Research Article

Amita Malik and Arindam Roy*

On the distribution of zeros of derivatives of the Riemann ξ-function

https://doi.org/10.1515/forum-2018-0081
Received April 12, 2018; revised October 7, 2018

Abstract: For the completed Riemann zeta function $\xi(s)$, it is known that the Riemann hypothesis for $\xi(s)$ implies the Riemann hypothesis for $\xi^{(m)}(s)$, where m is any positive integer. In this paper, we investigate the distribution of the fractional parts of the sequence $(\alpha\gamma_m)$, where α is any fixed non-zero real number and γ_m runs over the imaginary parts of the zeros of $\xi^{(m)}(s)$. We also obtain a zero density estimate and an explicit formula for the zeros of $\xi^{(m)}(s)$. In particular, all our results hold uniformly for $0 \le m \le g(T)$, where the function $g(T)$ tends to infinity with T and $g(T) = o(\log\log T)$.

Keywords: Riemann ξ-function, zeros, explicit formula, fractional parts, zero density

MSC 2010: Primary 11M26; secondary 11K38

Communicated by: Freydoon Shahidi

1 Introduction

The Riemann ξ-function is defined by

$$\xi(s) = H(s)\zeta(s), \tag{1.1}$$

where

$$H(s) := \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big), \tag{1.2}$$

and $\zeta(s)$ denotes the Riemann zeta function. The non-trivial zeros of $\zeta(s)$ are identical to the zeros of $\xi(s)$. It is well known that the real parts of the zeros of $\xi(s)$ lie in the critical strip $0 < \mathrm{Re}(s) < 1$. The Riemann hypothesis for $\xi(s)$ states that these zeros lie on the critical line $\mathrm{Re}(s) = 1/2$. Moreover, the Riemann hypothesis for $\xi(s)$ implies that the zeros of $\xi^{(m)}(s)$ also lie on the critical line $\mathrm{Re}(s) = 1/2$. In 1983, Conrey [4] showed that for $m \ge 0$, the real parts of the zeros of $\xi^{(m)}(s)$ also lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

There has also been a great interest in studying the vertical distribution of the zeros of $\xi(s)$. Under the assumption of the Riemann hypothesis, Rademacher [32] first proved that the sequence $(\alpha\gamma_0)$, where γ_0 denotes the imaginary part of a non-trivial zero of $\zeta(s)$ and α is any fixed non-zero real number, is uniformly distributed modulo one. Hlawka [19] proved this result unconditionally.

Let $\{x\}$ denote the fractional part of a real number x. Let $\rho_0 = \beta_0 + i\gamma_0$ denote a non-trivial zero of $\zeta(s)$. The discrepancy of the set $\{\{\alpha\gamma_0\} : 0 < \gamma_0 \le T\}$ is defined by

$$D_\alpha^*(T) := \sup_{0\le y\le 1}\left|\frac{\#\{0 \le \gamma_0 \le T;\ 0 \le \{\alpha\gamma_0\} < y\}}{N(T)} - y\right|,$$

where $N(T)$ denotes the number of zeros of $\zeta(s)$ such that $0 \le \beta_0 \le 1$ and $0 < \gamma_0 \le T$.

***Corresponding author: Arindam Roy,** Department of Mathematics and Statistics, University of North Carolina at Charlotte, 9201 University City Blvd., Charlotte, NC 28223, USA, e-mail: aroy15@uncc.edu
Amita Malik, Department of Mathematics, Rutgers University, New Brunswick, NJ, USA, e-mail: amita.malik@rutgers.edu

For any integer x, let $\alpha = \frac{\log x}{2\pi}$. In 1975, Hlawka [19] showed that

$$D^*_\alpha(T) \ll \frac{\log x}{\log T} \tag{1.3}$$

under the Riemann hypothesis, while

$$D^*_\alpha(T) \ll \frac{\log x}{\log\log T} \tag{1.4}$$

unconditionally. In 1993, Fujii [17] improved this bound and showed that

$$D^*_\alpha(T) \ll_\alpha \frac{\log\log T}{\log T}.$$

Recently, Ford and Zaharescu [15] investigated this result on discrepancy in more general settings. In particular, they showed that the discrepancy of the set $\{h(\alpha\gamma_0) : 0 < \gamma_0 \leq T\}$ is of the order $O(1/\log T)$ for a large class of functions $h\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. Also, Akbary and Murty [1] obtained similar results on the uniform distribution and the discrepancy for a large class of Dirichlet series on the assumption of average density hypothesis.

Another very important result in this direction is due to Montgomery. In [26], he studied the pair correlation of zeros of $\zeta(s)$. Odlyzko [30] showed that the distribution of consecutive spacing of imaginary parts of zeros would agree with the distribution of eigenvalue spacings in Gaussian unitary ensemble. These results were also extended for general L functions by Murty and Perelli [28], and Murty and Zaharescu [29]. In his work [26], Montgomery mentioned the connection between Landau–Siegel zeros and the gap between consecutive zeros of the Riemann zeta function. Conrey and Iwaniec [5] showed that the existence of Landau–Siegel zeros implies that the spacing of consecutive zeros of $\zeta(s)$ are close to multiples of half the average spacing.

The vertical distributions of zeros of $\xi'(s)$ have also been studied recently. In [11], Farmer, Gonek and Lee initiated the study of consecutive spacing of zeros of $\xi'(s)$. They investigated the pair correlation of the zeros of $\xi'(s)$ under the Riemann hypothesis. They obtained various estimates on the consecutive spacing and multiplicity of the zeros of $\xi'(s)$. Bui [2] improved some of their results on consecutive spacing of zeros of $\xi'(s)$.

One motivation of studying such distributions of $\xi^{(m)}(s)$ is to understand the distribution of zeros of an entire function under differentiation. From the functional equation

$$\xi(s) = \xi(1-s), \tag{1.5}$$

one can see that the entire function $\xi(1/2+it)$ is real on the real axis and has order one. Also, from the work of Craven, Csordas and Smith [7], Ki and Kim [22], and Kim [23], one may observe that for sufficiently large m, the Riemann hypothesis is true for $\xi^{(m)}(s)$ in a bounded region. Also, the zeros of $\xi^{(m)}(s)$ approach equal spacing as m tends to infinity. For details, readers are directed to the work of Farmer and Rhoades [12], Coffey [3], and Ki [21]. Since the small gaps between zeros become larger under differentiation, by the work of Conrey and Iwaniec [5], one may disprove the existence of Landau–Siegel zeros by showing the gap between consecutive zeros of $\xi^{(m)}(s)$ to be less than half of the average spacing for sufficiently many zeros; for details, also see [11].

In 2009, Ford, Soundararajan and Zaharescu [14] established some connections between Montgomery's pair correlation function and the distribution of the fractional parts of $(\alpha\gamma_0)$. So one might expect that the pair correlation result of Gonek, Farmer and Lee [11] would have connections with the distribution of fractional parts of $(\alpha\gamma_m)$, where $\rho_m = \beta_m + i\gamma_m$ denotes a complex zero of $\xi^{(m)}(s)$.

Although much information on the distribution of fractional parts of $(\alpha\gamma_0)$ is known, the authors cannot recall any results of the distribution of fractional parts of $(\alpha\gamma_m)$. The main goal of this paper to obtain some classical results on the distribution of the fractional parts of $(\alpha\gamma_m)$ analogous to the results of Rademacher [32] and Hlawka [19]. Our first result in this direction is stated below.

Theorem 1.1. *For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and a positive integer m, the sequence $(\alpha\gamma_m)$ is uniformly distributed modulo one, where γ_m runs over the imaginary parts of zeros of $\xi^{(m)}(s)$ and zeros are counted with multiplicity.*

Next, we are interested in the discrepancy of the sequence $(\alpha\gamma_m)$. Let $N_m(T)$ denote the number of zeros of $\xi^{(m)}(s)$ such that $0 \le \beta_m \le 1$ and $0 < \gamma_m \le T$. Conrey [4] proved that

$$N_m(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O_m(\log T). \tag{1.6}$$

Let $D_m^*(\alpha; T)$ denote the discrepancy

$$D_m^*(\alpha; T) := \sup_{0\le y\le 1}\left|\frac{\#\{0 \le \gamma_m \le T;\ 0 \le \{\alpha\gamma_m\} < y\}}{N_m(T)} - y\right|$$

of the set $\{\{\alpha\gamma_m\} : 0 < \gamma_m \le T\}$, and zeros are counted with multiplicity.

For $T > 10$, let us define

$$\mathcal{L} := \frac{\log\log T}{(\log\log\log T)^2}.$$

Then $\mathcal{L}$ tends to infinity with T. We have the following bound for $D_m^*(\alpha; T)$, which generalizes the results of Hlawka [19] for $\zeta(s)$.

Theorem 1.2. *Let $\alpha \ge \frac{\log 2}{2\pi}$ and let $m \le \mathcal{L}$ be a non-negative integer. Then*

$$D_m^*(\alpha; T) \le \frac{a_1\alpha}{\log\log T} + \frac{e^{a_2 m}\alpha}{\sqrt{\log T}}$$

as $T \to \infty$ and uniformly on $m \le \mathcal{L}$, where a_1 and a_2 are absolute constants. Moreover, under the assumption of the Riemann hypothesis,

$$D_m^*(\alpha; T) \le \frac{c_1\alpha}{\log T} + \exp\left(\frac{c_2 m\log T}{\log\log\log T}\right)\frac{\alpha}{\sqrt{T}}$$

as $T \to \infty$ and uniformly on $m \le \mathcal{L}$, where c_1 and c_2 are absolute constants.

Remark. From the above theorem, for $0 \le m \le \mathcal{L}$, the bound

$$D_m^*(\alpha; T) \ll \frac{\alpha}{\log\log T}$$

holds unconditionally and

$$D_m^*(\alpha; T) \ll \frac{\alpha}{\log T}$$

holds under the assumption of the Riemann hypothesis. Theorem 1.2 shows that the distribution of the sequence $(\alpha\gamma_m)$ depends on m. If we take $m \le \mathcal{L}$, then the discrepancy vanishes as m tend to infinity with T. This result can be compared with that of Ki [21] who showed that *there exist sequences A_n and C_n, with $C_n \to 0$ slowly, such that*

$$\lim_{n\to\infty} A_n\xi^{(2n)}\left(\frac{1}{2} + iC_n s\right) = \cos s$$

uniformly on compact subset of $\mathbb{C}$, which was conjectured by Farmer and Rhoades [12]. In other words, one can say that the zeros of derivatives become more well spaced as m increases.

