambda}} := \frac{|\hat{\lambda}+\hat{\rho}|^2}{2(k+g)} - \frac{|\rho|^2}{2g} \tag{255}$$

To get rid of this extra exponential lurking around, we define the **normalized affine characters**

$$\tilde{\chi}_{\hat{\lambda}} := e^{-s_{\hat{\lambda}}}\chi_{\hat{\lambda}} = \frac{\sum_{w\in W} \epsilon(w)\Theta_{w(\hat{\lambda}+\hat{\rho})}}{\sum_{w\in W} \epsilon(w)\Theta_{w(\hat{\rho})}} \tag{256}$$

We can evaluate characters at specific weights to get **specialized characters**. Namely, if we define $\hat{\mu} = (0, 2\pi i\tau, 0)$ for some number τ then we get from the basic definition (251) that

$$\chi_{\hat{\lambda}}(\tau) = \sum_{n=0}^{\infty} d_n q^n \tag{257}$$

where $q = e^{2\pi i\tau}$ and d_n is the dimension of the nth graded piece of the $R_{\hat{\lambda}}$ (the grading of a vector is defined as (negative) the sum of the subscripts of the operators in $\hat{\mathfrak{g}}$ applied to it). More generally, it is conventional to write a weight $\hat{\lambda}'$ as

$$\hat{\lambda}' = 2\pi i(\zeta, \tau, t) \tag{258}$$

We can obviously write any function of the weights $\hat{\lambda}'$ instead as a function of the inputs $\zeta \in \mathfrak{g}_0^*$, τ and t. This normalization convention makes it easier to work with modular transformations of weights; the weight space $\hat{\mathfrak{g}}_0^*$ carries an action of $PSL_2(\mathbb{Z})$ defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (\zeta, \tau, t) = (\frac{\zeta}{c\tau+d}, \frac{a\tau+b}{c\tau+d}, t - \frac{c|\zeta|^2}{c\tau+d}) \tag{259}$$

We recall from earlier sections that $PSL_2(\mathbb{Z})$ is generated by two matrices (or rather their equivalence classes) $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Since we have a group action on the set of weights, we get an associated group action for any function of the weights such as $\Theta_{\hat{\lambda}}$ or the characters χ. To compute the form of the group action on these functions, we only need to analyze the effect of S and T on them. For $\Theta_{\hat{\lambda}}$, we get

$$S\Theta_{\hat{\lambda}}(\zeta, \tau, t) = (-i\tau)^{\frac{r}{2}}|P/Q^\vee|^{-\frac{1}{2}}k^{-\frac{r}{2}} \sum_{\hat{\lambda}'\in\hat{P}_k/kQ^\vee} e^{-2\pi i(\hat{\lambda}',\hat{\lambda})k}\Theta_{\hat{\lambda}'}(\zeta, \tau, t) \tag{260}$$

$$T\Theta_{\hat{\lambda}} = e^{\frac{i\pi(\hat{\lambda},\hat{\lambda})}{k}}\Theta_{\hat{\lambda}}(\zeta, \tau, t) \tag{261}$$

All the symbols above have been defined besides $\hat{P}_k$ which is the set of affine weights with integer Dynkin labels at level k (a sublattice of the lattice $\hat{P}$ generated by the affine fundamental weights). Hence the sum is over representatives in this quotient of lattice (it doesn't matter which representatives are chosen). An interesting feature of (260) and (261) is that the modular transformations of the generalized Theta functions result in linear combinations of other generalized Theta functions at the same level k. From (260), (261) and (254) it is possible to compute the effect of a modular transformation on the characters or normalized characters. We just list the results here. It is easiest to work with the normalized characters $\tilde{\chi}$- everything can be translated back to the language of characters χ if need be. Firstly the normalized characters of integrable highest weight representations at a given level transform amongst themselves. We write

$$S\tilde{\chi}_{\hat{\lambda}} = \sum_{\hat{\lambda}' \in \hat{P}_k^+} S_{\hat{\lambda}\hat{\lambda}'} \tilde{\chi}_{\hat{\lambda}'} \tag{262}$$

$$T\tilde{\chi}_{\hat{\lambda}} = \sum_{\hat{\lambda}' \in \hat{P}_k^+} T_{\hat{\lambda}\hat{\lambda}'} \tilde{\chi}_{\hat{\lambda}'} \tag{263}$$

(264)

where $\hat{P}_k^+ \subset \hat{P}_k$ is the subset of dominant affine weights at level k. The explicit forms of these matrices are

$$S_{\hat{\lambda}\hat{\lambda}'} = (-1)^{|\Phi^+|} |P/Q^\vee|^{-\frac{1}{2}} (k+g)^{-\frac{1}{2}} \sum_{w \in W} exp(\frac{-2\pi i}{k+g}(w(\hat{\lambda}+\hat{\rho}), \hat{\lambda}'+\hat{\rho})) \tag{265}$$

$$T_{\hat{\lambda}\hat{\lambda}'} = e^{2\pi i s_{\hat{\lambda}}} \delta_{\hat{\lambda},\hat{\lambda}'} \tag{266}$$

It can be shown that the matrix $S_{\hat{\lambda},\hat{\lambda}'}$ is unitary and symmetric. Clearly the same is true of $T_{\hat{\lambda},\hat{\lambda}'}$. The matrices $S_{\hat{\lambda},\hat{\lambda}'}$ and $T_{\hat{\lambda},\hat{\lambda}'}$ do not constitute a representation of the modular group $PSL_2(\mathbb{Z})$ since they do not satisfy the defining properties $S^2 = 1$ and $(ST)^3 = 1$. Rather, they satisfy $S^4 = 1$, and $(ST)^6 = 1$, and so they give a representation of the two-fold covering $SL_2(\mathbb{Z})$ on the space of normalized characters instead. In fact, what we get is $S^2 = (ST)^3 = C$ where C is the **charge conjugation matrix** defined as $C_{\hat{\lambda},\hat{\lambda}'} = \delta_{\hat{\lambda},\hat{\lambda}^+}$ where $\hat{\lambda}^+$ is the highest weight of the conjugate representation to $R_{\hat{\lambda}}$ (explicitly $-w_0 \cdot \hat{\lambda} = \hat{\lambda}^+$ where w_0 is the longest element of the affine Weyl group).

