Research Article

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On the distribution of zeros of derivatives of the Riemann $\xi$-function

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Abstract: For the completed Riemann zeta function $\xi(s)$, it is known that the Riemann hypothesis for $\xi(s)$ implies the Riemann hypothesis for $\xi^{(m)}(s)$, where $m$ is any positive integer. In this paper, we investigate the distribution of the fractional parts of the sequence $(\alpha \gamma_m)$, where $\alpha$ is any fixed non-zero real number and $\gamma_m$ runs over the imaginary parts of the zeros of $\xi^{(m)}(s)$. We also obtain a zero density estimate and an explicit formula for the zeros of $\xi^{(m)}(s)$. In particular, all our results hold uniformly for $0 \leq m \leq g(T)$, where the function $g(T)$ tends to infinity with $T$ and $g(T) = o(\log \log T)$.

Keywords: Riemann $\xi$-function, zeros, explicit formula, fractional parts, zero density

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1 Introduction

The Riemann $\xi$-function is defined by

$$\xi(s) = H(s) \bar{\xi}(s),$$

where

$$H(s) := \frac{s}{2} (s - 1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right),$$

and $\bar{\xi}(s)$ denotes the Riemann zeta function. The non-trivial zeros of $\bar{\xi}(s)$ are identical to the zeros of $\xi(s)$. It is well known that the real parts of the zeros of $\bar{\xi}(s)$ lie in the critical strip $0 < \text{Re}(s) < 1$. The Riemann hypothesis for $\xi(s)$ states that these zeros lie on the critical line $\text{Re}(s) = 1/2$. Moreover, the Riemann hypothesis for $\xi(s)$ implies that the zeros of $\xi^{(m)}(s)$ also lie on the critical line $\text{Re}(s) = 1/2$. In 1983, Conrey [4] showed that for $m \geq 0$, the real parts of the zeros of $\xi^{(m)}(s)$ also lie in the critical strip $0 < \text{Re}(s) < 1$.

There has also been a great interest in studying the vertical distribution of the zeros of $\xi(s)$. Under the assumption of the Riemann hypothesis, Rademacher [32] first proved that the sequence $(\alpha \gamma_0)$, where $\gamma_0$ denotes the imaginary part of a non-trivial zero of $\bar{\xi}(s)$ and $\alpha$ is any fixed non-zero real number, is uniformly distributed modulo one. Hlawka [19] proved this result unconditionally.

Let $\{x\}$ denote the fractional part of a real number $x$. Let $\rho_0 = \beta_0 + i\gamma_0$ denote a non-trivial zero of $\xi(s)$. The discrepancy of the set $\{(\alpha \gamma_0) : 0 < \gamma_0 \leq T\}$ is defined by

$$D^*_\alpha(T) := \sup_{0 \leq y_0 \leq 1} \left| \frac{\# \{0 \leq \gamma_0 \leq T; 0 \leq \langle \alpha \gamma_0 \rangle < y\} - y}{N(T)} \right|,$$

where $N(T)$ denotes the number of zeros of $\xi(s)$ such that $0 \leq \beta_0 \leq 1$ and $0 < y_0 \leq T$.

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For any integer $x$, let $\alpha = \frac{\log x}{2\pi}$. In 1975, Hlawka [19] showed that
\[ D^*_a(T) \ll \frac{\log x}{\log T} \] (1.3)
under the Riemann hypothesis, while
\[ D^*_a(T) \ll \frac{\log x}{\log \log T} \] (1.4)
unconditionally. In 1993, Fujii [17] improved this bound and showed that
\[ D^*_a(T) \ll a \frac{\log \log T}{\log T}. \]

Recently, Ford and Zaharescu [15] investigated this result on discrepancy in more general settings. In particular, they showed that the discrepancy of the set $\{h(\alpha y) : 0 < y_0 \leq T\}$ is of the order $O(1/\log T)$ for a large class of functions $h : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$. Also, Akbary and Murty [1] obtained similar results on the uniform distribution and the discrepancy for a large class of Dirichlet series on the assumption of average density hypothesis.

Another very important result in this direction is due to Montgomery. In [26], he studied the pair correlation of zeros of $\xi(s)$. Odlyzko [30] showed that the distribution of consecutive spacing of imaginary parts of zeros would agree with the distribution of eigenvalue spacings in Gaussian unitary ensemble. These results were also extended for general $L$ functions by Murty and Perelli [28], and Murty and Zaharescu [29]. In his work [26], Montgomery mentioned the connection between Landau–Siegel zeros and the gap between consecutive zeros of the Riemann zeta function. Conrey and Iwaniec [5] showed that the existence of Landau–Siegel zeros implies that the spacing of consecutive zeros of $\xi(s)$ are close to multiples of half the average spacing.

The vertical distributions of zeros of $\xi'(s)$ have also been studied recently. In [11], Farmer, Gonek and Lee initiated the study of consecutive spacing of zeros of $\xi'(s)$. They investigated the pair correlation of the zeros of $\xi'(s)$ under the Riemann hypothesis. They obtained various estimates on the consecutive spacing and multiplicity of the zeros of $\xi'(s)$. Bui [2] improved some of their results on consecutive spacing of zeros of $\xi'(s)$.

One motivation of studying such distributions of $\xi^{(m)}(s)$ is to understand the distribution of zeros of an entire function under differentiation. From the functional equation
\[ \xi(s) = \xi(1 - s), \] (1.5)
one can see that the entire function $\xi(1/2 + it)$ is real on the real axis and has order one. Also, from the work of Craven, Csordas and Smith [7], Ki and Kim [22], and Kim [23], one may observe that for sufficiently large $m$, the Riemann hypothesis is true for $\xi^{(m)}(s)$ in a bounded region. Also, the zeros of $\xi^{(m)}(s)$ approach equal spacing as $m$ tends to infinity. For details, readers are directed to the work of Farmer and Rhoades [12], Coffey [3], and Ki [21]. Since the small gaps between zeros become larger under differentiation, by the work of Conrey and Iwaniec [5], one may disprove the existence of Landau–Siegel zeros by showing the gap between consecutive zeros of $\xi^{(m)}(s)$ to be less than half of the average spacing for sufficiently many zeros; for details, also see [11].

In 2009, Ford, Soundararajan and Zaharescu [14] established some connections between Montgomery’s pair correlation function and the distribution of the fractional parts of $(\alpha y_0)$. So one might expect that the pair correlation result of Gonek, Farmer and Lee [11] would have connections with the distribution of fractional parts of $(\alpha y_m)$, where $\rho_m = \beta_m + i\gamma_m$ denotes a complex zero of $\xi^{(m)}(s)$.

Although much information on the distribution of fractional parts of $(\alpha y_0)$ is known, the authors cannot recall any results of the distribution of fractional parts of $(\alpha y_m)$. The main goal of this paper to obtain some classical results on the distribution of the fractional parts of $(\alpha y_m)$ analogous to the results of Rademacher [32] and Hlawka [19]. Our first result in this direction is stated below.

**Theorem 1.1.** For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and a positive integer $m$, the sequence $(\alpha y_m)$ is uniformly distributed modulo one, where $y_m$ runs over the imaginary parts of zeros of $\xi^{(m)}(s)$ and zeros are counted with multiplicity.
Next, we are interested in the discrepancy of the sequence \((a, y_m)\). Let \(N_m(T)\) denote the number of zeros of \(\xi^{(m)}(s)\) such that \(0 \leq \beta_m \leq 1\) and \(0 < y_m \leq T\). Conrey [4] proved that

\[
N_m(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O_m(\log T).
\]

Let \(D_m^*(\alpha; T)\) denote the discrepancy

\[
D_m^*(\alpha; T) := \sup_{0 < y \leq 1} \left| \frac{\# \{0 \leq y_m \leq T; 0 \leq \alpha y_m < y \}}{N_m(T)} - \frac{1}{\alpha} \right|
\]

of the set \(\{\alpha y_m : 0 < y_m \leq T\}\), and zeros are counted with multiplicity.