Hlawka's discrepancy bounds (1.3) and (1.4) rely on the explicit formula of Landau [24]

$$\sum_{0<\gamma_0\le T} x^{\rho_0} = -\Lambda(x)\frac{T}{2\pi} + O(\log T), \tag{1.7}$$

where $\Lambda(n)$ is the von-Mangoldt function. Gonek [18] gave an explicit formula, similar to (1.7), which is uniform in both x and T. Fujii [16] also obtained a similar result independently. Gonek's explicit formula can be stated as follows:

$$\sum_{0<\gamma_0\le T} x^{\rho_0} = -\Lambda(x)\frac{T}{2\pi} + O\left(x\log^2(2xT) + \frac{\log 2T}{\log x}\right) + O\left(\log x \min\left(T, \frac{x}{\langle x\rangle}\right)\right), \tag{1.8}$$

where $\langle x\rangle$ is the distance to the nearest integer prime power other than x itself.

In order to prove Theorems 1.1 and 1.3, we also need an explicit formula for the zeros of $\xi^{(m)}(s)$. An essential ingredient in obtaining the explicit formulas (1.7) and (1.8) in the case of $\zeta(s)$ is the Dirichlet series representation of $\frac{\zeta'}{\zeta}(s)$ for $\operatorname{Re} s > 1$. However, there are no such Dirichlet series for $\frac{\xi^{(m+1)}}{\xi^{(m)}}(s)$. We give an explicit formula for $\xi^{(m)}(s)$.

Theorem 1.3. *Let $x, T > 1$ and let n_x be the nearest prime power to x. Let the zeros of $\xi^{(m)}(s)$ be counted with multiplicity. Then, for any number $\delta > 0$ and integer $K \geq 1$,*

$$\begin{aligned}\sum_{0\le\gamma_m\le T} x^{\rho_m} = &-\frac{\Lambda(n_x)}{2\pi}\delta_{x,T} + O\Big(T\frac{(2\log 2x)^{mK+1}}{(\log T)^{mK}}\Big) + O\Big(xT\Big(\frac{\log\log T}{\log T}\Big)^{\frac{K+1}{4}}\Big)\\ &+ O\Big(x\log(2xT)\log\log 2x + \log 2T\min\Big(T, \frac{1}{\log x}\Big)\Big)\\ &+ O\Big(x\log x(\log 2x)^{mK+1}\Big(\frac{\log\log T}{\delta(\log T)^{\delta}}\Big)^{mK} + x(\log T)^{\delta}e^{(\log T)^{\delta}}\Big)\end{aligned} \tag{1.9}$$

holds uniformly for $0 \leq m \leq \mathcal{L}$. Here $\delta_{x,T} = T$ if $x = n_x$, and

$$\delta_{x,T} \ll \min\Big(T, \frac{1}{|\log\frac{x}{n_x}|}\Big) \quad \textit{if } x \neq n_x.$$

The first error term in (1.9) can be written as a main term with some more efforts. This error term also disappears if x is not an integer, which could be proved by using the same method given in the proof of Theorem 1.3. Also the error term containing K and δ in (1.9) can be improved by a result of Erdős [9] for small values of K.

Differentiating (1.5) gives the functional equation

$$\xi^{(m)}(s) = (-1)^m\xi^{(m)}(1-s). \tag{1.10}$$

Since $\xi^{(m)}(s)$ is real-valued for real values of s, it is clear from (1.10) that the zeros of $\xi^{(m)}(s)$ are symmetric with respect to the line $\sigma = 1/2$. Therefore, for $0 < x < 1$, we have

$$\sum_{0\le\gamma_m\le T} x^{\rho_m} = \sum_{0\le\gamma_m\le T} x^{1-\bar\rho_m} = x\overline{\sum_{0\le\gamma_m\le T}\Big(\frac{1}{x}\Big)^{\rho_m}}. \tag{1.11}$$

If we choose

$$K = \Big\lfloor\frac{4\log T}{\log\log T}\Big\rfloor,$$

then for a fixed x and for T sufficiently large, one can show that (1.9) can be written as

$$\sum_{0\le\gamma_m\le T} x^{\rho_m} \ll Tx^{\epsilon} + xT^{\epsilon},$$

for $\epsilon > 0$, which may depend on T. Therefore, by the Riemann hypothesis, we find that

$$\sum_{0\le\gamma_m\le T} x^{i\gamma_m} \ll Tx^{-\frac12+\epsilon} + x^{\frac12}T^{\epsilon}, \tag{1.12}$$

which is non trivial for $2 \leq x \leq T^{2-\epsilon}$ by (1.6). Now, if one assumes that $\{x^{i\gamma}\}_\gamma$ behave like independent random variables, then we may expect that

$$\sum_{0\le\gamma_m\le T} x^{i\gamma_m} \ll T^{\frac12+\epsilon} \tag{1.13}$$

for all $x > 0$. Clearly, this is not true for every x.

By observing the bounds in (1.12) and (1.13), we have the following conjecture.

Conjecture 1.4. For all real numbers $x, T \geq 2$ and any $\epsilon > 0$,

$$\sum_{0\le\gamma_m\le T} x^{i\gamma_m} \ll Tx^{-\frac12+\epsilon} + T^{\frac12}x^{\epsilon}$$

holds uniformly for $0 \leq m \leq \mathcal{L}$.

To obtain the bounds in (1.3) and (1.4), another important result needed is to obtain a non-trivial upper bound for

$$\sum_{0<\gamma_0\le T}\Big|\beta_0-\frac{1}{2}\Big|.$$

In 1924, Littlewood [25] proved that

$$\sum_{0<\gamma_0\le T}\Big|\beta_0-\frac{1}{2}\Big| \ll T\log\log T,$$

which was later improved by Selberg [33] in 1942. In particular, he obtained

$$\int_{1/2}^{1} N_0(\sigma,T)\,d\sigma \ll T,$$

where $N_0(\sigma,T)$ denotes the number of zeros ρ_0 of $\zeta(s)$ such that $\beta_0>\sigma$ and $0<\gamma_0<T$.

We need a similar result for $\xi^{(m)}(s)$. For a fixed σ, let $N_m(\sigma,T)$ denote the number of zeros $\rho_m=\beta_m+i\gamma_m$ of $\xi^{(m)}(s)$, counted with multiplicity, such that $\beta_m>\sigma$ and $0<\gamma_m<T$. Our next result provides a zero density estimate for $\xi^{(m)}(s)$.

Theorem 1.5. *With $N_m(\sigma,T)$ defined as above, we have*

$$\int_{1/2}^{1} N_m(\sigma,T)\,d\sigma \le e^{(O(m))}T$$

for $0\le m\le\mathcal{L}$ uniformly.

Since the prior works suggest that the zeros of $\xi^{(m)}(s)$ migrate to the line $\sigma=\frac{1}{2}$, we have the following conjecture.

Conjecture 1.6. Let

$$C(m):=\limsup_{T\to\infty}\frac{1}{T}\int_{1/2}^{1} N_m(\sigma,T)\,d\sigma.$$

Then the function $C(m)$ is a decreasing function of m.

Remark. Note that for $\sigma>\frac{1}{2}$,

$$\int_{1/2}^{1} N_m(\sigma',T)\,d\sigma' \ge \Big(\sigma-\frac{1}{2}\Big)N_m(\sigma,T).$$

Therefore,

$$N_m(\sigma,T)=O_m\Big(\frac{T}{\sigma-\frac{1}{2}}\Big) \tag{1.14}$$

holds for $\frac{1}{2}<\sigma\le 1$. Combining (1.6) and (1.14) we find that the zeros of $\xi^{(m)}(s)$ are clustered near the line $\sigma=\frac{1}{2}$.

2 Auxiliary lemmas

For a positive real number θ, and $X=T^{\theta}$, define

$$M_X(s)=\sum_{n\le X}\frac{\mu(n)}{n^{s+R/\log T}}P\Big(1-\frac{\log n}{\log X}\Big), \tag{2.1}$$

where R is an absolute constant and P is a polynomial with $P(0)=0$ and $P(1)=1$. We have the following result from [6, p. 10].

Lemma 2.1. *Let* $V(s) = Q(-\frac{1}{\log T}\frac{d}{ds})\zeta(s)$ *for some polynomial* Q*, and let* $M_X(s)$ *be defined as in* (2.1). *For* $\theta < 4/7$,

$$\int_2^T \left|VM_X\Big(\frac{1}{2} - \frac{R}{\log T} + it\Big)\right|^2 dt \sim c(P, Q, R)T,$$

where $0 < R \ll 1$ *and*

$$c(P, Q, R) = |P(1)Q(0)|^2 + \frac{1}{\theta}\int_0^1\int_0^1 e^{2Ry}\left|Q(y)P'(x) + \theta Q'(y)P(x) + \theta RQ(y)P(x)\right|^2 dx\, dy.$$

For fixed P and R, one has

$$c(P, Q, R) \ll \Big|\max_{0\le x\le 1}(Q(x), Q'(x))\Big|^2. \tag{2.2}$$

As an application of the Faà di Bruno formula [13, p. 188], we obtain the following result.

Lemma 2.2. *For any non-zero analytic function* f*, we have*

$$\frac{f^{(n)}}{f}(s) = \sum_{\substack{\mu_1+2\mu_2+\cdots+k\mu_k=n\\ \mu_i\ge 0}} n!\prod_{i=1}^k \frac{1}{\mu_i!(i!)^{\mu_i}}\left(\Big(\frac{f'}{f}\Big)^{(i-1)}(s)\right)^{\mu_i}.$$

Let B_n denote the nth Bell number. Then we know

$$B_n = \sum_{\substack{\mu_1+2\mu_2+\cdots+k\mu_k=n\\ \mu_i\ge 0}} n!\prod_{i=1}^k \frac{1}{\mu_i!(i!)^{\mu_i}} \ll \Big(\frac{n}{\log n}\Big)^n. \tag{2.3}$$

We also need the following result. This is a uniform version of [4, Lemma 1].

Lemma 2.3. *Let* $H(s)$ *be as defined in* (1.2)*, and* $s = \sigma + it$*. Then the following hold:*

(i) *For* $|t| \ge 1$ *and any* σ*,*

$$\frac{H'(s)}{H(s)} = \frac{1}{2}\log\frac{s}{2\pi} + O\Big(\frac{1}{|t|}\Big)$$

and

$$\Big(\frac{H'(s)}{H(s)}\Big)^{(k)}(s) \ll \frac{(k!)}{|t|^k}$$

holds uniformly for $k \le \mathcal{L}$*. If* $|t| \le \sigma$*, then we replace* t *by* σ *in the error terms.*

(ii) *For* $|t| > 10$ *and* $|\sigma| < A\log\log T$*, where* A *is a constant,*

$$\frac{H^{(k)}(s)}{H(s)} = \Big(\frac{H'(s)}{H(s)}\Big)^k\Big(1 + O\Big(\frac{B_k}{|t\log t|}\Big)\Big) \tag{2.4}$$

holds uniformly for $k \le \mathcal{L}$*. If* $|t| \le \sigma$*, then we replace* t *by* σ *in the error terms.*

The above results follow from Lemma 2.2 and the following form of Stirling's formula

$$\log\Gamma(s) = \frac{1}{2}\log 2\pi + \Big(s - \frac{1}{2}\Big)\log s - s - \Omega(s),$$

where $\Omega(s) \ll 1/|t|$, and if $|t| \le \sigma$, then $\Omega(s) \ll 1/\sigma$. (See [31, Section 21].)