8 Wess Zumino Witten models

8.1 Sigma model and WZW model

Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$.

Definition 8.1 *The* ***sigma model*** *associated to G is the field theory with action*

$$S_0[g] = \frac{1}{4a^2} \int_{S^2} d^2z \, Tr'(\partial_\mu g^{-1} \partial^\mu g) \tag{267}$$

where $g(z,\bar{z}) : \hat{\mathbb{C}} = S^2 \to R(G)$ is a map from the 2-sphere to some unitary faithful matrix representation of G, and $Tr' := \frac{Tr}{x_R}$ where x_R is the Dynkin index of the representation on $\mathfrak{g}$ induced by the representation R on G. This normalization ensures that this action is independent of the particular representation that is chosen.

The sigma model has an important $G_L \times G_R$ symmetry given by $g(z,\bar{z}) \to g_L g(z,\bar{z}) g_R^{-1}$ for any $g_L \in G_L \cong G$ and $g_R \in G_R \cong G$. Since G is a Lie group, this is a continuous symmetry and from Noether we obtain a conserved current. If we just focus on the G_R part of the symmetry, the corresponding conserved current is $J_\mu = g^{-1}\partial_\mu g$. As we will see after computing the variation δS_0, the vanishing of $\partial^\mu J_\mu$ is actually the same statement as the classical equations of motion. When we move to complex coordinates, this equation reads $\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}^z = 0$ where we have defined $\tilde{J}_z := g^{-1}\partial_z g$ and $\tilde{J}_{\bar{z}} := g^{-1}\partial_{\bar{z}} g$ (we put tildes because we would like to reserve the notation $J_{z/\bar{z}}$ for currents in the full WZW model). In order for the quantum theory to have holomorphic factorization, we expect that each of the terms above ($\partial_z J^{\bar{z}}$ and $\partial_{\bar{z}} J^z$) should vanish on their own, but in fact this does not hold for the sigma model.
Classically the sigma model is conformally invariant, but when we move to the quantum theory beta function calculations show that the sigma model is no longer conformally invariant (CFTs are fixed points of RG flow and hence beta functions should vanish at a true (quantum) conformal field theory). We can add a term to this action to make it so that the resulting theory is classically and quantum mechanically conformally invariant.

Definition 8.2 *The* ***Wess-Zumino term*** *is*

$$\Gamma[g] := \frac{-i}{24\pi} \int_B d^3y \, \epsilon_{\alpha\beta\gamma} Tr(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g) \tag{268}$$

We will take as our total action $S_{WZW}[g] = S_0[g] + k\Gamma[g]$ for some k (k will turn out to be the (integer) level that we talked about when discussing integrable highest weight representations of affine Lie algebras). We need to explain what B is, and how $g : S^2 \to R(G)$ can be integrated over it. B is the closed 3 dimensional ball, i.e $B \cong \{(x,y,z)|x^2+y^2+z^2 \leq 1\}$ and the boundary $\partial B = \{(x,y,z)|x^2+y^2+z^2 = 1\}$ is identified with the S^2 on which g is defined. So we have g defined on the boundary of B, but we still need to clarify what values g takes inside the ball. The second homotopy group $\pi_2(G)$ of any Lie group vanishes, which tells us that every map $g : S^2 \to G$ is nullhomotopic. We can use this nullhomotopy $g_t : S^2 \times [0,1] \to G$ to construct an extension

$g^{ext}: B \to G$ of g by setting the restriction $g^{ext}\Big|_{\{(x,y,z)|x^2+y^2+z^2=t\}\cong S^2}$ equal to g_t (so g_0 should actually be the constant map and $g_1 = g$). Now we have an extension, and we would like to know if it is unique. In fact it is not; there are a $\mathbb{Z}'s$ worth of homotopy classes of such extensions if we put some conditions on G. Take two extensions g_1 and g_2 of g. Then we can define a map $h: S^3 \cong B_1 \cup B_2/(\partial B_1 \sim \partial B_2) \to G$ by letting h be g_i on B_i; this prescription makes sense because by definition the extensions g_1 and g_2 agree on their boundary. Maps from S^3 to G are of course classified up to homotopy by $\pi_3(G)$, and if we impose the further conditions on G (namely that G has to be compact and simple) we have from algebraic topology that $\pi_3(G) = \mathbb{Z}$. Taking two extensions g_1 and g_2 and constructing $h: S^3 \to G$ from them as above, if the class of h in $\pi_3(G)$ is n, then it can be shown that $\Gamma[g_1] = \Gamma[g_2] + 2\pi i n$. This is not an issue so long as k is an integer, because then the Euclidean path integral integrand $e^{-S_{WZ}}$ will be well defined. So k must be an integer, which agrees with the fact that k will eventually be identified as the level of a $\hat{\mathfrak{g}}$ representation.
We can compute the variation of $\Gamma[g]$ with respect to small homotopies of g rel the boundary S^2 of B, and the result is

$$\delta\Gamma = \frac{i}{8\pi}\int_{S^2} d^2z\, \epsilon_{\mu\nu} Tr'(g^{-1}\delta g \partial^\mu(g^{-1}\partial^\nu g)) \tag{269}$$