For \(T > 10\), let us define

\[
\mathcal{L} := \frac{\log \log T}{(\log \log T)^2}.
\]

Then \(\mathcal{L}\) tends to infinity with \(T\). We have the following bound for \(D_m^*(\alpha; T)\), which generalizes the results of Hlawka [19] for \(\zeta(s)\).

**Theorem 1.2.** Let \(\alpha \geq \log \frac{2}{\pi} \) and let \(m \leq \mathcal{L}\) be a non-negative integer. Then

\[
D_m^*(\alpha; T) \leq \frac{a_1 \alpha}{\log \log T} + \frac{e^{a_2 m} \alpha}{\sqrt{\log T}}
\]

as \(T \to \infty\) and uniformly on \(m \leq \mathcal{L}\), where \(a_1\) and \(a_2\) are absolute constants. Moreover, under the assumption of the Riemann hypothesis,

\[
D_m^*(\alpha; T) \leq \frac{c_1 \alpha}{\log T} + \exp\left(\frac{c_2 m \log T}{\log \log \log T}\right) \frac{\alpha}{\sqrt{T}}
\]

as \(T \to \infty\) and uniformly on \(m \leq \mathcal{L}\), where \(c_1\) and \(c_2\) are absolute constants.

**Remark.** From the above theorem, for \(0 \leq m \leq \mathcal{L}\), the bound

\[
D_m^*(\alpha; T) \ll \frac{a}{\log \log T}
\]

holds unconditionally and

\[
D_m^*(\alpha; T) \ll \frac{a}{\log T}
\]

holds under the assumption of the Riemann hypothesis. Theorem 1.2 shows that the distribution of the sequence \((a, y_m)\) depends on \(m\). If we take \(m \leq \mathcal{L}\), then the discrepancy vanishes as \(m\) tend to infinity with \(T\). This result can be compared with that of Ki [21] who showed that there exist sequences \(A_n\) and \(C_n\), with \(C_n \to 0\) slowly, such that

\[
\lim_{n \to \infty} A_n \xi^{(2n)} \left( \frac{1}{2} + i C_n s \right) = \cos s
\]

uniformly on compact subset of \(\mathbb{C}\), which was conjectured by Farmer and Rhoades [12]. In other words, one can say that the zeros of derivatives become more well spaced as \(m\) increases.

Hlawka’s discrepancy bounds (1.3) and (1.4) rely on the explicit formula of Landau [24]

\[
\sum_{0 < y \leq T} x^{\phi_0} = -\Lambda(x) \frac{T}{2\pi} + O(\log T),
\]

where \(\Lambda(n)\) is the von-Mangoldt function. Gonek [18] gave an explicit formula, similar to (1.7), which is uniform in both \(x\) and \(T\). Fujii [16] also obtained a similar result independently. Gonek’s explicit formula can be stated as follows:

\[
\sum_{0 < y \leq T} x^{\phi_0} = -\Lambda(x) \frac{T}{2\pi} + O\left( x \log^2(2xT) \right) + \frac{\log 2T}{\log x} + O\left( \log x \min\left( T, \frac{x}{\langle x \rangle} \right) \right),
\]

where \(\langle x \rangle\) is the distance to the nearest integer prime power other than \(x\) itself.
In order to prove Theorems 1.1 and 1.3, we also need an explicit formula for the zeros of \( \xi^{(m)}(s) \). An essential ingredient in obtaining the explicit formulas (1.7) and (1.8) in the case of \( \xi(s) \) is the Dirichlet series representation of \( \xi^\prime(s) \) for \( \Re s > 1 \). However, there are no such Dirichlet series for \( \xi^{(m+1)}(s) \). We give an explicit formula for \( \xi^{(m)}(s) \).

**Theorem 1.3.** Let \( x, T > 1 \) and let \( n_x \) be the nearest prime power to \( x \). Let the zeros of \( \xi^{(m)}(s) \) be counted with multiplicity. Then, for any number \( \delta > 0 \) and integer \( K \geq 1 \),

\[
\sum_{0 \leq \gamma_m \leq T} x^{\rho_m} = -\frac{\Lambda(n_x)}{2\pi} \delta_{x,T} + O\left(T \frac{\log 2x}{\log T} \right)^{\frac{\delta}{\log T}} + O\left(T \log(2xT) \log 2x + \log 2T \min\left(1, \frac{1}{\log x}\right)\right) + O\left(x \log x \left(\frac{\log T}{\log \log T}\right)^K + x \log \log T \right)\]

holds uniformly for \( 0 \leq m \leq \mathcal{L} \). Here \( \delta_{x,T} = T \) if \( x = n_x \), and

\[
\delta_{x,T} = \min\left(1, \frac{T}{\log x}\right) \quad \text{if} \quad x \neq n_x.
\]

The first error term in (1.9) can be written as a main term with some more efforts. This error term also disappears if \( x \) is not an integer, which could be proved by using the same method given in the proof of Theorem 1.3. Also the error term containing \( K \) and \( \delta \) in (1.9) can be improved by a result of Erdős [9] for small values of \( K \).

Differentiating (1.5) gives the functional equation

\[
\xi^{(m)}(s) = (-1)^m \xi^{(m)}(1-s) \tag{1.10}
\]

Since \( \xi^{(m)}(s) \) is real-valued for real values of \( s \), it is clear from (1.10) that the zeros of \( \xi^{(m)}(s) \) are symmetric with respect to the line \( \sigma = 1/2 \). Therefore, for \( 0 < x < 1 \), we have

\[
\sum_{0 \leq \gamma_m \leq T} x^{\rho_m} = \sum_{0 \leq \gamma_m \leq T} x^{1-\rho_m} = x \sum_{0 \leq \gamma_m \leq T} \left(\frac{1}{x}\right)^{\rho_m} \tag{1.11}
\]

If we choose

\[
K = \left\lfloor \frac{4 \log T}{\log \log T} \right\rfloor
\]

then for a fixed \( x \) and for \( T \) sufficiently large, one can show that (1.9) can be written as

\[
\sum_{0 \leq \gamma_m \leq T} x^{\rho_m} \ll Tx^{\epsilon} + xT^\epsilon,
\]

for \( \epsilon > 0 \), which may depend on \( T \). Therefore, by the Riemann hypothesis, we find that

\[
\sum_{0 \leq \gamma_m \leq T} x^{\gamma_m} \ll Tx^{-\frac{1}{2} + \epsilon} + x^{\frac{1}{2} + T^\epsilon}, \tag{1.12}
\]

which is non trivial for \( 2 \leq x \leq T^{2-\epsilon} \) by (1.6). Now, if one assumes that \( \{x^{\gamma_m}\} \) behave like independent random variables, then we may expect that

\[
\sum_{0 \leq \gamma_m \leq T} x^{\gamma_m} \ll T^\frac{3}{2} + \epsilon \tag{1.13}
\]

for all \( x > 0 \). Clearly, this is not true for every \( x \).

By observing the bounds in (1.12) and (1.13), we have the following conjecture.