We also need the following lemma from [18].

Lemma 2.4. *For* $x, T \ge 1$ *and* $c = 1 + \frac{1}{\log 2x}$,

$$\sum_{\substack{n=2\\ n\ne x}}^{\infty} \frac{\Lambda(n)}{n^c}\min\Big(T, \frac{1}{|\log x/n|}\Big) \ll \log 2x\log\log 2x + \log x\min\Big(\frac{T}{x}, \frac{1}{\langle x\rangle}\Big),$$

where $\langle x\rangle$ *is the distance to the nearest integer prime power other than* x *itself.*

Weyl's criterion [35] for uniformly distributed sequences is given by the following lemma.

Lemma 2.5. *A sequence $(x_n)_{n\geq 1}$, is uniformly distributed modulo one if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ikx_n}=0 \quad \textit{for all integers } k\neq 0.$$

The following inequality is due to Erdős and Turán [10].

Lemma 2.6. *Let D_N denote the discrepancy of a sequence $(x_n)_{n\geq 1}$ of real numbers. Then, for any positive integer M,*

$$D_N\leq\frac{C_1}{M+1}+C_2\sum_{k=1}^{M}\frac{1}{k}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ikx_n}\right|,$$

where C_1 and C_2 are absolute positive constants.

The following lemma is due to Montgomery and Vaughan [27].

Lemma 2.7. *If $\sum_{n=1}^{\infty}n|a_n|^2$ converges, then*

$$\int_0^T\left|\sum_{n=1}^{\infty}a_n n^{-it}\right|^2 dt=\sum_{n=1}^{\infty}|a_n|^2(T+O(n)).$$

The following lemma from [34, p. 213] will be used to bound the argument of an analytic function.

Lemma 2.8. *Let $f(s)$ be an analytic function except for a pole at $s=1$, which is real for real s. Let $0\leq a<b<2$. Suppose that T is not an ordinate of any zero of $f(s)$. Let $|f(\sigma+it)|\leq M$ for $\sigma\geq a$, $1\leq t\leq T+2$ and $\mathrm{Re}(f(2+it))\geq c>0$ for some $c\in\mathbb{R}$. Then, for $\sigma\geq b$,*

$$|\arg f(\sigma+iT)|\leq\frac{c}{\log\frac{2-a}{2-b}}\left(\log M+\log\frac{1}{c}\right)+\frac{3\pi}{2}.$$

Let Λ_k denote the generalized von-Mangoldt defined by

$$\Lambda_k(n):=\sum_{d|n}\mu(d)\log^k\frac{n}{d}.$$

Therefore, for $\mathrm{Re}(s)>1$,

$$\sum_{n=1}^{\infty}\frac{\Lambda_k(n)}{n^s}=(-1)^k\frac{\zeta^{(k)}(s)}{\zeta(s)}. \tag{2.5}$$

Let Λ_k^{*l} denote the l-fold convolutions of Λ_k, i.e.,

$$\Lambda_k^{*l}=\underbrace{\Lambda_k*\cdots*\Lambda_k}_{l\text{ times}}. \tag{2.6}$$

Then, we have the following inequality.

Lemma 2.9. *With the notation from* (2.5) *and* (2.6),

$$(\Lambda_k\log*\Lambda_{k_1}^{*l_1}*\cdots*\Lambda_{k_m}^{*l_m})(n)\leq(\log n)^{1+k+k_1+l_1+\cdots+k_n+l_n}.$$

Proof. From [20, p. 35], we have

$$\Lambda_k(n)\leq\log^k n.$$

Using the above inequality and (2.5), we find that

$$(\Lambda_k\log*\Lambda_{k_1})(n)=\sum_{ab=n}\Lambda_k(a)\log(a)\Lambda_{k_1}(b)\leq\Lambda_k(n)\log(n)(1*\Lambda_{k_1})(n)\leq\log^{k+k_1+1}n.$$

By repeating this argument, we complete the proof of the lemma. □

In [4], the details of the proof of (1.6) are omitted. Also, the error term in the asymptotic of $N_m(T)$ depends on m. We prove an asymptotic result for $N_m(T)$ where the dependence of m is explicit.

Lemma 2.10. *Let $N_m(T)$ denote the number of zeros of $\xi^{(m)}(s)$ with $0 < t < T$, and zeros are counted with multiplicity. Then*

$$N_m(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T + m\log m)$$

uniformly for $m \leq \mathcal{L}$. The constant in the error term is absolute.

Proof. Applying Leibnitz's rule in (1.1), we find that

$$\xi^{(m)}(s) = H^{(m)}(s)F_m(s), \tag{2.7}$$

where

$$F_m(s) := \zeta(s) + \sum_{j=1}^{m}\binom{m}{j}\frac{H^{(m-j)}(s)}{H^{(m)}(s)}\zeta^{(j)}(s). \tag{2.8}$$

Let T be a large number. Then, for $\sigma \geq \mathcal{L}$ and $k \leq m$,

$$|\zeta^{(k)}(s)| \geq \frac{\log^k 2}{2^\sigma} - \sum_{n=3}^{\infty}\frac{\log^k n}{n^\sigma} \geq \frac{\log^k 2}{2^{\sigma+1}}.$$

From (2.8), Lemma 2.3 and for $\sigma \geq \mathcal{L}$, we have

$$|F_m(s) - 1| \ll \frac{1}{2^{\sigma+1}}(\log 6)^m < \frac{1}{2} \tag{2.9}$$

when $m \leq \mathcal{L}$.

Choose $T > 0$ so that the line $t = T$ is free of zeros of $\xi^{(m)}(s)$. Let R be the rectangle, taken counterclockwise, with vertices $\mathcal{L} - iT$, $\mathcal{L} + iT$, $1 - \mathcal{L} + iT$, $1 - \mathcal{L} - iT$. By the argument principle, the functional equation (1.10) and relation (2.7),

$$\begin{aligned}
4\pi N_m(T) &= \operatorname{Im}\Big(\int_R \frac{d}{ds}\log\xi^{(m)}(s)\,ds\Big)\\
&= 2\Big(\int_C \frac{d}{ds}\log H^{(m)}(s)\,ds\Big) + 2\operatorname{Im}\Big(\int_{1/2-iT}^{\mathcal{L}-iT} + \int_{\mathcal{L}-iT}^{\mathcal{L}+iT} - \int_{1/2+iT}^{\mathcal{L}+iT}\frac{d}{ds}\log F_m(s)\,ds\Big)\\
&= 2\operatorname{Im}(I_1 + I_2 + I_3 - I_4),
\end{aligned} \tag{2.10}$$

where C denotes the lines from $1/2 - iT$ to $\mathcal{L} - iT$, $\mathcal{L} - iT$ to $\mathcal{L} + iT$ and then $\mathcal{L} + iT$ to $1/2 + iT$. From (2.3), $B_m \ll \log T$ for $m \leq \mathcal{L}$. Combining this with (2.4), we have

$$I_1 = \operatorname{Im}(\log H^{(m)}(s)|_C) = \big(\log H(s)|_C + m\log L(s)|_C\big) + O\Big(\frac{1}{T}\Big).$$

From [8, p. 98], we have

$$\operatorname{Im}(\log H(s)|_C) = T\log\frac{T}{2\pi} - T + O(m).$$

From Lemma 2.3, one finds

$$\operatorname{Im}(m\log L(s)|_C) = O(m)$$

for $m \leq \mathcal{L}$. Hence,

$$I_1 = T\log\frac{T}{2\pi} - T + O(m).$$

From (2.9), we have

$$I_3 = \operatorname{Im}\big(\log F_m(s)|_{\mathcal{L}-iT}^{\mathcal{L}+iT}\big) = O(m).$$

Next, we compute I_4. Let

$$\tilde{F}(z) = \frac{1}{2}\big(F_m(z + iT) + \overline{F_m(\bar{z} + iT)}\big).$$

Then for large T the function $\tilde{F}(z)$ is analytic in the disk $|z-\mathcal{L}|<4\mathcal{L}$. Let $n(x)$ denote the number of zeros of $\tilde{F}(z)$ in the disk $|z-\mathcal{L}|<x$. Then, by Jensen's theorem,

$$n(2\mathcal{L})\log 2 = n(2\mathcal{L})\int_{2\mathcal{L}}^{4\mathcal{L}}\frac{1}{t}\,dt \le \int_{0}^{4\mathcal{L}}\frac{n(t)}{t}\,dt \le \frac{1}{2\pi}\int_{0}^{2\pi}\log|\tilde{F}(4\mathcal{L}e^{i\theta}+2\mathcal{L})|\,d\theta - \log|\tilde{F}(2\mathcal{L})|. \tag{2.11}$$

From (2.9), we have

$$|\tilde{F}(2\mathcal{L})-1| = |\mathrm{Re}(F_m(2\mathcal{L}+iT))-1| \le |F_m(2\mathcal{L}+iT)-1| < \frac{1}{2}. \tag{2.12}$$

For $\mathrm{Re}(s)>0$, one has

$$\zeta(s) = s\int_{1}^{\infty}\frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,dx + \frac{1}{s-1} + \frac{1}{2}.$$

Therefore, for $\mathrm{Re}(s)>1/2$ and $|s-1|>A$, we have

$$\zeta^{(k)}(s) \ll k!t^{B} \tag{2.13}$$

for some $B>0$. From Stirling's formula, (2.8) and (2.13), we have

$$\tilde{F}(s) \ll (4m)^{m}|t|^{B} \tag{2.14}$$

in the disk $|z-\mathcal{L}|<4\mathcal{L}$. Equations (2.11), (2.12) and (2.14) give us

$$n(2\mathcal{L}) \ll \log T + m\log m$$

for $m\le\mathcal{L}$. Since the number of zeros of $\mathrm{Re}(F_m(s))$ on $(\mathcal{L}+iT, 1/2+iT)$ are bounded by $n(2\mathcal{L})$, we get

$$I_4 = \mathrm{Im}\big(\log F_m(s)|_{\mathcal{L}+iT}^{1/2+iT}\big) = O(\log T + m\log m).$$

Similarly, $I_2 = O(\log T + m\log m)$. By combining I_1, I_2, I_3, and I_4 in (2.10), we complete the proof of the lemma. □