We can also compute the variation of S_0 to get

$$\delta S_0 = \frac{1}{2a^2}\int_{S^2} d^2z\, Tr'(g^{-1}\delta g \partial_\mu(g^{-1}\partial^\mu g)) \tag{270}$$

which is where the equation of motion for the sigma model comes from. We can read the equation of motion off from $\delta S_{ZW} = \delta S_0 + k\delta\Gamma$, convert it into complex coordinates, and the result is

$$(1 + \frac{a^2 k}{4\pi})\partial_z(g^{-1}\partial_{\bar{z}} g) + (1 - \frac{a^2 k}{4\pi})\partial_{\bar{z}}(g^{-1}\partial_z g) = 0 \tag{271}$$

We choose $a^2 = \frac{4\pi}{k}$ so that this second term vanishes, and the EOM reads $\partial_z J_{\bar{z}} := \partial_z(g^{-1}\partial_{\bar{z}} g) = 0$. If we define $J_z := \partial_z g g^{-1}$ then the anti-holomorphicity of $J_{\bar{z}}$ implies the holomorphicity of J_z :

$$\begin{aligned}
&g^{-1}(\partial_{\bar{z}} J_z)g = g^{-1}(\partial_{\bar{z}}[(\partial_z g)g^{-1}])g = \\
&g^{-1}([\partial_{\bar{z}}\partial_z g]g^{-1} + \partial_z g \partial_{\bar{z}} g^{-1})g = \\
&g^{-1}[\partial_{\bar{z}}\partial_z g] + g^{-1}\partial_z g(-g^{-1}(\partial_z g)g^{-1})g = \\
&g^{-1}[\partial_{\bar{z}}\partial_z g] - g^{-1}(\partial_z g)(g^{-1}\partial_z g) = \partial_z(g^{-1}\partial_{\bar{z}} g) = \partial_z J_{\bar{z}} = 0
\end{aligned} \tag{272}$$

We used the relation $\partial_\mu g^{-1} = -g^{-1}(\partial_\mu g)g^{-1}$ twice in this calculation. We can normalize J_z and $J_{\bar{z}}$ to obtain

Definition 8.3 *The conserved currents of the WZW model are*

$$J := -kJ_z = -k(\partial_z g)g^{-1} \tag{273}$$
$$\bar{J} := kJ_z = kg^{-1}\partial_{\bar{z}} g \tag{274}$$

The WZW model still has the $G_L \times G_R$ symmetry of the sigma model, but this symmetry actually gets substantially enhanced in the WZW model: the WZW action is invariant under $g \to g_L(z)g(z,\bar{z})g_R(\bar{z})^{-1}$ for any *functions* g_L and g_R. This can be checked infinitesemally by taking $g_L(z) = 1 + \omega(z)$ and $g_R(\bar{z}) = 1 + \bar{\omega}(\bar{z})$ for $\omega, \bar{\omega}: S^2 \to \mathfrak{g}$. We obtain $\delta_\omega g = (1+\omega)g - g = \omega g$ and similarly $\delta_{\bar{\omega}} g = g(1-\bar{\omega}) - g = -\bar{\omega} g$. We can compute the variation of the action S_{WZ} with respect to $g \to g + \delta_{\omega,\bar{\omega}} g = g + \delta_\omega g + \delta_{\bar{\omega}} g$ and the result is that

$$\begin{aligned}
&\delta_{\omega,\bar{\omega}} S_{WZW} = \\
&\frac{k}{2\pi}\int d^2x Tr'[\omega(z)\partial_{\bar{z}}(\partial_z g g^{-1}) - \bar{\omega}(\bar{z})\partial_z(g^{-1}\partial_{\bar{z}} g)] \\
&\frac{-1}{2\pi}\int d^2x \partial_{\bar{z}} Tr'[\omega(z)J(z)] + \partial_z Tr[\bar{\omega}(\bar{z})\bar{J}(\bar{z})] = \\
&\frac{i}{4\pi}\oint dz Tr'[\omega(z)J(z)] - \frac{i}{4\pi}\oint d\bar{z} Tr'[\bar{\omega}(\bar{z})\bar{J}(\bar{z})] = \\
&-\frac{1}{2\pi i}\oint dz \omega^a J^a + \frac{1}{2\pi i}\oint d\bar{z}\bar{\omega}^a \bar{J}^a
\end{aligned} \tag{275}$$

The first of these expressions follows directly from varying the action. The second is the result of an integration by parts and substituting in J and $\bar{J}$. The third is moving to complex coordinates and using complex analysis to write a $dzd\bar{z}$ integral of derivatives into a sum of a holomorphic and anti-holomorphic contour integral. The last is writing $\omega = \omega^a t^a$ and $J = J^a t^a$ as a sum of basis elements t^a of (a representation of) $\mathfrak{g}$ and using the normalization condition $Tr'(t^a t^b) = 2\delta^{ab}$ (the factor x_R - the Dynkin index - in Tr' is put there specifically so that this conidtion holds). Some things to note about (275) are that it vanishes, which we apparently get by integrating the first expression by parts...

Remark 23 *I am yet to understand this although apparently it is obvious? I thought for a second that it just vanished because of $\partial_z \bar{J}$ and $\partial_{\bar{z}} J$ are present in the first expression, but current conservation is only guaranteed to hold on shell.*

The vanishing of (275) is good because it confirms at an infinitesimal level that we do indeed have an upgraded $G_L(z) \times G_R(\bar{z})$ symmetry in the WZW theory. The second thing to note is that we can write a Ward identity associated to this symmetry. Using the last expression of (275), it takes the form

$$\delta_{\omega,\bar{\omega}}\langle X\rangle = -\frac{1}{2\pi i}\oint dz\omega^a\langle J^a X\rangle + \frac{1}{2\pi i}\oint d\bar{z}\bar{\omega}^a\langle \bar{J}^a X\rangle \tag{276}$$

We can compute $\delta_\omega J$ from the fact that $J = -k\partial_z g g^{-1}$ and get

$$\delta_\omega J = -k(\partial_z(\delta_\omega g)+(\partial_z g)\delta_\omega g^{-1}) = -k(\partial_z(\omega)g+\omega\partial_z(g))+k(\partial_z g)(g^{-1}\omega g g^{-1}) = [\omega, J]-k\partial_z\omega \tag{277}$$