**Conjecture 1.4.** For all real numbers \( x, T \geq 2 \) and any \( \epsilon > 0 \),

\[
\sum_{0 \leq \gamma_m \leq T} x^{\gamma_m} \ll Tx^{-\frac{1}{2} + \epsilon} + T^\epsilon \tag{1.14}
\]

holds uniformly for \( 0 \leq m \leq \mathcal{L} \).
To obtain the bounds in (1.3) and (1.4), another important result needed is to obtain a non-trivial upper bound for
\[ \sum_{0 < y_0 \leq T} \left| \beta_0 - \frac{1}{2} \right|. \]

In 1924, Littlewood [25] proved that
\[ \sum_{0 < y_0 \leq T} \left| \beta_0 - \frac{1}{2} \right| \ll T \log \log T, \]
which was later improved by Selberg [33] in 1942. In particular, he obtained
\[ \int_{1/2}^{1} N_0(\sigma, T) \, d\sigma \ll T, \]
where \( N_0(\sigma, T) \) denotes the number of zeros \( \rho_0 \) of \( \zeta(s) \) such that \( \beta_0 > \sigma \) and \( 0 < y_0 < T \).

We need a similar result for \( \xi^{(m)}(s) \). For a fixed \( \sigma \), let \( N_m(\sigma, T) \) denote the number of zeros \( \rho_m = \beta_m + i\gamma_m \) of \( \xi^{(m)}(s) \), counted with multiplicity, such that \( \beta_m > \sigma \) and \( 0 < \gamma_m < T \). Our next result provides a zero density estimate for \( \xi^{(m)}(s) \).

**Theorem 1.5.** With \( N_m(\sigma, T) \) defined as above, we have
\[ \int_{1/2}^{1} N_m(\sigma, T) \, d\sigma \leq e^{O(m)} T \]
for \( 0 \leq m \leq L \) uniformly.

Since the prior works suggest that the zeros of \( \xi^{(m)}(s) \) migrate to the line \( \sigma = \frac{1}{2} \), we have the following conjecture.

**Conjecture 1.6.** Let
\[ C(m) := \limsup_{T \to \infty} \frac{1}{T} \int_{1/2}^{1} N_m(\sigma, T) \, d\sigma. \]
Then the function \( C(m) \) is a decreasing function of \( m \).

**Remark.** Note that for \( \sigma > \frac{1}{2} \),
\[ \int_{1/2}^{1} N_m(\sigma', T) \, d\sigma' \geq \left( \sigma - \frac{1}{2} \right) N_m(\sigma, T). \]
Therefore,
\[ N_m(\sigma, T) = O_m\left( \frac{T}{\sigma - \frac{1}{2}} \right) \]  \hspace{1cm} (1.14)
holds for \( \frac{1}{2} < \sigma \leq 1 \). Combining (1.6) and (1.14) we find that the zeros of \( \xi^{(m)}(s) \) are clustered near the line \( \sigma = \frac{1}{2} \).

### 2 Auxiliary lemmas

For a positive real number \( \theta \), and \( X = T^{\theta} \), define
\[ M_X(s) = \sum_{n \leq X} \mu(n) \frac{1}{n^{s+\theta} \log T} P\left( 1 - \frac{\log n}{\log X} \right), \] \hspace{1cm} (2.1)
where \( R \) is an absolute constant and \( P \) is a polynomial with \( P(0) = 0 \) and \( P(1) = 1 \). We have the following result from [6, p. 10].
Lemma 2.1. Let \( V(s) = Q( - \frac{1}{\log T} \frac{d}{ds} \zeta(s) ) \) for some polynomial \( Q \), and let \( M_X(s) \) be defined as in (2.1). For \( \theta < 4/7 \),
\[
\int_2^T |VM_X(\frac{1}{2} - \frac{R}{\log T} + it)|^2 \, dt \sim c(P, Q, R) T,
\]
where \( 0 < R \ll 1 \) and
\[
c(P, Q, R) = |P(1)Q(0)|^2 + \frac{1}{9} \int_0^1 \int_0^1 e^{2Ry} |Q(y)P'(x) + \theta Q'(y)P(x) + \theta RQ(y)P(x)|^2 \, dx \, dy.
\]
For fixed \( P \) and \( R \), one has
\[
c(P, Q, R) \ll \max_{0 \leq x \leq 1} |Q(x), Q'(x)|^2.
\]
As an application of the Faà di Bruno formula [13, p. 188], we obtain the following result.

Lemma 2.2. For any non-zero analytic function \( f \), we have
\[
\frac{f^{(n)}}{f}(s) = \sum_{\mu_i + 2\mu_2 + \cdots + k\mu_k = n} n! \prod_{i=1}^k \frac{1}{\mu_i!} (\frac{f}{f}(s))^{(i-1)} \mu_i.
\]
Let \( B_n \) denote the \( n \)th Bell number. Then we know
\[
B_n = \sum_{\mu_i + 2\mu_2 + \cdots + k\mu_k = n} n! \prod_{i=1}^k \frac{1}{\mu_i!} \mu_i \ll \left( \frac{n}{\log n} \right)^n.
\]
We also need the following result. This is a uniform version of [4, Lemma 1].

Lemma 2.3. Let \( H(s) \) be as defined in (1.2), and \( s = \sigma + it \). Then the following hold:
(i) For \( |t| \geq 1 \) and any \( \sigma \),
\[
\frac{H'(s)}{H(s)} = \frac{1}{2} \log \frac{s}{2\pi} + O\left( \frac{1}{|t|} \right)
\]
and
\[
\left( \frac{H'(s)}{H(s)} \right)^{(k)} \ll \frac{(k)!}{|t|^k}
\]
holds uniformly for \( k \leq \mathcal{L} \). If \( |t| \leq \sigma \), then we replace \( t \) by \( \sigma \) in the error terms.
(ii) For \( |t| > 10 \) and \( |\sigma| < A \log \log T \), where \( A \) is a constant,
\[
\frac{H^{(k)}(s)}{H(s)} = \left( \frac{H'(s)}{H(s)} \right)^k \left( 1 + O\left( \frac{B_k}{|t| \log |t|} \right) \right)
\]
holds uniformly for \( k \leq \mathcal{L} \). If \( |t| \leq \sigma \), then we replace \( t \) by \( \sigma \) in the error terms.

The above results follow from Lemma 2.2 and the following form of Stirling’s formula
\[
\log \Gamma(s) = \frac{1}{2} \log 2\pi + \left( s - \frac{1}{2} \right) \log s - s - \Omega(s),
\]
where \( \Omega(s) \ll 1/|t| \), and if \( |t| \leq \sigma \), then \( \Omega(s) \ll 1/\sigma \). (See [31, Section 21].)

We also need the following lemma from [18].

Lemma 2.4. For \( x, T \geq 1 \) and \( c = 1 + \frac{1}{\log 2x} \),
\[
\sum_{n=2}^{n \leq T} \frac{\Lambda(n)}{n^c} \min\left( T, \frac{1}{\log x/n} \right) \ll \log 2x \log \log 2x + \log x \min\left( \frac{T}{x}, \frac{1}{(x)} \right),
\]
where \( \langle x \rangle \) is the distance to the nearest integer prime power other than \( x \) itself.
Lemma 2.5. A sequence \((x_n)_{n \geq 1}\), is uniformly distributed modulo one if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0 \quad \text{for all integers } k \neq 0.
\]

The following inequality is due to Erdős and Turán [10].

Lemma 2.6. Let \(D_N\) denote the discrepancy of a sequence \((x_n)_{n \geq 1}\) of real numbers. Then, for any positive integer \(M\),

\[
D_N \leq \frac{C_1}{M + 1} + C_2 \sum_{k=1}^{M} \frac{1}{k} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \right|,
\]

where \(C_1\) and \(C_2\) are absolute positive constants.

The following lemma is due to Montgomery and Vaughan [27].

Lemma 2.7. If \(\sum_{n=1}^{\infty} n|a_n|^2\) converges, then

\[
\left| \sum_{n=1}^{T} a_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 (T + O(n))
\]

The following lemma from [34, p. 213] will be used to bound the argument of an analytic function.