3 Proof of the explicit formula

Let $m\le\mathcal{L}$ be any positive integer and let $t>10$. Then, by Lemma 2.3, one finds that $\frac{H^{(m)}(s)}{H(s)}$ is non-zero and $H(s)$ never vanishes. Therefore, $H^{(m)}(s)$ does not have any complex zero for $t>10$. By (2.7), the complex zeros of $F_m(s)$ are the only zeros of $\xi^{(m)}(s)$. The logarithmic derivative of (2.8) yields

$$\frac{F_m'(s)}{F_m(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{E_m'(s)}{E_m(s)}, \tag{3.1}$$

where

$$E_m(s) = 1 + \sum_{j=1}^{m}\binom{m}{j}\frac{H^{(m-j)}(s)}{H^{(m)}(s)}\frac{\zeta^{(j)}(s)}{\zeta(s)}. \tag{3.2}$$

Also,

$$\begin{aligned}\frac{d}{ds}E_m(s) &= \sum_{j=1}^{m}\binom{m}{j}\Big(\frac{H^{(m-j)}(s)}{H^{(m)}(s)}\frac{\zeta^{(j)}(s)}{\zeta(s)}\Big)' \\ &= \sum_{j=1}^{m}\binom{m}{j}\Big(\frac{\zeta^{(j)}(s)}{\zeta(s)}\Big)'\frac{H^{(m-j)}(s)}{H^{(m)}(s)} + \sum_{j=1}^{m}\binom{m}{j}\Big(\frac{H^{(m-j)}(s)}{H^{(m)}(s)}\Big)'\frac{\zeta^{(j)}(s)}{\zeta(s)} \\ &=: E_{1,m}(s) + E_{2,m}(s).\end{aligned} \tag{3.3}$$

Let $c = 1 + \frac{1}{\log 2x}$ and $T_\delta = \exp((\log T)^\delta)$ for any fixed $\delta > 0$. Consider the rectangle $\mathcal{R}$ defined by the vertices $1 - c + iT_\delta$, $c + iT_\delta$, $c + iT$ and $1 - c + iT$. It is known that the zeros of $\xi^{(m)}(s)$ lie in the vertical strip $0 < \sigma < 1$. Then, by the residue theorem and Lemma 2.10,

$$\begin{aligned}\sum_{0 \le \gamma_m \le T} x^{\rho_m} + O(xT_\delta \log T_\delta) &= \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{F_m'(s)}{F_m(s)} x^s \, ds \\ &= \frac{1}{2\pi i}\Bigg(\int_{1-c+iT_\delta}^{c+iT_\delta} + \int_{c+iT_\delta}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+iT_\delta} \Bigg) \frac{F_m'(s)}{F_m(s)} x^s \, ds \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.4}$$

In the next three subsections, we will compute the integrals I_1 and I_3, I_2, and I_4, respectively.

3.1 Computation of the horizontal line integrals I_1 and I_3

As an application of Lemma 2.3, we deduce that

$$\frac{H^{(m-j)}}{H^{(m)}}(s) \ll \frac{2^j}{\log^j t} \tag{3.5}$$

and

$$\Big(\frac{H^{(m-j)}}{H^{(m)}}(s) \Big)' \ll \frac{2^{j+1}}{\log^{j+1} t} \tag{3.6}$$

for $t > T_\delta$ and $m \le \mathcal{L}$. Let $t \ge T_\delta$. Then, by (3.2) and (3.5), we have

$$|E_m(2 + it) - 1| \ll \frac{m 2^m}{\log t} < \frac{\mathcal{L} 2^{\mathcal{L}}}{\log T_\delta} < \frac{1}{2}. \tag{3.7}$$

From (3.3), (3.5) and (3.6), we find

$$|E_m'(2 + it)| \ll \frac{m 2^m}{\log t} < \frac{\mathcal{L} 2^{\mathcal{L}}}{\log T_\delta} < \frac{1}{2}. \tag{3.8}$$

Combining (3.1), (3.7) and (3.8), we find

$$\frac{F_m'}{F_m}(2 + it) \ll 1 \tag{3.9}$$

for $t \ge T_\delta$. Since $\xi^{(m)}(s)$ is an entire function of order 1, by Hadamard's factorization theorem, one can rewrite it as

$$\xi^{(m)}(s) = e^{A+Bs} \prod_{\rho_m} \Big(1 - \frac{s}{\rho_m}\Big) e^{-s/\rho_m},$$

where the product runs over all the zeros of $\xi^{(m)}(s)$, and A, B are certain constants. By logarithmic differentiation, (2.7), (3.9) and Lemma 2.3, we obtain

$$\frac{F_m'}{F_m}(s) = \sum_{\rho_m} \Big(\frac{1}{s - \rho_m} - \frac{1}{\rho_m} \Big) + O(\log t) = \sum_{\rho_m} \Big(\frac{1}{s - \rho_m} - \frac{1}{2 + it - \rho_m} \Big) + O(\log t). \tag{3.10}$$

Now, we consider the terms in the sum on the right side of (3.10) for which $|\gamma_m - t| \ge 1$. From Lemma 2.10, we have

$$N_m(t + 1) - N_m(t) \ll \log t. \tag{3.11}$$

Using (3.11), we find that

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{n \le |\gamma_m - t| < n+1} \frac{2 - \sigma}{(s - \rho_m)(2 + it - \rho_m)} &\ll \sum_{n=1}^{\infty} \sum_{n \le |\gamma_m - t| < n+1} \frac{1}{(\gamma_m - t)^2} \\ &\ll \sum_{n=1}^{\infty} \sum_{n \le |\gamma_m - t| < n+1} \frac{1}{n^2} \\ &\ll \sum_{n=1}^{\infty} \frac{\log(t + n)}{n^2} \ll \log t. \end{aligned} \tag{3.12}$$

By (3.11), we have

$$\sum_{|\gamma_m - t|<1} \frac{1}{2 + it - \rho_m} \ll \log t. \tag{3.13}$$

Invoking (3.12) and (3.13) in (3.10), we obtain

$$\frac{F_m'}{F_m}(s) = \sum_{|\gamma_m - t|<1} \frac{1}{s - \rho_m} + O(\log t) \tag{3.14}$$

for $t \geq T_\delta$. From (3.14), in (3.4) the integral I_3 can be written as

$$\sum_{|\gamma_m - T|<1} \int_{c+iT}^{1-c+iT} \frac{x^s}{s - \rho_m}\, ds + O\bigg(\log 2T \int_{1-c}^{c} x^\sigma\, d\sigma\bigg) =: \sum_{|\gamma_m - T|<1} I_{\gamma_m} + O\bigg(x \frac{\log 2T}{\log 2x}\bigg). \tag{3.15}$$

In order to compute I_{γ_m}, we shift the line of integration from $\operatorname{Im} s = T$ to $\operatorname{Im} s = T + 1$. For $|\gamma_m - T| < 1$, by the residue theorem, we see that

$$\begin{aligned}
I_{\gamma_m} &= \bigg(\int_{c+i(T+1)}^{1-c+i(T+1)} + \int_{c+iT}^{c+i(T+1)} - \int_{1-c+iT}^{1-c+i(T+1)}\bigg) \frac{x^s}{s - \rho_m}\, ds + O(1) \\
&\ll 1 + \int_{1-c}^{c} \frac{x^\sigma}{\sqrt{(\sigma - \beta_m)^2 + (T + 1 - \gamma_m)^2}}\, d\sigma + x \int_{T}^{T+1} \frac{dt}{\sqrt{(c - \beta_m)^2 + (t - \gamma_m)^2}} + \frac{x^{1-c}}{\beta_m - 1 + c} \\
&\ll x \log\log 2x.
\end{aligned} \tag{3.16}$$

Note that the sum on the right side of (3.15) has $\log(2T)$ terms. Therefore, the contribution from the top horizontal integral is

$$I_3 = \frac{1}{2\pi i} \int_{c+iT}^{1-c+iT} \frac{F_m'(s)}{F_m(s)} x^s\, ds \ll x \log(2T) \log\log(2x). \tag{3.17}$$

Computing similarly, as in (3.15) and (3.16), the contribution from the integral along the lower horizontal of the rectangle $\mathcal{R}$ in (3.4) is given by

$$I_1 = \frac{1}{2\pi i} \int_{c+iT_\delta}^{1-c+iT_\delta} \frac{F_m'(s)}{F_m(s)} x^s\, ds \ll x \log(2T_\delta) \log\log(2x). \tag{3.18}$$

3.2 Computation of the right vertical integral I_2

Next, we compute the integral on the right vertical line of the rectangle $\mathcal{R}$ in (3.4). From (3.1), one has

$$\begin{aligned}
I_2 = \int_{c+iT_\delta}^{c+iT} \frac{F_m'}{F_m}(s) x^s\, ds &= \int_{c+iT_\delta}^{c+iT} \frac{\zeta'}{\zeta}(s) x^s\, ds + \int_{c+iT_\delta}^{c+iT} \frac{E_{1,m}}{E_m}(s) x^s\, ds + \int_{c+iT_\delta}^{c+iT} \frac{E_{2,m}}{E_m}(s) x^s\, ds \\
&=: I_{1,2} + I_{2,2} + I_{3,2}.
\end{aligned}$$

The integral $I_{3,2}$. From [34, Section 6.19], we have the following bound for the Riemann zeta function:

$$\frac{\zeta'}{\zeta}(\sigma + it) \ll \log^{\frac{2}{3}} t \log^{\frac{1}{3}} \log t,$$

which holds uniformly on $\sigma > 1 - A \log^{-\frac{2}{3}} t \log^{-\frac{1}{3}} \log t$, where A is an absolute constant. Using the Cauchy integral formula, for any positive integer n, we obtain

$$\bigg(\frac{\zeta'}{\zeta}(\sigma + it)\bigg)^{(n)} \ll n! \log^{\frac{2n}{3}} t \log^{\frac{n}{3}} \log t,$$

which holds uniformly on $\sigma > 1 - A\log^{-\frac{2}{3}} t \log^{-\frac{1}{3}} \log t$. Hence, by Lemma 2.2,

$$\frac{\zeta^{(n)}}{\zeta}(\sigma + it) \ll B_n^2 \log^{\frac{2n}{3}} t \log^{\frac{n}{3}} \log t \tag{3.19}$$

and

$$\Big(\frac{\zeta^{(n)}}{\zeta}(\sigma + it)\Big)' \ll B_n^2 \log^{\frac{2n+2}{3}} t \log^{\frac{n+1}{3}} \log t \tag{3.20}$$

for $\sigma > 1 - A\log^{-\frac{2}{3}} t \log^{-\frac{1}{3}} \log t$. Combining (2.3), (3.19) and (3.5) with (3.2), we find that

$$|E_m(\sigma + it) - 1| \ll B_m^2 \sum_{j=1}^{m} \binom{m}{j} \frac{\log^{\frac{2j}{3}} t \log^{\frac{j}{3}} \log t}{\log^j t} \ll B_m^3 \frac{\log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t} < \frac{1}{2} \tag{3.21}$$

for large $t \geq T_\delta$ and uniformly for $\sigma > 1$ and $m \leq \mathcal{L}$. Using (3.6), (3.19) and (3.21) in (3.3), we have

$$\frac{E_{2,m}}{E_m}(\sigma + it) \ll B_m^2 \sum_{j=1}^{m} \binom{m}{j} \frac{\log^{\frac{2j}{3}} t \log^{\frac{j}{3}} \log t}{t \log^{j+1} t} \ll B_m^3 \frac{\log^{\frac{2}{3}} t \log^{\frac{1}{3}} \log t}{t \log^2 t} \ll \frac{1}{t \log^{\frac{7}{6}} t} \tag{3.22}$$

for $t \geq T_\delta$ and uniformly for $\sigma > 1$. Therefore, integrating by parts, and using (3.21) and (3.22), one deduces that

$$I_{3,2} \ll x. \tag{3.23}$$

The integral $I_{2,2}$. Rewrite $I_{2,2}$ as

$$I_{2,2} = \sum_{k=0}^{K-1} (-1)^k \int_{c+iT_\delta}^{c+iT} E_{1,m}(s)(E_m(s) - 1)^k x^s \, ds + \int_{c+iT_\delta}^{c+iT} \frac{E_{1,m}(s)(E_m(s) - 1)^K}{E_m(s)} x^s \, ds =: J_1 + J_2. \tag{3.24}$$