Writing (277) in terms of the components J^a of J gives

$$\delta_\omega J^a = if_{abc}\omega^b J^c - k\partial_z\omega^a \tag{278}$$

On the other hand we can use (276) to get

$$\delta_\omega\langle J^b(w)\rangle = \frac{1}{2\pi i}\oint dz\omega^a\langle J^a(z)J^b(w)\rangle \tag{279}$$

Using Cauchy and comparing (278) with (279) gives us the $J^a J^b$ OPE:

$$J^a(z)J^b(w) = \frac{k\delta_{ab}}{(z-w)^2} + \frac{if_{abc}J^c(w)}{z-w} + ... \tag{280}$$

Remark 24 *Each J^a turns out to have conformal dimension 1. Classically this is clear by computing the dimension of g from the action, but the fact that this dimension receives no quantum corrections apparently follows because (according to Lorenz Eberhardts notes) 'holomorphic quantities are protected.' I need to learn what this means, I am not sure at the moment.*

We can mode expand each J^a as

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n \tag{281}$$

As always, knowledge of the singular part of the OPE is equivalent to knowledge of commutators between all modes. This yields for (280) and (281) the defining relations of the affine Lie algebra $\hat{\mathfrak{g}}$

$$[J^a_n, J^b_m] = if_{abc}J^c + kn\delta_{n,-m}\delta_{ab} \tag{282}$$

Since k takes a particular value here and is not just some central element, we see that the modes form a representation at level k. We can of course write this all in a particular basis, i.e a Cartan Weyl basis or even more specifically a Chevalley-Serre basis, and then the commutation rules will take the form that we are familiar with from the last two sections. Let's note that we do not yet have an element D adjoined so it is a bit of a lie to say we have $\hat{\mathfrak{g}}$; we just have the central extension of the loop algebra over $\mathfrak{g}$ so far. D will turn out to be $-L_0$, where the Virasoro modes will be defined via the Sugawara construction in the next section. This definition of D will actually force us to consider non-zero third components of affine weights $\hat{\lambda}(D) \neq 0$ because we won't be able to just translate D by some constant to set the eigenvalue to 0 anymore.

8.2 Sugawara's construction of the energy momentum tensor

So far we do not have Virasoro algebra anywhere in sight, nor have we discussed the energy momentum tensor of the WZW model. We now remedy this situation.

Definition 8.4 *The **Sugawara energy momentum tensor** is the field*

$$T(z) := \gamma(J^a J^a)(z) \tag{283}$$

There is an implied sum $a = 1, ..., \dim\mathfrak{g}$ and γ is a constant that is to be determined later.

Before we jump into the analysis of OPE's that arise from this definition, lets take a second to think what this says algebraically. The modes of the stress energy tensor form the Virasoro algebra of a conformal field theory. This proposed tensor therefore must satisfy the standard OPE

$$T(z)T(w) \sim \frac{\mathbf{c}}{2(z-w)^4} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

which is of course completely equivalent to its modes satisfying the Virasoro algebra. So we need to show that this OPE holds. This is possible to do because we know the $J^a(z)J^b(w)$ OPE and T is defined in terms of the J's. Assuming that we have verified this, let's recall equation (67) which gave us the modes of the normal ordering of two fields in terms of quadratic expressions in the modes of the original fields. This tells us that quadratic expressions in the modes of the J^a currents (which are elements of the universal enveloping algebra of $\hat{\mathfrak{g}}$) form a Virasoro algebra, and hence every affine Lie algebra contains a Virasoro algebra in its universal enveloping algebra.

Using Wick contractions, we can compute the OPE of $T(z)$ with $J^a(w)$ and with itself. These computations will help us fix the normalization constant γ. We begin by writing the Wick rule

$$\overbracket{A(z)(BC)}(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}[\overbracket{A(z)B}(x)C(w) + B(x)\overbracket{A(z)C}(w)] \tag{284}$$

which is a consequence of formal calculus. Applying this rule to $A(z) = J^a(z)$ and $B = C = J^b$ and inserting the OPE (280) yields

$$\begin{aligned}
&\overbracket{J^a(z)(J^bJ^b)}(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}[\overbracket{J^a(z)J^b}(x)J^b(w) + J^b(x)\overbracket{J^a(z)J^b}(w)] = \\
&\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}([\frac{k\delta_{ab}}{(z-x)^2} + \frac{if_{abc}J^c(x)}{z-x}]J^b(w) + J^b(x)[\frac{k\delta_{ab}}{(z-w)^2} + \frac{if_{abc}J^c(w)}{z-w}]) = \\
&\frac{1}{2\pi i}\oint \frac{dx}{x-w}\Big[\frac{k\delta_{ab}J^b(w)}{(z-x)^2} + \frac{if_{abc}}{z-x}[\frac{k\delta_{cb}}{(x-w)^2} + \frac{if_{cbd}J^d(w)}{x-w} + (J^cJ^b)(w) + ...]\Big] + \\
&\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\Big[\frac{k\delta_{ab}J^b(x)}{(z-w)^2} + \frac{if_{abc}}{z-w}[\frac{k\delta_{bc}}{(x-w)^2} + \frac{if_{cbd}J^d(w)}{(x-w)} + (J^bJ^c)(w) + ...]\Big]
\end{aligned} \tag{285}$$

The third and fourth lines of (285) are the summands corresponding to the contractions $\overbracket{J^a(z)J^b}(x)$ and $\overbracket{J^a(z)J^b}(w)$respectively. They look very similar, but there is actually an asymmetry between these terms caused by the fact that we are integrating with respect to x. The first term on the fourth line of (285) can be evaluated using Cauchy's integral formula to get

$$\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\frac{k\delta_{ab}J^b(x)}{(z-w)^2} = \frac{k\delta_{ab}J^b(w)}{(z-w)^2} \tag{286}$$