Lemma 2.8. Let \(f(s)\) be an analytic function except for a pole at \(s = 1\), which is real for real \(s\). Let \(0 \leq a < b < 2\). Suppose that \(T\) is not an ordinate of any zero of \(f(s)\). Let \(|f(\sigma + it)| \leq M\) for \(\sigma \geq a\), \(1 \leq t \leq T + 2\) and \(\text{Re}(f(2 + it)) \geq c > 0\) for some \(c \in \mathbb{R}\). Then, for \(\sigma \geq b\),

\[
|\arg f(\sigma + iT)| \leq \frac{c}{\log \frac{2-a}{2-b}} \left( \log M + \log \frac{1}{c} \right) + \frac{3\pi}{2}.
\]

Let \(\Lambda_k\) denote the generalized von-Mangoldt defined by

\[
\Lambda_k(n) := \sum_{d|n} \mu(d) \log^k \frac{n}{d}.
\]

Therefore, for \(\text{Re}(s) > 1\),

\[
\sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s} = (-1)^k \sum_{l=0}^{k} \frac{\zeta(s)}{\zeta(s)}.
\]

(2.5)

Let \(\Lambda_k^l\) denote the \(l\)-fold convolutions of \(\Lambda_k\), i.e.,

\[
\Lambda_k^l = \Lambda_k \ast \cdots \ast \Lambda_k
\]

(2.6)

Then, we have the following inequality.

Lemma 2.9. With the notation from (2.5) and (2.6),

\[
(\Lambda_k \log * \Lambda_k^{l_1} \ast \cdots \ast \Lambda_k^{l_m})(n) \leq (\log n)^{1+k+l_1+\cdots+k_n+l_n}.
\]

Proof. From [20, p. 35], we have

\[
\Lambda_k(n) \leq \log^k n.
\]

Using the above inequality and (2.5), we find that

\[
(\Lambda_k \log * \Lambda_k)(n) = \sum_{ab=n} \Lambda_k(a) \log(a) \Lambda_k(b) \leq \Lambda_k(n) \log(n) (1 \ast \Lambda_k)(n) \leq \log^{k+2} n.
\]

By repeating this argument, we complete the proof of the lemma.
In [4], the details of the proof of (1.6) are omitted. Also, the error term in the asymptotic of $N_m(T)$ depends on $m$. We prove an asymptotic result for $N_m(T)$ where the dependence of $m$ is explicit.

**Lemma 2.10.** Let $N_m(T)$ denote the number of zeros of $\xi^{(m)}(s)$ with $0 < t < T$, and zeros are counted with multiplicity. Then

$$N_m(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T + m \log m)$$

uniformly for $m \leq \mathcal{L}$. The constant in the error term is absolute.

**Proof.** Applying Leibnitz’s rule in (1.1), we find that

$$\xi^{(m)}(s) = H^{(m)}(s)F_m(s), \quad \text{(2.7)}$$

where

$$F_m(s) := \zeta(s) + \sum_{j=1}^{m} \binom{m}{j} \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \xi^{(j)}(s). \quad \text{(2.8)}$$

Let $T$ be a large number. Then, for $\sigma \geq \mathcal{L}$ and $k \leq m$,

$$|\xi^{(k)}(s)| \geq \frac{\log^2 k}{2^\sigma} - \sum_{n=3}^{\infty} \frac{\log^k n}{n^\sigma} \geq \frac{\log^2 k}{2^{\sigma+1}}.$$ 

From (2.8), Lemma 2.3 and for $\sigma \geq \mathcal{L}$, we have

$$|F_m(s) - 1| \ll \frac{1}{2^\sigma T} (\log 6)^m \ll \frac{1}{2} \quad \text{(2.9)}$$

when $m \leq \mathcal{L}$.

Choose $T > 0$ so that the line $t = T$ is free of zeros of $\xi^{(m)}(s)$. Let $R$ be the rectangle, taken counterclockwise, with vertices $\mathcal{L} - iT, \mathcal{L} + iT, 1 - \mathcal{L} + iT, 1 - \mathcal{L} - iT$. By the argument principle, the functional equation (1.10) and relation (2.7),

$$4\pi N_m(T) = \text{Im} \left( \int_R \frac{d}{ds} \log \xi^{(m)}(s) \, ds \right)$$

$$= 2 \left( \int_C \frac{d}{ds} \log H^{(m)}(s) \, ds \right) + 2 \text{Im} \left( \int_{1/2-iT}^{\mathcal{L}-iT} + \int_{\mathcal{L}+iT}^{\mathcal{L}+iT} - \int_{1/2+iT}^{\mathcal{L}+iT} \frac{d}{ds} \log F_m(s) \, ds \right)$$

$$= 2 \text{Im}(I_1 + I_2 + I_3 - I_4), \quad \text{(2.10)}$$

where $C$ denotes the lines from $1/2 - iT$ to $\mathcal{L} - iT$, $\mathcal{L} - iT$ to $\mathcal{L} + iT$ and then $\mathcal{L} + iT$ to $1/2 + iT$. From (2.3), $B_m \ll \log T$ for $m \leq \mathcal{L}$. Combining this with (2.4), we have

$$I_1 = \text{Im}(\log H^{(m)}(s)|_C) = (\log H(s)|_C + m \log L(s)|_C) + O\left(\frac{1}{T}\right).$$

From [8, p. 98], we have

$$\text{Im}(\log H(s)|_C) = T \log \frac{T}{2\pi} - T + O(m).$$

From Lemma 2.3, one finds

$$\text{Im}(m \log L(s)|_C) = O(m)$$

for $m \leq \mathcal{L}$. Hence,

$$I_1 = T \log \frac{T}{2\pi} - T + O(m).$$

From (2.9), we have

$$I_3 = \text{Im}(\log F_m(s)|_{\mathcal{L}+iT}) = O(m).$$

Next, we compute $I_4$. Let

$$\tilde{F}(z) = \frac{1}{2} (F_m(z + iT) + \overline{F_m(z + iT)}).$$
Then for large $T$ the function $\tilde{F}(z)$ is analytic in the disk $|z - L| < 4L$. Let $n(x)$ denote the number of zeros of $\tilde{F}(z)$ in the disk $|z - L| < x$. Then, by Jensen’s theorem,

$$n(2L) \log 2 = n(2L) \int_{2L}^{4L} \frac{1}{t} dt \leq \int_{0}^{2L} \frac{n(t)}{t} dt \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |\tilde{F}(4Le^{i\theta} + 2L)| d\theta - \log|\tilde{F}(2L)|.$$  \hfill (2.11)

From (2.9), we have

$$|\tilde{F}(2L) - 1| = |\text{Re}(F_m(2L + iT)) - 1| \leq |F_m(2L + iT) - 1| < \frac{1}{2}. \hfill (2.12)$$

For $\text{Re}(s) > 0$, one has

$$\zeta(s) = s \int_{1}^{\infty} \frac{|x| - x + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{s-1} + \frac{1}{2}.$$  \hfill (3.1)

Therefore, for $\text{Re}(s) > 1/2$ and $|s - 1| > A$, we have

$$\zeta^{(k)}(s) \ll k 1^B \hfill (3.2)$$

for some $B > 0$. From Stirling’s formula, (2.8) and (2.13), we have

$$\tilde{F}(s) \ll (4m)^m |t|^B \hfill (3.3)$$

in the disk $|z - L| < 4L$. Equations (2.11), (2.12) and (2.14) give us

$$n(2L) \ll \log T + m \log m$$

for $m \leq L$. Since the number of zeros of $\text{Re}(F_m(s))$ on $(L + iT, 1/2 + iT)$ are bounded by $n(2L)$, we get

$$I_0 = \text{Im}(\log F_m(s))_{L+iT}^{1/2+iT} = O(\log T + m \log m).$$

Similarly, $I_2 = O(\log T + m \log m)$. By combining $I_1$, $I_2$, $I_3$, and $I_4$ in (2.10), we complete the proof of the lemma.