Firstly, we compute J_2. From (3.3), (3.5) and (3.20), we find that

$$E_{1,m}(\sigma + it) \ll B_m^2 \sum_{j=1}^{m} \binom{m}{j} \frac{\log^{\frac{2j}{3}} t \log^{\frac{j}{3}} \log t}{\log^j t} \ll B_m^3 \frac{\log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t}. \tag{3.25}$$

Hence, from (3.21), (3.25), and the definition of J_2 in (3.24), we have

$$J_2 \ll xT\Big(\frac{B_m^3 \log^{\frac{1}{3}} \log T}{\log^{\frac{1}{3}} T}\Big)^{K+1} \ll xT\Big(\frac{\log^{\frac{1}{4}} \log T}{\log^{\frac{1}{4}} T}\Big)^{K+1}, \tag{3.26}$$

where the implied constant in the bound is absolute.

Now, we compute J_1. From (3.2) and (3.3), we have

$$E_{1,m}(s)(E_m(s) - 1)^k = \sum_{j=1}^{m} \sum_{l_1 + l_2 + \cdots + l_m = k} \binom{m}{j} \Big(\frac{\zeta^{(j)}(s)}{\zeta(s)}\Big)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} k! \prod_{i=1}^{m} \frac{1}{l_i!} \Big(\binom{m}{i} \frac{\zeta^{(i)}(s)}{\zeta(s)} \frac{H^{(m-i)}(s)}{H^{(m)}(s)}\Big)^{l_i}. \tag{3.27}$$

Let $L := l_1 + 2l_2 + \cdots + ml_m + j$ and $\sigma > 1$. Then, by Lemma 2.3 and (2.5), we find

$$\begin{aligned}\Big(\frac{\zeta^{(j)}(s)}{\zeta(s)}\Big)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \prod_{i=1}^{m} \Big(\frac{\zeta^{(i)}(s)}{\zeta(s)} \frac{H^{(m-i)}(s)}{H^{(m)}(s)}\Big)^{l_i} &= \frac{2^L}{\log^L(s/2\pi)} \Big(\frac{\zeta^{(j)}(s)}{\zeta(s)}\Big)' \prod_{i=1}^{m} \Big(\frac{\zeta^{(i)}(s)}{\zeta(s)}\Big)^{l_i} \Big(1 + O\Big(\frac{1}{t \log t}\Big)\Big) \\ &= \frac{(-2)^L}{\log^L(s/2\pi)} \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s} \Big(1 + O\Big(\frac{1}{t \log t}\Big)\Big),\end{aligned} \tag{3.28}$$

where

$$b_L(n) = (\Lambda_j \log * \Lambda^{l_1} * \Lambda_2^{*l_2} * \cdots * \Lambda_m^{*l_m})(n).$$

From Lemma 2.9, we find that

$$b_L(n) \leq \log^{L+1} n.$$

Therefore, from (3.24), (3.27) and (3.28), we deduce that

$$J_1 = \sum_{k=0}^{K-1}(-1)^k \sum_{j=1}^{m} \sum_{l_1+l_2+\cdots+l_m=k} \binom{m}{j} k! \prod_{i=1}^{m} \frac{1}{l_i!}\left(\binom{m}{i}\right)^{l_i} \sum_{n=1}^{\infty} \int\limits_{c+iT_\delta}^{c+iT} \frac{(-2)^L b_L(n)}{\log^L(s/2\pi)}\left(\frac{x}{n}\right)^s \left(1+O\left(\frac{1}{t\log t}\right)\right) ds. \tag{3.29}$$

Let n' be the nearest integer to x. Then

$$\int\limits_{c+iT_\delta}^{c+iT} \frac{1}{\log^L(s/2\pi)}\left(\frac{x}{n'}\right)^s ds \ll \left(\frac{x}{n'}\right)^c \int\limits_{T_\delta}^{T} \frac{1}{\log^L t}\,dt \ll \frac{T}{\log^L T}.$$

If x is not an integer, then, by integrating by parts, we obtain

$$\int\limits_{c+iT_\delta}^{c+iT} \frac{1}{\log^L(s/2\pi)}\left(\frac{x}{n}\right)^s ds \ll \frac{x^c}{(\log^L T_\delta) n^c \log(x/n)}.$$

Therefore,

$$\sum_{n=1}^{\infty} b_L(n) \int\limits_{c+iT_\delta}^{c+iT} \frac{1}{\log^L(s/2\pi)}\left(\frac{x}{n}\right)^s ds \ll \frac{b_L(n')T}{\log^L T} + \sum_{\substack{n=1\\ n\neq n'}}^{\infty} \frac{x^c b_L(n)}{(\log^L T_\delta) n^c \log(x/n)}. \tag{3.30}$$

Also,

$$\begin{aligned}\sum_{1\le n\le n'/2} \frac{x^c b_L(n)}{(\log^L T_\delta) n^c \log(x/n)} + \sum_{n\ge 2n'} \frac{x^c b_L(n)}{(\log^L T_\delta) n^c \log(x/n)} &\le \frac{x}{(\log^L T_\delta)} \sum_{n=1}^{\infty} \frac{b_L(n)}{n^c}\\ &\ll \frac{x(j+1)!\prod_{i=1}^{m}(i!)^{l_i}}{(\log^L T_\delta)(c-1)^{L+1}}\\ &\ll x\log x\left(\frac{\log\log T_\delta\ \log x}{\log T_\delta}\right)^L. \end{aligned} \tag{3.31}$$

In the penultimate step, we have used the fact that $(i!/(\log\log T_\delta)^i \ll 1$ for $i \le m \le \mathcal{L}$.

For the remaining terms in the sum on the right side of (3.30), we have

$$\sum_{\substack{n'/2\le n\le 2n'\\ n\neq n'}} \frac{x^c b_L(n)}{(\log^L T_\delta) n^c \log(x/n)} \ll \frac{\log^{L+1} x}{\log^L T_\delta} \sum_{\substack{n'/2\le n\le 2n'\\ n\neq n'}} \frac{1}{\log(x/n)}. \tag{3.32}$$

Since

$$\log\frac{x}{n} \ge \log\frac{n'}{n} = -\log\left(1-\frac{n'-n}{n'}\right) \ge \frac{|n-n'|}{n'},$$

we have

$$\sum_{\substack{n'/2\le n\le 2n'\\ n\neq n'}} \frac{1}{\log(x/n)} \le \sum_{\substack{n'/2\le n\le 2n'\\ n\neq n'}} \frac{n'}{|n-n'|} \ll x\log 2x. \tag{3.33}$$

For the error term in (3.29), we have

$$\sum_{n=1}^{\infty} b_L(n) \int\limits_{c+iT_\delta}^{c+iT} \frac{1}{\log^L(s/2\pi)}\left(\frac{x}{n}\right)^s O\left(\frac{1}{t\log t}\right) ds \le \frac{x}{(\log^L T_\delta)} \sum_{n=1}^{\infty} \frac{b_L(n)}{n^c} \ll x\log x\left(\frac{\log\log T_\delta\ \log x}{\log T_\delta}\right)^L. \tag{3.34}$$

From the definition of L, we have

$$\sum_{k=0}^{K-1}\sum_{j=1}^{m} \sum_{l_1+l_2+\cdots+l_m=k} \binom{m}{j} k! \prod_{i=1}^{m} \frac{1}{l_i!}\left(\binom{m}{i}\right)^{l_i} (Y)^L \le (1+Y)^{mK}. \tag{3.35}$$

Substituting (3.30), (3.31), (3.32), (3.33) and (3.34) in (3.29) and then use (3.35), we deduce that

$$\begin{aligned} J_1 &\ll \sum_{k=0}^{K-1}\sum_{j=1}^{m}\sum_{l_1+l_2+\cdots+l_m=k}\binom{m}{j}k!x\log x\Bigg(\prod_{i=1}^{m}\frac{1}{l_i!}\bigg(\binom{m}{i}\bigg)^{l_i}\Big(\frac{2\log\log T_\delta\ \log x}{\log T_\delta}\Big)^{L} \\ &\quad+\log 2x\prod_{i=1}^{m}\frac{1}{l_i!}\bigg(\binom{m}{i}\bigg)^{l_i}\frac{(2\log x)^L}{\log^L T_\delta}+\frac{T}{x}\prod_{i=1}^{m}\frac{1}{l_i!}\bigg(\binom{m}{i}\bigg)^{l_i}\frac{(2\log x)^L}{\log^L T}\Bigg) \\ &\ll x\log x\Bigg(\Big(\frac{2\log\log T_\delta\ \log 2x}{\log T_\delta}\Big)^{mK}+\log 2x\frac{(2\log 2x)^{mK}}{(\log T_\delta)^{mK}}+\frac{T}{x}\frac{(2\log 2x)^{mK}}{(\log T)^{mK}}\Bigg) \\ &\ll x\log x(\log 2x)^{mK+1}\Big(\frac{\log\log T_\delta}{\log T_\delta}\Big)^{mK}+T\frac{(2\log 2x)^{mK+1}}{(\log T)^{mK}}. \end{aligned} \tag{3.36}$$

Combining (3.26) and (3.36), we find that

$$I_{2,2}\ll x\log x(\log 2x)^{mK+1}\Big(\frac{\log\log T_\delta}{\log T_\delta}\Big)^{mK}+T\frac{(2\log 2x)^{mK+1}}{(\log T)^{mK}}+xT\Big(\frac{\log^{\frac14}\log T}{\log^{\frac14}T}\Big)^{K+1}. \tag{3.37}$$

The integral $I_{1,2}$. Let n_x be the nearest prime power to x. Then, by Lemma 2.4,

$$\begin{aligned} I_{1,2} &= \int_{c+iT_\delta}^{c+iT}\frac{\zeta'}{\zeta}(s)x^s\,ds=-\int_{c+iT_\delta}^{c+iT}\sum_{n=2}^{\infty}\Lambda(n)\Big(\frac{x}{n}\Big)^s\,ds \\ &= -i\Lambda(n_x)\int_{T_\delta}^{T}\Big(\frac{x}{n_x}\Big)^{it}\,dt+O\Bigg(x^c\sum_{\substack{n=2\\ n\neq n_x}}^{\infty}\frac{\Lambda(n)}{n^c\log(x/n_x)}\Bigg) \\ &= -i\Lambda(n_x)\delta_{x,T}+O(x\log(2x)\log\log(2x))+O(T_\delta\log x), \end{aligned} \tag{3.38}$$

where

$$\delta_{x,T}=\int_0^T\Big(\frac{x}{n_x}\Big)^{it}\,dt.$$

Clearly, $\delta_{x,T}\ll T$. If $x=n_x$, then

$$\delta_{x,T}=T,$$

otherwise

$$\delta_{x,T}=\frac{(\frac{x}{n_x})^{iT}-1}{i\log\frac{x}{n_x}}\ll\Big|\log\frac{x}{n_x}\Big|^{-1}.$$

Notice that the first term on the right side of (3.38) disappears if x is not an integer.