The rest of the fourth line has all of its x dependence in the powers of $(x-w)$; by Cauchy again, we conclude that only the term with a first order pole at $x = w$ could survive. Higher order poles in the integrand would give us derivatives with respect to x of a constant with respect to x, and the quantities in the ellipses are all order $x - w$, so even after dividing them by the $\frac{1}{x-w}$ out front they will be holomorphic and integrate to 0. The order 1 pole at $x = w$ is the term

$$\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\frac{if_{abc}}{z-w}(J^bJ^c)(w) = \frac{if_{abc}(J^bJ^c)(w)}{z-w} \tag{287}$$

So the total contribution from the fourth line is

$$\frac{if_{abc}(J^bJ^c)(w)}{z-w} + \frac{k\delta_{ab}J^b(w)}{(z-w)^2}$$

The third line of (285) is a bit more involved. A simple application of Cauchy's integral theorem for the first term on the third line gives us

$$\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\frac{k\delta_{ab}J^b(w)}{(z-x)^2} = \frac{k\delta_{ab}J^b(w)}{(z-w)^2} \tag{288}$$

which we notice is identical to the first contribution (286) from the fourth line. The rest of the (non-zero) terms on the third line read

$$\frac{1}{2\pi i}\oint_w dx\frac{if_{abc}}{z-x}\Big[\frac{k\delta_{cb}}{(x-w)^3} + \frac{if_{cbd}J^d(w)}{(x-w)^2} + \frac{(J^cJ^b)(w)}{x-w}\Big]$$

we could integrate these terms individually, but it is quicker to realize that

- the first one dies since $f_{abc}\delta_{cb} = f_{acc} = 0$ since Lie algebra structure constants are antisymmetric.
- the last one integrates to $\frac{if_{abc}(J^cJ^b)(w)}{z-w}$. If we sum this up with the contribution (287), by the symmetricity of $(J^bJ^c)(w)+(J^cJ^b)(w)$ under $b \iff c$ and the antisymmetricity of f_{abc} these two terms cancel each other.
- the middle term integrates to $\frac{if_{abc}if_{cbd}J^d(w)}{(z-w)^2}$.

So the final result is

$$\overbracket{J^a(z)(J^bJ^b)}(w) = \frac{2k\delta_{ab}J^b(w)}{(z-w)^2} + \frac{-f_{abc}f_{cbd}J^d(w)}{(z-w)^2} \tag{289}$$

We have implied sums on c, b and d in this second term. If we do the summation on b and c first, then using the identity $-f_{abc}f_{cbd} = 2g\delta_{ad}$ where g is the previously defined dual Coxeter number of $\mathfrak{g}$. So we get that

$$\overbracket{J^a(z)T}(w) = \gamma\overbracket{J^a(z)(J^bJ^b)}(w) = \frac{2\gamma k\delta_{ab}J^b(w)}{(z-w)^2} + \frac{2\gamma g\delta_{ad}J^d(w)}{(z-w)^2} = 2\gamma(g+k)\frac{J^a(w)}{(z-w)^2} \tag{290}$$

By swapping the variables z and w and using commutativity of fields in a contraction and then Taylor expanding, this gives us

$$\overbrace{T(z)J^a}(w) = 2\gamma(k+g)\frac{J^a(z)}{(z-w)^2} = 2\gamma(k+g)\Big[\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\Big] \tag{291}$$

We would like J^a to be a primary field of our theory so we take the previously unspecificed constant γ to now be

$$\gamma = \frac{1}{2(k+g)} \tag{292}$$

We can verify that such a choice for γ gives the $T(z)T(w)$ OPE its proper form $\frac{\mathbf{c}}{2(z-w)^4} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$, and from this calculation we can also obtain the value $\mathbf{c}$ for our theory. We just use Wick's theorem again, and now employ our knowledge of the TJ^a, J^aJ^a and ∂J^aJ^a OPEs (the last follows from just taking derivatives of the J^aJ^a OPE).

$$\begin{aligned}\overbrace{T(z)T}(w) =& \frac{1}{2(k+g)}\overbrace{T(z)(J^aJ^a})(w) = \\ & \frac{1}{2(k+g)}\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\Big\{\frac{1}{(z-x)^2}[\frac{k\dim\mathfrak{g}}{(x-w)^2} + 0 + (J^aJ^a)(w) + ...]+ \\ & \frac{1}{z-x}[\frac{-2k\dim\mathfrak{g}}{(x-w)^3} + (\partial J^aJ^a)(w) + ...]+ \\ & \frac{1}{(z-w)^2}[\frac{k\dim\mathfrak{g}}{(x-w)^2} + 0 + (J^aJ^a)(w) + ...] \\ & \frac{1}{z-w}[\frac{2k\dim g}{(x-w)^3} + (\partial J^aJ^a)(x) + ...]\Big\}\end{aligned} \tag{293}$$