### 3 Proof of the explicit formula

Let $m \leq L$ be any positive integer and let $t > 10$. Then, by Lemma 2.3, one finds that $\frac{H^{(m)}(s)}{H(s)}$ is non-zero and $H(s)$ never vanishes. Therefore, $H^{(m)}(s)$ does not have any complex zero for $t > 10$. By (2.7), the complex zeros of $F_m(s)$ are the only zeros of $\zeta^{(m)}(s)$. The logarithmic derivative of (2.8) yields

$$\frac{F'_m(s)}{F_m(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{E_m(s)}{E_m(s)},$$  \hfill (3.4)

where

$$E_m(s) = 1 + \sum_{j=1}^{m} \frac{m}{j} \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \frac{\zeta^{(j)}(s)}{\zeta(s)}. \hfill (3.5)$$

Also,

$$\frac{d}{ds} E_m(s) = \sum_{j=1}^{m} \frac{m}{j} \left( \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \frac{\zeta^{(j)}(s)}{\zeta(s)} \right)'$$

$$= \sum_{j=1}^{m} \frac{m}{j} \left( \frac{\zeta^{(j)}(s)}{\zeta(s)} \right)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} + \sum_{j=1}^{m} \frac{m}{j} \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \frac{\zeta^{(j)}(s)}{\zeta(s)}$$

$$= E_{1,m}(s) + E_{2,m}(s).$$  \hfill (3.6)
Let \( c = 1 + \frac{1}{\log T} \) and \( T_\delta = \exp((\log T)^\delta) \) for any fixed \( \delta > 0 \). Consider the rectangle \( R \) defined by the vertices \( 1 - c + iT_\delta, c + iT_\delta, c + iT \) and \( 1 - c + iT \). It is known that the zeros of \( \xi^{(m)}(s) \) lie in the vertical strip \( 0 < \sigma < 1 \). Then, by the residue theorem and Lemma 2.10,

\[
\sum_{0 \leq y_n \leq T} x^{\theta_n} + O(xT_\delta \log T_\delta) = \frac{1}{2\pi i} \int_{c - iT}^{c + iT} \frac{F'_m(s)}{F_m(s)} x^s \, ds
\]

\[
= \frac{1}{2\pi i} \left( \int_{1 - c + iT}^{c - iT} + \int_{c + iT}^{1 - c + iT} \right) \frac{F'_m(s)}{F_m(s)} x^s \, ds
\]

\[= I_1 + I_2 + I_3 + I_4. \tag{3.4} \]

In the next three subsections, we will compute the integrals \( I_1 \) and \( I_3, I_2, \) and \( I_4 \), respectively.

### 3.1 Computation of the horizontal line integrals \( I_1 \) and \( I_3 \)

As an application of Lemma 2.3, we deduce that

\[
\frac{H^{(m-j)}}{H^{(m)}}(s) \ll \frac{2^j}{\log t} \tag{3.5}
\]

and

\[
\left( \frac{H^{(m-j)}}{H^{(m)}}(s) \right)' \ll \frac{2^{j+1}}{\log^{j+1} t} \tag{3.6}
\]

for \( t > T_\delta \) and \( m \leq L \). Let \( t \geq T_\delta \). Then, by (3.2) and (3.5), we have

\[
|E_m(2 + it) - 1| \ll \frac{m^2}{\log t} < \frac{L \log T_\delta}{2} < \frac{1}{2}. \tag{3.7}
\]

From (3.3), (3.5) and (3.6), we find

\[
|E'_m(2 + it)| \ll \frac{m^2}{\log t} < \frac{L \log T_\delta}{2} < \frac{1}{2}. \tag{3.8}
\]

Combining (3.1), (3.7) and (3.8), we find

\[
\frac{F'_m}{F_m}(2 + it) \ll 1 \tag{3.9}
\]

for \( t \geq T_\delta \). Since \( \xi^{(m)}(s) \) is an entire function of order 1, by Hadamard’s factorization theorem, one can rewrite it as

\[
\xi^{(m)}(s) = e^{A + B s} \prod_{\rho_m} \left( 1 - \frac{s}{\rho_m} \right) e^{-s/\rho_m},
\]

where the product runs over all the zeros of \( \xi^{(m)}(s) \), and \( A, B \) are certain constants. By logarithmic differentiation, (2.7), (3.9) and Lemma 2.3, we obtain

\[
\frac{F'_m}{F_m}(s) = \sum_{\rho_m} \left( \frac{1}{s - \rho_m} - \frac{1}{\rho_m} \right) + O(\log t) = \sum_{\rho_m} \left( \frac{1}{s - \rho_m} - \frac{1}{2 + it - \rho_m} \right) + O(\log t). \tag{3.10}
\]

Now, we consider the terms in the sum on the right side of (3.10) for which \( |y_m - t| \geq 1 \). From Lemma 2.10, we have

\[
N_m(t + 1) - N_m(t) \ll \log t. \tag{3.11}
\]

Using (3.11), we find that

\[
\sum_{n=1}^{\infty} \sum_{n \leq |2 - \sigma| < n + 1} \frac{2 - \sigma}{(s - \rho_m)(2 + it - \rho_m)} \ll \sum_{n=1}^{\infty} \sum_{n \leq |y_m - t| < \sqrt{n + 1}} \frac{1}{(y_m - t)^2}
\]

\[
\ll \sum_{n=1}^{\infty} \frac{\log(t + n)}{n^2} \ll \log t. \tag{3.12}
\]
By (3.11), we have
\[
\sum_{|y_m - T| < 1} \frac{1}{2 + it - \rho_m} \ll \log t. \tag{3.13}
\]
Invoking (3.12) and (3.13) in (3.10), we obtain
\[
\frac{F_m^t}{F_m}(s) = \sum_{|y_m - T| < 1} \frac{1}{s - \rho_m} + O(\log t) \tag{3.14}
\]
for \(t \geq T_\delta\). From (3.14), in (3.4) the integral \(I_3\) can be written as
\[
\sum_{|y_m - T| < 1} \frac{1 - c + iT}{c + iT} \int_{c+iT}^{c+i(T+1)} \frac{x^{\sigma}}{s - \rho_m} ds + O\left(\log 2T \int_{1-c}^{c} x^\sigma d\sigma\right) = \sum_{|y_m - T| < 1} I_{y_m} + O\left(x \log 2T \log 2x\right). \tag{3.15}
\]
In order to compute \(I_{y_m}\), we shift the line of integration from \(\text{Im } s = T\) to \(\text{Im } s = T + 1\). For \(|y_m - T| < 1\), by the residue theorem, we see that
\[
I_{y_m} = \left(\int_{c+iT}^{c+i(T+1)} + \int_{c+i(T+1)}^{c+iT} - \int_{1-c+iT}^{c+iT} \right) \frac{x^\sigma}{s - \rho_m} ds + O(1)
\ll 1 + \int_{1-c}^{c} \frac{x^\sigma}{\sqrt{(\sigma - \beta_m)^2 + (T + 1 - y_m)^2}} d\sigma + x \int_{T}^{T+1} \frac{dt}{\sqrt{(c - \beta_m)^2 + (t - y_m)^2}} + \frac{x^{1-c}}{\beta_m - 1 + c}
\ll x \log \log 2x. \tag{3.16}
\]
Note that the sum on the right side of (3.15) has \(\log(2T)\) terms. Therefore, the contribution from the top horizontal integral is
\[
I_3 = \frac{1}{2\pi i} \int_{c+iT}^{c+i(T+1)} \frac{F_m^t(s)}{F_m(s)} x^\sigma ds \ll x \log(2T) \log \log(2x). \tag{3.17}
\]
Computing similarly, as in (3.15) and (3.16), the contribution from the integral along the lower horizontal of the rectangle \(R\) in (3.4) is given by
\[
I_1 = \frac{1}{2\pi i} \int_{c+iT_\delta}^{c+iT} \frac{F_m^t(s)}{F_m(s)} x^\sigma ds \ll x \log(2T_\delta) \log \log(2x). \tag{3.18}
\]
### 3.2 Computation of the right vertical integral \(I_2\)