Finally, Combining (3.23), (3.37) and (3.38), we arrive at

$$\begin{aligned} I_2 &= \int_{c+iT_\delta}^{c+iT}\frac{F_m'}{F_m}(s)x^s\,ds=-i\Lambda(n_x)\delta_{x,T}+O(x\log(2x)\log\log(2x)+T_\delta\log x) \\ &\quad+O\Bigg(T\frac{(2\log 2x)^{mK+1}}{(\log T)^{mK}}+xT\Big(\frac{\log^{\frac14}\log T}{\log^{\frac14}T}\Big)^{K+1}\Bigg) \\ &\quad+O\Bigg(x\log x(\log 2x)^{mK+1}\Big(\frac{\log\log T_\delta}{\log T_\delta}\Big)^{mK}\Bigg). \end{aligned} \tag{3.39}$$

3.3 Computation of the left vertical integral I_4

Now, we move on to estimate the integral along the left vertical side of the rectangle $\mathcal{R}$ in (3.4). From the functional equation (1.10), one can derive

$$\frac{F_m'}{F_m}(s)=(-1)^{m+1}\frac{F_m'}{F_m}(1-s)+(-1)^{m+1}\frac{H^{(m+1)}}{H^{(m)}}(1-s)-\frac{H^{(m+1)}}{H^{(m)}}(s).$$

Thus, for the integral along the left vertical line, we have

$$\begin{aligned} I_4 &= \int_{1-c+iT_\delta}^{1-c+iT} \frac{F'_m}{F_m}(s)x^s\,ds \\ &= (-1)^{m+1} \int_{1-c+iT_\delta}^{1-c+iT} \frac{F'_m}{F_m}(1-s)x^s\,ds + \int_{1-c+iT_\delta}^{1-c+iT} \Big((-1)^{m+1}\frac{H^{(m+1)}}{H^{(m)}}(1-s) - \frac{H^{(m+1)}}{H^{(m)}}(s)\Big)x^s\,ds \\ &=: I_{1,4} + I_{2,4}. \end{aligned}$$

Firstly, we estimate $I_{2,4}$. Integrating by parts and employing Lemma 2.3, we find that

$$I_{2,4} \ll \frac{\log 2T}{\log x}.$$

Also, trivially, we have

$$I_{2,4} \ll T\log 2T.$$

Hence,

$$I_{2,4} \ll \log 2T \min\Big\{T, \frac{1}{\log x}\Big\}.$$

Now, we estimate $I_{1,4}$. We rewrite the integral $I_{1,4}$ above as

$$\begin{aligned} I_{1,4} = &\int_{1-c+iT_\delta}^{1-c+iT} \frac{\zeta'}{\zeta}(1-s)x^s\,ds + \sum_{k=0}^{K-1}\int_{1-c+iT_\delta}^{1-c+iT} E'_{m1}(1-s)(E_m(1-s)-1)^k x^s\,ds \\ &+ \int_{1-c+iT_\delta}^{1-c+iT} \frac{E'_{m1}(1-s)(E_m(1-s)-1)^K}{E_m(1-s)}x^s\,ds + \int_{1-c+iT_\delta}^{1-c+iT} \frac{E'_{m2}}{E_m}(1-s)x^s\,ds \\ =:& J_{41} + J_{42} + J_{43} + J_{44}. \end{aligned}$$

Now, we compute J_{41} defined above as follows:

$$\begin{aligned} J_{41} &= \int_{1-c+iT_\delta}^{1-c+iT} \frac{\zeta'}{\zeta}(1-s)x^s\,ds = ix^{1-c}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^c}\int_{T_\delta}^{T}(nx)^{it}\,dt \\ &\ll \Big(x^{1-c}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^c\log(xn)}\Big) \ll \Big(\frac{x^{1-c}}{c-1}\Big) \ll \log x. \end{aligned}$$

Proceeding in a similar fashion as for J_1 earlier and using (3.29), we have

$$J_{42} = \sum_{k=0}^{K-1}(-1)^k\sum_{j=1}^{m}\sum_{l_1+l_2+\cdots+l_m=k}\binom{m}{j}k!\prod_{i=1}^{m}\frac{1}{l_i!}\Big(\binom{m}{i}\Big)^{l_i}\sum_{n=1}^{\infty}\int_{1-c+iT_\delta}^{1-c+iT}\frac{(-2)^L b_L(n)}{n\log^L(s/2\pi)}(nx)^s\Big(1+O\Big(\frac{1}{t\log t}\Big)\Big)\,ds,$$

where

$$\sum_{n=1}^{\infty}\frac{b_L(n)}{n}\int_{1-c+iT_\delta}^{1-c+iT}\frac{(nx)^s}{\log^L(s/2\pi)}\Big(1+O\Big(\frac{1}{t\log t}\Big)\Big)\,ds \ll \sum_{n=1}^{\infty}\frac{x^{1-c}b_L(n)}{(\log T_\delta))^L n^c\log(nx)} \ll \log^{L+1}x.$$

A similar computation as used for (3.36) gives

$$J_{42} \ll \log x(2\log 2x)^{mK+1}.$$

Proceeding similarly, as we did for J_2 and $I_{3,2}$, we arrive at

$$J_{43} \ll T\Big(\frac{\log^{\frac14}\log T}{\log^{\frac14} T}\Big)^{K+1} \quad \text{and} \quad J_{44} \ll 1.$$

Thus, the contribution from the integral along the left vertical side of the rectangle $\mathcal{R}$ in (3.4) becomes

$$I_4 = \int_{1-c+iT_\delta}^{1-c+iT} \frac{F_m'}{F_m}(s)x^s\,ds = O\bigg(\log x(2\log 2x)^{mK+1} + T\bigg(\frac{\log^{\frac{1}{4}}\log T}{\log^{\frac{1}{4}} T}\bigg)^{K+1}\bigg) + O\bigg(\min\bigg(T\log 2T, \frac{\log 2T}{\log x}\bigg)\bigg). \tag{3.40}$$

By using the estimates from (3.17), (3.18), (3.39), and (3.40) in (3.4), we complete the proof of Theorem 1.3.

4 Proof of the zero density estimates: Theorem 1.5

As discussed earlier in the previous section, since the complex zeros of $\xi^{(m)}(s)$ are identical to those of $F_m(s)$, we prove the theorem for $F_m(s)$ instead. Let

$$f(s) := M_X(s)F_m(s) - 1, \tag{4.1}$$

where M_X is defined by (2.1). Consider

$$h(s) := 1 - f^2(s). \tag{4.2}$$

Here $h(s)$ is analytic except for the pole at $s = 1$. Let $P(x) = x$ in Lemma 2.1. Then, for $0 < \theta < 1$ and $X = T^\theta$, we have

$$M_X(s) = \sum_{n\le X} \frac{\mu(n)}{n^s}\bigg(1 + O\bigg(\frac{\log n}{\log T}\bigg)\bigg)\bigg(1 - \frac{\log n}{\log X}\bigg) = \sum_{n\le X} \frac{\mu(n)}{n^s}\bigg(1 + O\bigg(\frac{\log n}{\log T}\bigg)\bigg).$$

Let $\sigma \ge 2$. Then, from (2.8), (3.5), (4.1), and (4.3),

$$f(s) \ll \bigg|\zeta(s)\sum_{n\le X}\frac{\mu(n)}{n^s} - 1\bigg| + \frac{e^{O(m)}}{\log T} \ll \sum_{n\ge X}\frac{d(n)}{n^\sigma} + \frac{e^{O(m)}}{\log T} \ll \frac{1}{\sqrt{X}} + \frac{e^{O(m)}}{\log T}$$

for $\frac{T}{2} \le t < T$. Therefore, for some $X > X_0$, $T > T_0$, $m \le \mathcal{L}$, and $\sigma \ge 2$,

$$|f(s)| < \frac{1}{2}. \tag{4.3}$$

Combining (4.2) and (4.3), we find that $h(2 + it) \ne 0$ for $t > T_0$ and $X \ge X_0$. Let $\nu(\sigma', T)$ denote the number of zeros of $h(s)$ in the rectangle $\sigma > \sigma'$ and $0 < t \le T$. By the Hardy–Littlewood lemma (see [34, p. 221]), one has

$$\begin{aligned} 2\pi\int_{\sigma_0}^{2} \nu\bigg(\sigma, \frac{T}{2}, T\bigg)\,d\sigma &= \int_{T/2}^{T} \log|h(\sigma_0 + it)|\,dt - \int_{T/2}^{T} \log|h(2 + it)|\,dt \\ &\quad + \int_{\sigma_0}^{2} \arg h(\sigma_0 + iT)\,d\sigma - \int_{\sigma_0}^{2} \arg h(\sigma_0 + iT/2)\,d\sigma, \end{aligned} \tag{4.4}$$

where $\nu(\sigma, \frac{T}{2}, T) = \nu(\sigma, T) - \nu(\sigma, \frac{T}{2})$ and $\sigma_0 \ge \frac{1}{2}$ is fixed.