As with the last OPE calculation, on each of the last two lines here only one term can contribute, and those are the terms with no $x-w$ power i.e $(J^aJ^a)(w)$ and $\partial J^aJ^a(w)$. These terms, once integrated, then contribute a total $\frac{1}{2(k+g)}[\frac{(J^aJ^a)(w)}{(z-w)^2} + \frac{(\partial J^aJ^a)(w)}{z-w}]$. So we just need to calculate the contributions from the second and third lines. The pieces with $k\dim g$'s on them are evaluated using Cauchy's integral therem applied to $\frac{1}{(z-x)^2}$ and $\frac{1}{z-x}$. The result is that those terms integrate to $\frac{1}{2(k+g)}[\frac{3k\dim\mathfrak{g}}{(z-w)^4} - \frac{2k\dim\mathfrak{g}}{(z-w)^4}] = \frac{1}{2(k+g)}\frac{k\dim\mathfrak{g}}{(z-w)^4}$. Furthermore, these are the only terms in (293) that give a $(z-w)^{-4}$ contribution, and hence we identify $\frac{\mathbf{c}}{2} = \frac{k\dim\mathfrak{g}}{2(k+g)}$ or in other words

$$\mathbf{c} = \frac{k\dim\mathfrak{g}}{k+g} \tag{294}$$

The other terms can be similarly integrated to get the standard $\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$ part of the TT OPE. All of this analysis could have been done at the mode level using equation (67) and the known relations of $\hat{\mathfrak{g}}$. We note that (67) for T can be written as

$$L_n = \frac{1}{2(k+g)}\sum_m : J^a_m J^a_{m-n} : \tag{295}$$

where the normal ordering on modes here works simply by putting the positive one on the right. We already know the commutation relations of $\hat{\mathfrak{g}}$ and of the Virasoro algebra, but using this relation we can calculate how the Virasoro modes interact with $\hat{\mathfrak{g}}$:

$$[L_n, J^a_m] = -mJ^a_{n+m} \tag{296}$$

8.3 WZW primaries

Let's recall our discussion in section 6 of extended symmetry. We are in exactly that situation now; the (chiral half of the) symmetry algebra of our theory is $\mathcal{W} = \hat{\mathfrak{g}} \oplus \mathfrak{vir}$, a semi direct sum whose complications are delineated by (296).

Remark 25 *I am a little bit unsure about the vector space decomposition $\mathcal{W} = \hat{\mathfrak{g}} \oplus \mathfrak{vir}$, but that is how it is written in Fuchs. It seems like these two subspaces both share $D = -L_0$, the element of the universal enveloping algebra of $(\mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\hat{k})$ and so do not intersect trivially. I wonder if it should really be $\mathcal{W} = (\mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\hat{k}) \oplus \mathfrak{vir}$ and we can take the 1D subspace spanned by L_0 in $\mathfrak{vir}$ and direct sum it with $\mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\hat{k}$ to get the full affine Lie algebra $\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}(-L_0)$ as a subalgebra of $\mathcal{W}$. In either case we still have a $\hat{\mathfrak{g}}$ and a $\mathfrak{vir}$ subalgebra, they just intersect nontrivially in the way that I have written it.*

Hence it makes sense to talk about fields and states that are primary with respect to this larger symmetry algebra $\mathcal{W}$, or various subalgebras of it such as $\hat{\mathfrak{g}}$ or $\mathfrak{vir}$. The W^i_n, $i \neq 0$ will now correspond to modes J^a_n for $a = 1, ..., \dim\mathfrak{g}$ and by definition for $i = 0$ we have $W^0_n = L_n$. k is central. We can give this $\mathcal{W}$ a triangular decomposition just as in section 6. Let's recall how this goes, in the specific case of $\mathcal{W} = \hat{\mathfrak{g}} \oplus \mathfrak{vir}$. First, $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^0 \oplus \mathcal{W}^-$ where $\mathcal{W}^+ = \hat{\mathfrak{g}}^+ \oplus \mathfrak{vir}^+$, $\mathcal{W}^- = \hat{\mathfrak{g}}^- \oplus \mathfrak{vir}^-$ and W^0

is the subspace spanned by W_0^i's (so the horizontal subalgebra of $\hat{g}$ and L_0) and the central terms k. Then we find a maximal abelian subalgebra of $\mathcal{W}^0$ that contains L_0, and call it $\mathcal{W}_0$. One such choice is $\mathcal{W}_0$ spanned by the Cartan subalgebra of the horizontal subalgebra of $\hat{\mathfrak{g}}$ along with L_0, and k. The commutativity of the Cartan subalgebra of the horizontal subalgebra of $\hat{\mathfrak{g}}$ is inherent, and all of these horizontal Cartan elements also commute with L_0 as is apparent from (296). k commutes with everything. The corresponding $\mathcal{W}_+$ and $\mathcal{W}_-$ that give us our triangular decomposition are then given by $\mathcal{W}_+$ being the subspace of $\mathcal{W}^0 \oplus \mathcal{W}^+$ spanned by $\mathcal{W}^+$ and the step operators corresponding to positive roots of the horizontal algebra in $\hat{\mathfrak{g}}$ and similarly for $\mathcal{W}_-$. So we have $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_0 \oplus \mathcal{W}_-$ is the triangular decomposition. The triangular decomposition allows us to form Verma modules of $\mathcal{W}$ and quotient them to get arbitrary highest weight modules.
Highest weight modules for $\mathcal{W}$ can be labeled by an affine $\hat{\mathfrak{g}}$ weight $\hat{\lambda}$, because the eigenvalues of L_0 and $\mathbf{c}$ on a level k state $|\hat{\lambda}\rangle$ are completely fixed by (294) and a relation that we will derive shortly that gives the conformal weight $h_{\hat{\lambda}}$ of $|\hat{\lambda}\rangle$. To every primary state $|\hat{\lambda}\rangle$ generating its own (affine) sector, by the state operator correspondence we get a primary field $\phi_{\hat{\lambda}}$. In the next section when we analyze the Gepner Witten equation we will see that fields corresponding to weights that are not dominant integrable automatically decouple from the other fields of the theory - that is, the correlation functions involving that field or its descendants all vanish. Hence the only relevant fields to the model are the ones corresponding to level k dominant integrable weights $\hat{\lambda} \in P_k^+$. We know there are only finitely many such $\hat{\lambda}$ for a given level k, those integral weights satisfying $(\lambda, \theta) \leq k$. Hence the WZW model for $\mathfrak{g}$ at level k is an example of a rational conformal field theory (defined briefly before, but it means a theory that has finitely many sectors with respect to its maximal symmetry algebra).
In our discussion from part 6 we talked about the conditions necessary for a field ϕ to be a primary field of some subalgebra of the maximal algebra $\mathcal{W}$. The examples we have encountered thus far are

- The $\mathfrak{sl}(2)$-primary fields are those fields ϕ that satisfy $[L_1, \phi(0)] = 0$. Equivalently, $L_1 |\phi\rangle = L_1\phi(0)|0\rangle = 0$.
- The $\mathfrak{vir}$ primary fields are those fields ϕ that satisfy $[L_n, \phi(0)] = 0$ for all $n > 0$. Equivalently, $L_n |\phi\rangle = L_n\phi(0)|0\rangle = 0$.