Next, we compute the integral on the right vertical line of the rectangle \(R\) in (3.4). From (3.1), one has
\[
I_2 = \int_{c+iT_\delta}^{c+iT} \frac{F_m^t(s)}{F_m(s)} x^\sigma ds = \frac{\zeta'}{\zeta} (\sigma + it) x^\sigma ds + \int_{c+iT_\delta}^{c+iT} \frac{E_{1,m}}{E_m}(s) x^\sigma ds + \int_{c+iT_\delta}^{c+iT} \frac{E_{2,m}}{E_m}(s) x^\sigma ds
= I_{1,2} + I_{2,2} + I_{3,2}.
\]

The integral \(I_{3,2}\). From [34, Section 6.19], we have the following bound for the Riemann zeta function:
\[
\frac{\zeta'}{\zeta} (\sigma + it) \ll \log^{\frac{1}{2}} t \log \log t,
\]
which holds uniformly on \(\sigma > 1 - A \log^{\frac{1}{2}} t \log^{-\frac{1}{2}} \log t\), where \(A\) is an absolute constant. Using the Cauchy integral formula, for any positive integer \(n\), we obtain
\[
\left(\frac{\zeta'}{\zeta} (\sigma + it)\right)^{(n)} \ll n! \log^{\frac{n}{2}} t \log \log t.
\]
which holds uniformly on \( \sigma > 1 - A \log^{-\frac{1}{4}} t \log^{\frac{1}{4}} \log t \). Hence, by Lemma 2.2,

\[
\frac{\zeta(n)}{\zeta}(\sigma + it) \ll B_n^2 \log^{\frac{2n}{\rho}} t \log^{\frac{n}{\rho}} \log t
\]

and

\[
\left( \frac{\zeta(n)}{\zeta}(\sigma + it) \right)' \ll B_n^2 \log^{\frac{2n+2}{\rho}} t \log^{\frac{n+1}{\rho}} \log t
\]

for \( \sigma > 1 - A \log^{-\frac{1}{4}} t \log^{\frac{1}{4}} \log t \). Combining (2.3), (3.19) and (3.5) with (3.2), we find that

\[
|E_m(\sigma + it) - 1| \ll B_m^2 \sum_{j=1}^m \left( \frac{m}{j} \right) \frac{\log^j t \log^j \log t}{\log^j t} \ll B_m^3 \log^{\frac{j}{\rho}} t \log^{\frac{j}{\rho}} \log t \ll \frac{1}{t \log^{\frac{j}{\rho}} t}
\]

for large \( t \geq T_\delta \) and uniformly for \( \sigma > 1 \) and \( m \leq \zeta \). Using (3.6), (3.19) and (3.21) in (3.3), we have

\[
E_{2,m}(\sigma + it) \ll B_m^2 \sum_{j=1}^m \left( \frac{m}{j} \right) \frac{\log^j t \log^j \log t}{t \log^{j+1} t} \ll B_m^3 \log^{\frac{j}{\rho}} t \log^{\frac{j}{\rho}} \log t \ll \frac{1}{t \log^{\frac{j}{\rho}} t}
\]

for \( t \geq T_\delta \) and uniformly for \( \sigma > 1 \). Therefore, integrating by parts, and using (3.21) and (3.22), one deduces that

\[
I_{1,2} \ll x.
\]

The integral \( I_{2,2} \). Rewrite \( I_{2,2} \) as

\[
I_{2,2} = \sum_{k=0}^{K-1} (-1)^k \int_{c+iT}^{c+iT+it} E_{1,m}(s)(E_m(s) - 1)^k x^s ds + \int_{c+iT}^{c+iT+it} E_{1,m}(s)(E_m(s) - 1)^K x^s ds =: I_1 + I_2.
\]

Firstly, we compute \( I_2 \). From (3.3), (3.5) and (3.20), we find that

\[
E_{1,m}(\sigma + it) \ll B_m^2 \sum_{j=1}^m \left( \frac{m}{j} \right) \frac{\log^j t \log^j \log t}{\log^j t} \ll B_m^3 \log^{\frac{j}{\rho}} t \log^{\frac{j}{\rho}} \log t.
\]

Hence, from (3.21), (3.25), and the definition of \( I_2 \) in (3.24), we have

\[
I_2 \ll xT\left( \frac{B_m^3 \log^{\frac{1}{\rho}} t \log T}{\log^{\frac{1}{\rho}} T} \right)^{K+1} \ll xT\left( \frac{\log^{\frac{1}{\rho}} t \log T}{\log^{\frac{1}{\rho}} T} \right)^{K+1},
\]

where the implied constant in the bound is absolute.

Now, we compute \( I_1 \). From (3.2) and (3.3), we have

\[
E_{1,m}(s)(E_m(s) - 1)^k = \sum_{j=1}^m \sum_{l_1, \ldots, l_m = 1} \left( \frac{m}{j} \right) \left( \frac{\zeta^{(j)}(s)}{\zeta(s)} \right)^l \left( \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \right) \left( \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \right)^{l_1}.
\]

Let \( L := l_1 + 2l_2 + \cdots + ml_m + j \) and \( \sigma > 1 \). Then, by Lemma 2.3 and (2.5), we find

\[
\left( \frac{\zeta^{(j)}(s)}{\zeta(s)} \right)^l \left( \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \right) \approx \frac{2^L}{\log^l(s/2m)} \left( \frac{\zeta^{(j)}(s)}{\zeta(s)} \right)^l \left( \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \right)^l \left( 1 + O\left( \frac{1}{t \log t} \right) \right).
\]

where

\[
b_L(n) = \left( \Lambda_j \log \Lambda_1^+ + \Lambda_2^+ + \cdots + \Lambda_m^+ \right) (n).
\]

From Lemma 2.9, we find that

\[
b_L(n) \leq \log^{L+1} n.
\]
Therefore, from (3.24), (3.27) and (3.28), we deduce that

\[
J_1 = \sum_{k=0}^{K-1} (-1)^k \sum_{j_1+j_2+\cdots+j_m=k} \binom{m}{j} k! \prod_{i=1}^{m} \frac{1}{i!} \int_{c+iT_0}^{c+iT} \frac{(-2)^L b_L(m)}{\log^2(s/2\pi)} \left( \frac{x}{n} \right)^s \left( 1 + O\left( \frac{1}{t \log t} \right) \right) ds. \tag{3.29}
\]

Let \( n' \) be the nearest integer to \( x \). Then,

\[
\int_{c+iT_0}^{c+iT} \frac{1}{\log^2(s/2\pi)} \left( \frac{x}{n'} \right)^s ds \ll \int_{\frac{1}{T}}^{T} \frac{1}{\log^2 t} dt \ll \frac{T}{\log^4 T}.
\]

If \( x \) is not an integer, then, by integrating by parts, we obtain

\[
\int_{c+iT_0}^{c+iT} \frac{1}{\log^2(s/2\pi)} \left( \frac{x}{n'} \right)^s ds \ll \frac{x^c}{(\log^L T) n^c \log(x/n)}.
\]