From (4.2) and (4.3), we deduce that

$$\mathrm{Re}(h(2 + it)) \ge \frac{1}{2}$$

for $t \ge T_0$ and $x \ge X_0$. From (2.13), $\zeta^{(k)}(s) \ll k!t^B$ for some constant B. Then

$$h(\sigma + it) \ll e^{Cm\log m}X^A t^A$$

for some constant C, $\sigma \ge 0$ and sufficiently large t. Therefore, from Lemma 2.8, we have

$$\arg h(\sigma + iT) - \arg h\bigg(\sigma + i\frac{T}{2}\bigg) \ll \log X + \log T + m\log m$$

for $\sigma \geq \sigma_0$. This gives

$$\int_{\sigma_0}^{2} \arg h(\sigma + iT)\, d\sigma - \int_{\sigma_0}^{2} \arg h\Big(\sigma + i\frac{T}{2}\Big)\, d\sigma \ll \log X + \log T + m \log m \ll \log T \tag{4.5}$$

for $0 < \theta < 1$, $m \leq \mathcal{L}$ and $X = T^{\theta}$.

From (2.8), (4.3), and for $\operatorname{Re} s > 1$,

$$\begin{aligned} M_X(s)F_m(s) &= \zeta(s)M_X(s) + \sum_{j=1}^{m} \binom{m}{j} \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \zeta^{(j)}(s) M_X(s) \\ &= \sum_{n=1}^{\infty} \frac{a_X(n)}{n^s} + \sum_{j=1}^{m} \binom{m}{j} \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \sum_{n=2}^{\infty} \frac{b_{j,X}(n)}{n^s}, \end{aligned} \tag{4.6}$$

where $a_X(1) = 1$,

$$a_X(n) = \sum_{d|n} \mu(d)\Big(1 + O\Big(\frac{\log d}{\log T}\Big)\Big) \ll \begin{cases} \frac{b(n)}{\log T} & \text{if } 2 \leq n < X, \\ d(n) + \frac{b(n)}{\log T} & \text{if } n \geq X, \end{cases} \tag{4.7}$$

and

$$b_{j,X}(n) = \sum_{d|n} \log^j\Big(\frac{n}{d}\Big)\mu(d)\Big(1 + O\Big(\frac{\log d}{\log T}\Big)\Big) \ll \begin{cases} \Lambda_j(n) + \frac{c(n)}{\log T} & \text{if } 2 \leq n < X, \\ c_1(n) + \frac{c(n)}{\log T} & \text{if } n \geq X. \end{cases} \tag{4.8}$$

Here, $d(n)$ denotes the divisor function,

$$b(n) = \sum_{d|n} \mu^2(d) \log d, \quad c_1(n) = \sum_{d|n} \log^j\Big(\frac{n}{d}\Big)\mu^2(d) \quad \text{and} \quad c(n) = \sum_{d|n} \log^j\Big(\frac{n}{d}\Big)\mu^2(d) \log d. \tag{4.9}$$

Therefore, for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s)\Big(\frac{\zeta(s)}{\zeta(2s)}\Big)', \quad \sum_{n=1}^{\infty} \frac{c_1(n)}{n^s} = \zeta^{(j)}(s)\frac{\zeta(s)}{\zeta(2s)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \zeta^{(j)}(s)\Big(\frac{\zeta(s)}{\zeta(2s)}\Big)'.$$

Since $h(s)$ is analytic for $\sigma \geq 2$ and $h(s) \to 1$ as $\sigma \to \infty$, by the residue theorem,

$$\int_{T/2}^{T} \log h(2 + it)\, dt = \int_{2}^{\infty} \log h\Big(\sigma + i\frac{T}{2}\Big)\, d\sigma - \int_{2}^{\infty} \log h(\sigma + iT)\, d\sigma. \tag{4.10}$$

One has

$$\log|h(s)| \leq \log(1 + |f(s)|^2) \leq |f(s)|^2 \leq |f(\sigma)|^2 \tag{4.11}$$

and

$$\log|h(s)| = \operatorname{Re}(\log h(s)).$$

Using this along with (4.10) and (4.11), we have

$$\begin{aligned} \int_{T/2}^{T} \log|h(2 + it)|\, dt &= \operatorname{Re}\Big(\int_{T/2}^{T} \log h(2 + it)\, dt\Big) \\ &\ll \int_{2}^{\infty} \log\Big|h\Big(\sigma + i\frac{T}{2}\Big)\Big|\, d\sigma + \int_{2}^{\infty} \log|h(\sigma + iT)|\, d\sigma \\ &\ll \int_{2}^{\infty} |f(\sigma)|^2\, d\sigma. \end{aligned} \tag{4.12}$$

From (4.1), (4.6), (4.7) and (4.8), we have

$$(f(s))^2 = \sum_{n=2}^{\infty} \frac{a(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$ and $a(n) \ll e^{O(m)}(d(n))^3 \log^{2m}(n)$. Thus, from (4.12),

$$\int_{T/2}^{T} \log|h(2+it)|\,dt = e^{O(m)}.$$

Thus, it remains to estimate only the first integral in (4.4), which is done by using the convexity theorem. From (4.1), we find that

$$I_1 := \int_{T/2}^{T} \left| f\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 dt \ll \int_{T/2}^{T} \left| M_X F_m\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 dt + T.$$

From Lemma 2.3 and integrating by parts, we have

$$\int_{T/2}^{T} \left| M_X F_m\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 dt \sim \int_{T/2}^{T} \left| M_X V\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 dt,$$

where

$$V(s) = \zeta(s) + \sum_{k=1}^{m} \binom{m}{k} \frac{2^k}{\log^k T} \zeta^k(s).$$

Let $Q(x) = (1+2x)^m$, $R \ll 1$ be fixed and $P(x) = x$. Then, by Lemma 2.1 and (2.2),

$$\int_{T/2}^{T} \left| M_X V\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 dt \sim cT,$$

where

$$c \ll \Big| \max_{0 \le x \le 1} (Q(x), Q'(x)) \Big|^2 \ll 3^m.$$

Next, we compute the integral

$$I_2 := \int_{T/2}^{T} |f(2+it) - 1|^2\,dt = \int_{T/2}^{T} |M_X F_m(2+it) - 1|^2\,dt.$$

From (3.5) and (4.6),

$$I_2 \ll m \int_{T/2}^{T} \left| \sum_{n=2}^{\infty} \frac{a_X(n)}{n^{2+it}} \right|^2 dt + \sum_{j=1}^{m} \left(\binom{m}{j}\right)^2 \frac{4^j}{\log^{2j} T} \int_{T/2}^{T} \left| \sum_{n=2}^{\infty} \frac{b_{j,X}(n)}{n^{2+it}} \right|^2 dt.$$

Employing Lemma 2.7, (4.7), (4.8), and (4.9), we have

$$I_2 = e^{O(m)} \frac{T}{\log^2 T}.$$

From an easy modification of the classical convexity theorem (see [34, p. 233]), one can deduce that

$$\int_{T/2}^{T} |f(\sigma_0 + it)|^2\,dt = e^{O(m)} T \log^{\frac{1-2\sigma_0}{3}} T \tag{4.13}$$

uniformly for $\frac{1}{2} - \frac{R}{\log T} \le \sigma_0 \le 2$. From (4.11) and (4.13), we find that

$$\int_{T/2}^{T} \log|h(\sigma_0 + it)|\,dt = e^{O(m)} T \log^{\frac{1-2\sigma_0}{3}} T. \tag{4.14}$$

Combining (4.4), (4.5), (4.12), (4.14), and the inequality

$$\int_{\sigma_0}^{2} \nu\Big(\sigma, \frac{T}{2}, T\Big)\, d\sigma \geq \int_{\sigma_0}^{1} N_m(\sigma, T)\, d\sigma - \int_{\sigma_0}^{1} N_m\Big(\sigma, \frac{T}{2}\Big)\, d\sigma,$$

which follows from (4.2), we obtain

$$\int_{\sigma_0}^{1} N_m(\sigma, T)\, d\sigma - \int_{\sigma_0}^{1} N_m\Big(\sigma, \frac{T}{2}\Big)\, d\sigma = e^{O(m)} T \log^{\frac{1-2\sigma_0}{3}} T$$

uniformly for $\frac{1}{2} \leq \sigma_0 \leq 1$. Now, we replace T by $T/2^n$, $n \geq 0$, in the above estimate, and sum over n for $0 \leq n \leq \infty$ to complete the proof of Theorem 1.5.

5 Uniform distribution and Discrepancy Bounds: Proofs of Theorems 1.1 and 1.2

5.1 Proof of Theorem 1.1

We start with the identity

$$\sum_{0 \leq \gamma_m \leq T} x^{i\gamma_m} = \sum_{0 \leq \gamma_m \leq T} x^{\rho_m - 1/2} + \sum_{0 \leq \gamma_m \leq T} (x^{i\gamma_m} - x^{\rho_m - 1/2}), \tag{5.1}$$

which holds for any x. Let $x = e^{2\pi\alpha}$, where $\alpha > 0$ is any fixed real number. From (1.10), it can be shown that the non-trivial zeros of $\xi^{(m)}(s)$ are symmetric with respect to the line $\sigma = 1/2$. Therefore,

$$\sum_{0 \leq \gamma_m \leq T} (x^{i\gamma_m} - x^{\rho_m - 1/2}) \ll \sum_{\substack{0 \leq \gamma_m \leq T \\ \beta_m > 1/2}} |1 - x^{\beta_m - 1/2}| \ll \sqrt{x} \log x \sum_{\substack{0 \leq \gamma_m \leq T \\ \beta_m > 1/2}} (\beta_m - 1/2) = \sqrt{x} \log x \int_{1/2}^{1} N_m(\sigma, T)\, d\sigma,$$

where in the penultimate step, we use the mean value theorem. Combining this with Theorem 1.5, we find that

$$\sum_{0 \leq \gamma_m \leq T} (x^{i\gamma_m} - x^{\rho_m - 1/2}) \ll e^{O(m)} \sqrt{x} T \log x. \tag{5.2}$$

Let T be large enough such that

$$\log 2x \leq (\log T)^{1-\epsilon}.$$

for some $\epsilon > 0$. For $x > 1$, from Theorem 1.3, we have

$$\begin{aligned}\sum_{0 \leq \gamma_m \leq T} x^{\rho_m - 1/2} \ll{}& \frac{T \log x}{\sqrt{x}} + \sqrt{x} \log(2xT) \log x + \frac{T}{\sqrt{x}} \frac{(2 \log 2x)^{mK+1}}{(\log T)^{mK}} \\ &+ \sqrt{x} (\log T)^{\delta} e^{(\log T)^{\delta}} + \sqrt{x} T \Big(\frac{\log \log T}{\log T}\Big)^{\frac{K+1}{4}} \\ &+ \sqrt{x} \log x (2 \log 2x)^{mK+1} \Big(\frac{\log \log T}{\delta (\log T)^{\delta}}\Big)^{mK}.\end{aligned} \tag{5.3}$$

Hence, from Lemma 2.10 and estimates (5.1), (5.2) and (5.3), we have

$$\frac{1}{N_m(T)} \sum_{0 \leq \gamma_m \leq T} x^{i\gamma_m} = o(1)$$

as $T \to \infty$ and uniformly for $m \leq \mathcal{L}$. A similar result also holds for $0 < x < 1$. In this case, we first use (1.11) on the left side of (5.3), and then apply Theorem 1.3.