The analog of this for a general subalgebra V of $\mathcal{W}$ is that a V primary field needs to have trivial commutators with all of the modes in the positive part of the triangular decomposition of V. Instead of calling a field a $\hat{\mathfrak{g}}$ primary field, we use the terminology **WZW primary field**. Indeed every field that is a $\hat{\mathfrak{g}}$ primary field is automatically a $\mathfrak{vir}$ primary field (this can be seen from the expression (295)), and hence is a primary field of the entire $\mathcal{W} = \hat{\mathfrak{g}} \oplus \mathfrak{vir}$, so this definition makes good sense.

Remark 26 *I have read that the only modular invariants that can be formed for a theory with maximal symmetry algebra are the diagonal ones. So for the WZW models with non-diagonal spectra and modular invariant partition functions, it seems like the true symmetry algebra of the theory is larger than just $\hat{\mathfrak{g}} \oplus \mathfrak{vir}$, and therefore maybe calling $\hat{\mathfrak{g}}$-primary fields WZW primary fields is a bit of a misnomer. Its just terminology anyways, so it doesn't matter.*

A WZW primary field $\phi_{\lambda,\mu}$ is labeled by two dominant integrable highest weights, one for the chiral half and the other for the anti-chiral half. Notice that we can write λ instead of $\hat{\lambda}$ without ambiguity: we are talking about level k weights, and using the relation (239) that $\lambda_0 = k - (\lambda, \theta)$ we can recover the last Dynkin label λ_0 of $\hat{\lambda}$, and the D eigenvalue is just $-h_\lambda$ (h_λ defined below in equation (301)). It also is good to write the labels for ϕ in terms of weights of $\mathfrak{g}$ instead of affine weights of $\hat{\mathfrak{g}}$ because ϕ is a field that transforms in the representations R_λ and R_μ of $\mathfrak{g}$. We can see this explicitly in the form that the $J^a\phi_{\lambda,\mu}$ and $\bar{J}^a\phi_{\lambda,\mu}$ OPEs take

$$J^a(z)\phi_{\lambda,\mu}(w,\bar{w}) \sim \frac{-t^a_\lambda \phi_{\lambda,\mu}(w,\bar{w})}{z-w} \tag{297}$$

$$\bar{J}^a(\bar{z})\phi_{\lambda,\mu}(w,\bar{w}) \sim \frac{\phi_{\lambda,\mu}(w,\bar{w})t^a_\mu}{\bar{z}-\bar{w}} \tag{298}$$

This is a *matrix* equation because $\phi_{\lambda,\mu}$ is valued in the representation space $R_\lambda \otimes R_\mu$ of $\mathfrak{g}$. t^a_R denotes the image of t^a under a representation homomorphism $\mathfrak{g} \xrightarrow{R} End(V)$ (we recall that t^a is the basis of $\mathfrak{g}$ used to obtain the components J^a from the current J). There are various other ways to express that ϕ is a primary field.

Remark 27 *We restrict attention to the chiral half of the theory so we write things like ϕ_λ from now on.*

One equivalent way to express that ϕ_λ is a WZW primary is

$$(J^a_n\phi_\lambda) = \begin{cases} -t^a_\lambda\phi_\lambda & n = 0 \\ 0 & n > 0 \end{cases} \tag{299}$$

where $J^a_n(w)$ is the field we get from mode expanding J^a away from 0 and the LHS of the above is the normal ordered product. If we make the usual definition of the state associated to a field

$|\phi_\lambda\rangle := \phi_\lambda(0)\,|0\rangle$, then another way we can express the condition that ϕ_λ is a WZW primary is

$$J^a_n\,|\phi_\lambda\rangle = \begin{cases} -t^a_\lambda\,|\phi_\lambda\rangle & n = 0 \\ 0 & n > 0 \end{cases} \tag{300}$$

This last characterization of WZW primaries allows us to see that a WZW primary is also a $\mathfrak{vir}$ primary from the formula (295) applied when n is positive.
From the section on affine Lie algebras we remember that $[\lambda_0, ..., \lambda_r]$ is not actually enough labels to describe $\hat{\lambda}$ - we still need $\hat{\lambda}(D)$, but in fact we know it in this case since we are taking D to be a particular element of the universal enveloping algebra $\mathfrak{U}(\mathcal{L}(\mathfrak{g})\oplus\mathbb{C}k)$, namely $D = -L_0 = \frac{1}{2(k+g)}\sum_n :J^a_{-n}J^a_n:$. We can calculate using (300)

$$h_\lambda\,|\phi_\lambda\rangle := L_0\,|\phi_\lambda\rangle = \frac{1}{2(k+g)}J^a_0 J^a_0\,|\phi_\lambda\rangle = \frac{1}{2(k+g)}t^a_\lambda t^a_\lambda\,|\phi_\lambda\rangle = \frac{(\lambda, \lambda + 2\rho)}{2(k+g)}\,|\phi_\lambda\rangle \tag{301}$$

There are a lot more terms in the expansion of L_0 in terms of the modes of J^a's, but these are the only ones that don't annhilate $|\phi_\lambda\rangle$ via (300). In the last equality we have recognized the appearance of the quadratic Casimir operator for the representation R_λ of $\mathfrak{g}$, and used (199) which tells us the value that it takes in a particular irrep.
From the OPE (280) we know that J is not itself a WZW primary field, although from the OPE (291) we know that it IS a $\mathfrak{vir}$ primary field. So we see that all WZW primaries are $\mathfrak{vir}$ primaries, but not all $\mathfrak{vir}$ primaries are WZW primaries. Indeed there will be infinitely many Virasoro primary fields, obtained by acting with the on the true WZW primaries with the elements of $\mathcal{W}_-$ that are in $\mathcal{W}^0$. In terms of the state space, this means that even though there are finitely many sectors in the $\mathcal{W}$ decomposition, in the $\mathfrak{vir}$ decomposition there will be infinitely many sectors.