Therefore,

\[
\sum_{n=1}^{\infty} b_L(n) \int_{c+iT_0}^{c+iT} \frac{1}{\log^2(s/2\pi)} \left( \frac{x}{n} \right)^s ds \ll \frac{b_L(n') T}{\log^L T} + \sum_{n=1}^{\infty} \frac{x^c b_L(n)}{(\log^L T) n^c \log(x/n)}.
\tag{3.30}
\]

Also,

\[
\sum_{1 \leq n \leq n'/2} \frac{x^c b_L(n)}{(\log^L T) n^c \log(x/n)} + \sum_{n \geq n'/2} \frac{x^c b_L(n)}{(\log^L T) n^c \log(x/n)} \leq \frac{x}{(\log^L T)} \sum_{n=1}^{\infty} \frac{b_L(n)}{n^c}
\leq \frac{x (j+1)! \prod_{i=1}^{m} (ii)!}{(\log^L T) (c-1)^{L+1}} \ll x \log \left( \frac{\log \log T \log x}{\log T} \right)^L. \tag{3.31}
\]

In the penultimate step, we have used the fact that \( (il)/(\log \log T)^l \ll 1 \) for \( i \leq m \leq L \).

For the remaining terms in the sum on the right side of (3.30), we have

\[
\sum_{n'/2 < n \leq 2n'} \frac{x^c b_L(n)}{(\log^L T) n^c \log(x/n)} \ll \frac{\log^{L+1} x}{\log^L T} \sum_{n'/2 < n \leq 2n'} \frac{1}{\log(x/n)}. \tag{3.32}
\]

Since

\[ \log \frac{x}{n} \geq \log \frac{n'}{n} = -\log (1 - \frac{n' - n}{n'}) \geq \frac{|n - n'|}{n'}, \]
we have

\[
\sum_{n'/2 < n \leq 2n'} \frac{1}{\log(x/n)} \leq \sum_{n'/2 < n \leq 2n'} \frac{n'}{|n - n'|} \ll x \log 2x. \tag{3.33}
\]

For the error term in (3.29), we have

\[
\sum_{n=1}^{\infty} b_L(n) \int_{c+iT_0}^{c+iT} \frac{1}{\log^2(s/2\pi)} \left( \frac{x}{n} \right)^s \left( \frac{1}{t \log t} \right) ds \leq \frac{x}{(\log^L T)} \sum_{n=1}^{\infty} \frac{b_L(n)}{n^c} \ll x \log x \left( \frac{\log \log T \log x}{\log T} \right)^L. \tag{3.34}
\]

From the definition of \( L \), we have

\[
\sum_{k=0}^{K-1} \sum_{j_1+j_2+\cdots+j_m=k} \binom{m}{j} k! \prod_{i=1}^{m} \frac{1}{i!} \left( \frac{m}{i} \right)^{l_i} \left( \frac{Y}{L} \right) \leq (1 + Y)^{mK}. \tag{3.35}
\]
Substituting (3.30), (3.31), (3.32), (3.33) and (3.34) in (3.29) and then use (3.35), we deduce that
\[ I_1 \ll \sum_{k=0}^{K-1} \sum_{l_1+\ldots+l_m=k} \cdots \ll \frac{\log x}{\log T} \log x \cdot \left( \frac{\log T}{\log x} \right)^L. \]

Combining (3.26) and (3.36), we find that
\[ I_{2,2} \ll x \log x \left( \frac{\log T}{\log T} \right)^m + T \left( \frac{\log x}{\log T} \right)^{m+1}. \]

The integral $I_{1,2}$. Let $n_x$ be the nearest prime power to $x$. Then, by Lemma 2.4,
\[ I_{1,2} = \int_{c+iT \leq s \leq c+iT_\delta} F(s) x^s \, ds = -\int_{c+iT \leq s \leq c+iT_\delta} \sum_{n=2}^{\infty} \Lambda(n) \left( \frac{X}{n} \right)^s \, ds \]
\[ = -i\Lambda(n_x) \int_{T_\delta}^T \left( \frac{X}{n_x} \right)^s \, dt + O \left( x^c \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c \log(x/n_x)} \right) \]
\[ = -i\Lambda(n_x) \delta_{x,T} + O(x \log(2x) \log(2x)) + O(T \log x), \]
where
\[ \delta_{x,T} = \left( \frac{X}{n_x} \right)^s dt. \]

Clearly, $\delta_{x,T} \ll T$. If $x = n_x$, then
\[ \delta_{x,T} = T, \]
otherwise
\[ \delta_{x,T} = \left( \frac{X}{n_x} \right)^s \ll \left| \log \frac{x}{n_x} \right|^{-1}. \]

Notice that the first term on the right side of (3.38) disappears if $x$ is not an integer.

Finally, Combining (3.23), (3.37) and (3.38), we arrive at
\[ I_2 = \int_{c+iT \leq s \leq c+iT_\delta} F_m(s) x^s \, ds = -i\Lambda(n_x) \delta_{x,T} + O(x \log(2x) \log(2x) + T \log x) \]
\[ + O \left( T \left( \frac{\log x}{\log T} \right)^{m+1} + xT \left( \frac{\log x}{\log T} \right)^{K+1} \right) \]
\[ + O \left( x \log x (\log 2x) \left( \frac{\log T}{\log T} \right)^m \right). \]

### 3.3 Computation of the left vertical integral $I_4$

Now, we move on to estimate the integral along the left vertical side of the rectangle $\Re(s)$ in (3.4). From the functional equation (1.10), one can derive
\[ \frac{F_m(s)}{F_m'} = (-1)^{m+1} \frac{F_m'(s)}{F_m'(1-s)} + (-1)^m \frac{H^{(m+1)}(1-s)}{H^{(m)}} - \frac{H^{(m+1)}(s)}{H^{(m)}}. \]
Thus, for the integral along the left vertical line, we have
\[
I_a = \int_{1-c+iT}^{1-c+IT} \frac{F'_m(s)}{F_m(s)} x^s \, ds
\]
\[
= (-1)^{m+1} \int_{1-c+IT}^{1-c+IT} \frac{F'_m(1-s)}{F_m(1-s)} x^s \, ds + \int_{1-c+IT}^{1-c+IT} \left( (-1)^{m+1} \frac{H^{(m+1)}}{H^{(m)}} (1-s) - \frac{H^{(m+1)}}{H^{(m)}} (s) \right) x^s \, ds
\]
\[
=: I_{1,A} + I_{2,A}.
\]

Firstly, we estimate \( I_{2,A} \). Integrating by parts and employing Lemma 2.3, we find that
\[
I_{2,A} \ll \frac{\log 2T}{\log x}.
\]

Also, trivially, we have
\[
I_{2,A} \ll T \log 2T.
\]

Hence,
\[
I_{2,A} \ll \log 2T \min \left\{ T, \frac{1}{\log x} \right\}.
\]

Now, we estimate \( I_{1,A} \). We rewrite the integral \( I_{1,A} \) above as
\[
I_{1,A} = \int_{1-c+IT}^{1-c+IT} \sum_{\zeta} r(1-s)x^s \, ds + \sum_{k=0}^{K-1} \int_{1-c+IT}^{1-c+IT} E'_m(1-s)(E_m(1-s) - 1)^k x^s \, ds
\]
\[
+ \int_{1-c+IT}^{1-c+IT} E_{m1}(1-s)(E_m(1-s) - 1)^K x^s \, ds + \int_{1-c+IT}^{1-c+IT} E'_m(1-s) x^s \, ds
\]
\[
=: J_{a1} + J_{a2} + J_{a3} + J_{a4}.
\]