Invoking the Weyl criterion, Lemma 2.5, we conclude that the sequence $(\alpha\gamma_m)$ is uniformly distributed modulo one. This completes the proof of Theorem 1.1.

5.2 Proof of Theorem 1.2

Case (i). Assuming the Riemann hypothesis. Let $\rho_m = \frac{1}{2} + i\gamma_m$. Also, note that $\Lambda(n_x)\delta_{x,T} \ll T\log x$. Then, from Theorem 1.3 and Lemma 2.6, we have

$$\begin{aligned}
D_m^*(\alpha; T) &\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \frac{1}{k} \Big| \sum_{0\le\gamma_m\le T} x^{ik\gamma_m} \Big| \\
&= \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \frac{1}{k} \Big| \sum_{0\le\gamma_m\le T} (x^k)^{\rho_m - 1/2} \Big| \\
&\ll \frac{1}{M+1} + \frac{1}{T\log T} \sum_{k=1}^{M} \Big\{ \frac{T\log x}{x^{k/2}} + x^{k/2} k \log(2xT)\log x \\
&\quad + \frac{T}{x^{k/2}} \frac{(2k\log 2x)^{mK+1}}{k(\log T)^{mK}} + x^{k/2}\log x (2k\log 2x)^{mK+1} \Big(\frac{\log\log T}{\delta(\log T)^{\delta}}\Big)^{mK} \\
&\quad + \frac{x^{k/2}}{k} (\log T)^{\delta} e^{(\log T)^{\delta}} + \frac{x^{k/2}}{k} T\Big(\frac{\log\log T}{\log T}\Big)^{\frac{K+1}{4}} \Big\} \\
&\ll \frac{1}{M+1} + \frac{1}{T\log T} \Big\{ \frac{T\log x}{x^{1/2}} + x^{\frac{M+1}{2}} M\log(2xT)\log x \\
&\quad + \frac{T}{x^{1/2}} \frac{M^{mK}(2\log 2x)^{mK+1}}{(\log T)^{mK}} + x^{\frac{M+1}{2}} \log x (2M\log 2x)^{mK+1}\Big(\frac{\log\log T}{\delta(\log T)^{\delta}}\Big)^{mK} \\
&\quad + x^{\frac{M+1}{2}} (\log T)^{\delta} e^{(\log T)^{\delta}} + x^{\frac{M+1}{2}} T \Big(\frac{\log\log T}{\log T}\Big)^{\frac{K+1}{4}} \Big\}. \qquad (5.4)
\end{aligned}$$

Now, we set

$$\delta = 1 - \frac{\log 10}{\log\log T}, \quad M+1 = \Big\lfloor \frac{\log T}{2\log 2x} \Big\rfloor \quad \text{and} \quad K = \Big\lfloor \frac{4\log T}{\log\log T} \Big\rfloor.$$

Therefore, we obtain

$$D_m^*(\alpha; T) \le \frac{c_1\log x}{\log T} + \exp\Big(\frac{c_2 m\log T}{\log\log\log T}\Big)\frac{\log x}{\sqrt{T}},$$

where c_1 and c_2 are absolute constants, and this holds uniformly for $0 \le m \le \mathcal{L}$. This completes the proof of Theorem 1.2.

Case (ii). Unconditional bound. From (5.1), we have

$$\begin{aligned}
D_m^*(\alpha; T) &\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \frac{1}{k} \Big| \sum_{0\le\gamma_m\le T} x^{ik\gamma_m} \Big| \\
&\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \frac{1}{k} \Big(\Big| \sum_{0\le\gamma_m\le T} (x^k)^{\rho_m-1/2} \Big| + \Big| \sum_{0\le\gamma_m\le T} (x^{ik\gamma_m} - (x^k)^{\rho_m-1/2}) \Big| \Big). \qquad (5.5)
\end{aligned}$$

From (5.4) and (5.2), we have

$$\begin{aligned}
D_m^*(\alpha; T) &\ll \frac{1}{M+1} + \frac{1}{T\log T} \Big\{ \frac{T\log x}{x^{1/2}} + x^{\frac{M+1}{2}} M\log(2xT)\log x \\
&\quad + \frac{T}{x^{1/2}} \frac{M^{mK}(2\log 2x)^{mK+1}}{(\log T)^{mK}} + x^{\frac{M+1}{2}} \log x (2M\log 2x)^{mK+1}\Big(\frac{\log\log T}{\delta(\log T)^{\delta}}\Big)^{mK} \\
&\quad + x^{\frac{M+1}{2}} (\log T)^{\delta} e^{(\log T)^{\delta}} + x^{\frac{M+1}{2}} T \Big(\frac{\log\log T}{\log T}\Big)^{\frac{K+1}{4}} + e^{O(m)} x^{\frac{M+1}{2}} T\log x \Big\}.
\end{aligned}$$

Now, we set

$$M+1 = \Big\lfloor \frac{\log\log T}{\log 2x} \Big\rfloor.$$

Hence, we deduce that

$$D_m^*(\alpha; T) \le \frac{a_1\log x}{\log\log T} + \frac{e^{a_2 m}\log x}{\sqrt{\log T}},$$

where a_1 and a_2 are absolute constants, and this holds uniformly for $0 \le m \le \mathcal{L}$.

Acknowledgment: The authors thank the referee for valuable comments and corrections which have improved the quality of this paper. The first author would also like to thank the Mathematics Department at Rice University for their hospitality where part of this research was completed.

References

[1] A. Akbary and M. R. Murty, Uniform distribution of zeros of Dirichlet series, in: *Anatomy of Integers*, CRM Proc. Lecture Notes 46, American Mathematical Society, Providence (2008), 143–158.
[2] H. M. Bui, Gaps between zeros of the derivative of the Riemann ξ-function, *J. Théor. Nombres Bordeaux* **22** (2010), no. 2, 287–305.
[3] M. W. Coffey, Asymptotic estimation of $\xi^{(2n)}(1/2)$: On a conjecture of Farmer and Rhoades, *Math. Comp.* **78** (2009), no. 266, 1147–1154.
[4] B. Conrey, Zeros of derivatives of Riemann's ξ-function on the critical line, *J. Number Theory* **16** (1983), no. 1, 49–74.
[5] B. Conrey and H. Iwaniec, Spacing of zeros of Hecke L-functions and the class number problem, *Acta Arith.* **103** (2002), no. 3, 259–312.
[6] J. B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, *J. Reine Angew. Math.* **399** (1989), 1–26.
[7] T. Craven, G. Csordas and W. Smith, The zeros of derivatives of entire functions and the Pólya–Wiman conjecture, *Ann. of Math. (2)* **125** (1987), no. 2, 405–431.
[8] H. Davenport and H. Montgomery, *Multiplicative Number Theory*, Grad. Texts Math. 74, Springer, New York, 2013.
[9] P. Erdös, On the integers having exactly K prime factors, *Ann. of Math. (2)* **49** (1948), 53–66.
[10] P. Erdös and P. Turán, On a problem in the theory of uniform distribution. I, *Nederl. Akad. Wetensch., Proc.* **51** (1948), 1146–1154; *Indagationes Math.* **10**, 370–378 (1948).
[11] D. W. Farmer, S. M. Gonek and Y. Lee, Pair correlation of the zeros of the derivative of the Riemann ξ-function, *J. Lond. Math. Soc. (2)* **90** (2014), no. 1, 241–269.
[12] D. W. Farmer and R. C. Rhoades, Differentiation evens out zero spacings, *Trans. Amer. Math. Soc.* **357** (2005), no. 9, 3789–3811.
[13] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University, Cambridge, 2009.
[14] K. Ford, K. Soundararajan and A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function. II, *Math. Ann.* **343** (2009), no. 3, 487–505.
[15] K. Ford and A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function, *J. Reine Angew. Math.* **579** (2005), 145–158.
[16] A. Fujii, On a theorem of Landau, *Proc. Japan Acad. Ser. A Math. Sci.* **65** (1989), no. 2, 51–54.
[17] A. Fujii, Some problems of Diophantine approximation in the theory of the Riemann zeta function. III, *Comment. Math. Univ. St. Pauli* **42** (1993), no. 2, 161–187.
[18] S. M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta-function, in: *A Tribute to Emil Grosswald: Number Theory and Related Analysis*, Contemp. Math. 143, American Mathematical Society, Providence (1993), 395–413.
[19] E. Hlawka, Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen, *Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II* **184** (1975), no. 8–10, 459–471.
[20] A. Ivić, *The Riemann Zeta-function. The Theory of the Riemann Zeta-function with Applications*, John Wiley & Sons, New York, 1985.
[21] H. Ki, The Riemann Ξ-function under repeated differentiation, *J. Number Theory* **120** (2006), no. 1, 120–131.
[22] H. Ki and Y.-O. Kim, On the number of nonreal zeros of real entire functions and the Fourier–Pólya conjecture, *Duke Math. J.* **104** (2000), no. 1, 45–73.
[23] Y.-O. Kim, Critical points of real entire functions and a conjecture of Pólya, *Proc. Amer. Math. Soc.* **124** (1996), no. 3, 819–830.
[24] E. Landau, Über die Nullstellen der Zetafunktion, *Math. Ann.* **71** (1912), no. 4, 548–564.
[25] J. E. Littlewood, On the Riemann zeta-function, *Proc. Lond. Math. Soc. (2)* **24** (1925), no. 3, 175–201.
[26] H. L. Montgomery, The pair correlation of zeros of the zeta function, in: *Analytic Number Theory* (St. Louis 1972), Proc. Sympos. Pure Math. 24, American Mathematical Society, Providence (1973), 181–193.
[27] H. L. Montgomery and R. C. Vaughan, Hilbert's inequality, *J. Lond. Math. Soc. (2)* **8** (1974), 73–82.
[28] M. R. Murty and A. Perelli, The pair correlation of zeros of functions in the Selberg class, *Int. Math. Res. Not. IMRN* **1999** (1999), no. 10, 531–545.
[29] R. Murty and A. Zaharescu, Explicit formulas for the pair correlation of zeros of functions in the Selberg class, *Forum Math.* **14** (2002), no. 1, 65–83.

[30] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function, *Math. Comp.* **48** (1987), no. 177, 273–308.
[31] H. Rademacher, *Topics in Analytic Number Theory*, Grundlehren Math. Wiss. 169, Springer, New York, 1973.
[32] H. Rademacher, *Collected Papers of Hans Rademacher. Vol. II*, MIT Press, Cambridge, 1974.
[33] A. Selberg, On the zeros of Riemann's zeta-function, *Skr. Norske Vid. Akad. Oslo I.* **10** (1942), 1–59.
[34] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed., The Clarendon Press, New York, 1986.
[35] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.* **77** (1916), no. 3, 313–352.