8.4 Null vector equations

We have already encountered differential equations that arose as a consequence of a null vector. These were the BPZ equations which follow from the existence of null vectors of level $N = rs$ in the (Virasoro)-Verma module $V(\mathbf{c}, h_{r,s}(\mathbf{c}))$. In the WZW model, there are two important types of null vector equations. The first is called the **Knizhnik-Zamolodchikov equation (KZ equation)** which arises from a null vector that expresses the relation between the Virasoro algebra of our theory and the affine Lie algebra $\hat{\mathfrak{g}}$. Namely, using equation (295) and (299) we get

$$\begin{aligned} L_{-1}\,|\phi_\lambda\rangle =& \frac{1}{k+g}J^a_{-1}J^a_0\,|\phi_\lambda\rangle = \frac{-1}{k+g}J^a_{-1}t^a_\lambda\,|\phi_\lambda\rangle \implies \\ &(L_{-1} + \frac{1}{k+g}J^a_{-1}t^a_\lambda)\,|\phi_\lambda\rangle = 0 \end{aligned} \tag{302}$$

This is the null vector that yields the KZ equation when inserted into correlation functions. To get the explicit differential equation, we first compute (using the contour definition of normal ordering, reversing the contour tightly wound around a point to be negative the sum of contours around all the other points, and the $J^a\phi_\lambda$ OPE)

$$\langle\phi_1(z_1)....(J^a_{-1}\phi_i(z_i))...\phi_n(z_n)\rangle = \sum_{j\neq i}\frac{t^a_j}{z_i - z_j}\langle\phi_1(z_1)...\phi_n(z_n)\rangle \tag{303}$$

Here we write ϕ_j in place of ϕ'_{λ_j}, a horizontal descendant of ϕ_{λ_j}, for simplicity. Using this and the equivalence between $\mathcal{L}_{-1}$ and ∂_{z_i} that was established in the section on normal ordering we get

$$[\partial_{z_i} + \frac{1}{k+g}\sum_{j\neq i}\frac{t^a_i\otimes t^a_j}{z_i - z_j}]\langle\phi_1(z_1)...\phi_n(z_n)\rangle \tag{304}$$

which is the KZ equation. The tensor symbol should be understand to mean that t_i acts on ϕ_i and t_j acts on ϕ_j. We won't dive further into the KZ equation, but like the BPZ equations it is interesting and fruitful to try to solve for the correlation functions that satisfy them.
The second null vector equation we will consider is the **Gepner Witten equation**. This is the equation that arises because of the null vector associated to the step operator of the simple root $\alpha_0 = (-\theta, 0, 1)$, namely $(E^\theta_{-1})^{k-(\theta,\lambda)+1}\,|\phi_\lambda\rangle = 0$. The existence of such a null vector is presupposing that $|\phi_\lambda\rangle$ belongs to an integrable representation, which we know is the case for at least the vacuum vector. Clearly raising E^θ_{-1} to a higher power will still give us a null vector, so for $p \geq k - (\theta, \lambda) + 1$ we get

$$\langle(E^\theta_{-1})^p\phi_\lambda(z)\phi_1(z_1)...\phi_n(z_n)\rangle = 0 \tag{305}$$

We can come up with an expression for the correlator by the standard procedure, writing the normal ordered product as a contour integral, using known OPE's, and doing residue calculus. The result is

$$\sum_{\substack{\ell_i=0 \\ \ell_1+...+\ell_n=p}}^{p}\frac{p!}{\ell_1!...\ell_n!}\frac{1}{(z-z_1)^{\ell_1}...(z-z_n)^{\ell_n}}\times\langle\phi_\lambda(z)\prod_{i=1}^{n}(E^\theta_0)^{\ell_i}\phi_i)(z_i)\rangle = 0 \tag{306}$$

This is the Gepner-Witten equation. As mentioned, we need $\lambda \in \mathfrak{g}_0^*$ to yield a dominant integrable weight $\hat{\lambda} \in \hat{\mathfrak{g}}_0^*$ in order for this analysis to go through. The identity field of the theory corresponds to the dominant integrable level k weight $k\hat{\omega}_0 = (0, k, 0)$, and hence is an example of such a primary field. Letting ϕ_λ be the identity, and integrating (306) times $(z - z_n)^{p-1}$ with respect to z we pick up the $\ell_i = p\delta_{in}$ term in the above sum which gives us

$$\langle \phi_1(z_1)....(E^{\theta}_{-1})^p \phi_n(z_n)\rangle = 0 \tag{307}$$

9 GKO coset models

9.1 Virasoro modes

The WZW model of the last section was constructed particularly for $\mathfrak{g}$ simple, but for $\mathfrak{g}$ semisimple we can take the energy momentum tensor of theory to be the sum of the Sugawara energy momentum tensors for each of the irreducible Lie algebras that $\mathfrak{g}$ is composed of. This immediately enlarges the amount of WZW theories that we have at our disposal, and this section will introduce a tool that allows for the construction of many other CFT's as a sort of quotient of WZW theories. Specifically, we start with an embedding of Lie algebras $\mathfrak{p}$

9.2 Identification group and field identification

9.3 Fixed points

9.4 Twining characters and orbit Lie algebras

9.5 Maverick cosets