Now, we compute \( J_{a1} \) defined above as follows:
\[
J_{a1} = \int_{1-c+IT}^{1-c+IT} \sum_{\zeta} r(1-s)x^s \, ds = i x^{1-c} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c} \sum_{t=0}^{T} (\log(xn))^t \, dt
\]
\[
\ll (x^{1-c} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c \log(xn)}) \ll (x^{1-c} \frac{1}{c-1}) \ll \log x.
\]

Proceeding in a similar fashion as for \( J_1 \) earlier and using (3.29), we have
\[
J_{a2} = \sum_{k=0}^{K-1} (-1)^k \sum_{j=1}^{m} \sum_{1+1+1+\ldots+1=m} \binom{m}{j} \binom{m}{i} \binom{m}{i} \sum_{n=1}^{\infty} \frac{(-2)^j b_j(n)}{n \log(s/2\pi)} (nx)^{s} \left( 1 + O\left( \frac{1}{T \log t} \right) \right) ds,
\]
where
\[
\sum_{n=1}^{\infty} \frac{b_j(n)}{n} \frac{1}{1-c+IT} \sum_{1-c+IT}^{1-c+IT} (nx)^{s} \left( 1 + O\left( \frac{1}{T \log t} \right) \right) ds \ll \sum_{n=1}^{\infty} \frac{x^{1-c} b_j(n)}{\log(T\delta)} e^{c} \log(nx) \ll \log^{L+1} x.
\]

A similar computation as used for (3.36) gives
\[
J_{a2} \ll \log x (2 \log 2x)^{mK+1}.
\]

Proceeding similarly, as we did for \( J_2 \) and \( I_{3,2} \), we arrive at
\[
J_{a3} \ll T \left( \frac{\log \frac{1}{T}}{\log T} \right)^{K+1} \quad \text{and} \quad J_{a4} \ll 1.
\]
Thus, the contribution from the integral along the left vertical side of the rectangle $R$ in (3.4) becomes
\[
I_\lambda = \int_{1-c+iT}^{1+c+iT} \frac{F'_m(s)}{F_m(s)} x^s \, ds = O\left( \log x (2 \log 2x)^{nK+1} + T \left( \frac{\log^2 \log T}{\log \frac{1}{2} T} \right)^{K+1} \right) + O\left( \min\left( T \log 2T, \frac{\log 2T}{\log x} \right) \right). \tag{3.40}
\]

By using the estimates from (3.17), (3.18), (3.39), and (3.40) in (3.4), we complete the proof of Theorem 1.3.

4 Proof of the zero density estimates: Theorem 1.5

As discussed earlier in the previous section, since the complex zeros of $\xi^{(m)}(s)$ are identical to those of $F_m(s)$, we prove the theorem for $F_m(s)$ instead. Let
\[
f(s) := M_X(s) F_m(s) - 1, \tag{4.1}
\]
where $M_X$ is defined by (2.1). Consider
\[
h(s) := 1 - f^2(s). \tag{4.2}
\]
Here $h(s)$ is analytic except for the pole at $s = 1$. Let $P(x) = x$ in Lemma 2.1. Then, for $0 < \theta < 1$ and $X = T^\theta$, we have
\[
M_X(s) = \sum_{n \leq X} \frac{\mu(n)}{n^s} \left( 1 + O \left( \frac{\log n}{\log T} \right) \right) \left( 1 - \frac{\log n}{\log X} \right) = \sum_{n \leq X} \frac{\mu(n)}{n^s} \left( 1 + O \left( \frac{\log n}{\log T} \right) \right).
\]
Let $\sigma \geq 2$. Then, from (2.8), (3.5), (4.1), and (4.3),
\[
f(s) \ll \left| \int (s) \sum_{n \leq X} \frac{\mu(n)}{n^s} - 1 \right| + e^{O(m)} \frac{\log T}{\log T} \ll \int \frac{d(n)}{n^\sigma} + e^{O(m)} \frac{\log T}{\log T} \ll \frac{1}{\sqrt{X}} + e^{O(m)} \log T
\]
for $\frac{T}{2} \leq t < T$. Therefore, for some $X > X_0$, $T > T_0$, $m \leq \mathcal{L}$, and $\sigma \geq 2$,
\[
|f(s)| < \frac{1}{2}. \tag{4.3}
\]
Combining (4.2) and (4.3), we find that $h(2 + it) \neq 0$ for $t > T_0$ and $X \geq X_0$. Let $\nu(\sigma', T)$ denote the number of zeros of $h(s)$ in the rectangle $\sigma > \sigma'$ and $0 < t \leq T$. By the Hardy–Littlewood lemma (see [34, p.221]), one has
\[
2\pi \int_{\delta_0}^{\delta_0 + \frac{T}{2}} \nu(\sigma, T) \, d\sigma = \int_{T/2}^{T} \log |h(\sigma_0 + it)| \, dt - \int_{T/2}^{T} \log |h(2 + it)| \, dt
+ \int_{\delta_0}^{\frac{2}{T} + \frac{T}{2}} \log |h(\sigma_0 + iT)| \, d\sigma - \int_{\delta_0}^{\frac{2}{T} + \frac{T}{2}} \log |h(\sigma_0 + iT/2)| \, d\sigma, \tag{4.4}
\]
where $\nu(\sigma, T) = \nu(\sigma, T) - \nu(\sigma, \frac{T}{2})$ and $\sigma_0 \geq \frac{1}{2}$ is fixed.

From (4.2) and (4.3), we deduce that
\[\text{Re}(h(2 + it)) \geq \frac{1}{2}\]
for $t \geq T_0$ and $X \geq X_0$. From (2.13), $\xi^{(k)}(s) \ll k t T$ for some constant $B$. Then
\[
h(\sigma + it) \leq e^{C m \log m X A t A}
\]
for some constant $C$, $\sigma \geq 0$ and sufficiently large $t$. Therefore, from Lemma 2.8, we have
\[
\arg h(\sigma + iT) - \arg h\left( \sigma + \frac{T}{2} \right) \ll \log X + \log T + m \log m
\]
for \( \sigma \geq \sigma_0 \). This gives
\[
\int_{\sigma_0}^{2} \arg h(\sigma + iT) \, d\sigma - \int_{\sigma_0}^{2} \arg h\left(\sigma + i\frac{T}{2}\right) \, d\sigma \ll \log X + \log T + m \log m \ll \log T \tag{4.5}
\]
for \( 0 < \theta < 1, m \leq L \) and \( X = T^{\theta} \).

From (2.8), (4.3), and for \( Re \, s > 1 \),
\[
M_{X}(s)F_{m}(s) = \zeta(s)M_{X}(s) + \sum_{j=1}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) H^{(m-j)}(s) \zeta^{(j)}(s)M_{X}(s)
\]
\[
= \sum_{n=1}^{\infty} \frac{a_{X}(n)}{n^{s}} + \sum_{j=1}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) H^{(m-j)}(s) \sum_{n=2}^{\infty} \frac{b_{j,X}(n)}{n^{s}}, \tag{4.6}
\]
where \( a_{X}(1) = 1 \),
\[
a_{X}(n) = \sum_{d|n} \mu(d) \left( 1 + O\left( \frac{\log d}{\log T} \right) \right) \ll \begin{cases} \frac{b(n)}{\log T} & \text{if } 2 \leq n < X, \\ \frac{c(n)}{\log T} & \text{if } n \geq X, \end{cases}
\]
and
\[
b_{j,X}(n) = \sum_{d|n} \log \left( \frac{n}{d} \right) \mu(d) \left( 1 + O\left( \frac{\log d}{\log T} \right) \right) \ll \begin{cases} \Lambda_{j}(n) + \frac{c(n)}{\log T} & \text{if } 2 \leq n < X, \\ c_{1}(n) + \frac{c(n)}{\log T} & \text{if } n \geq X. \end{cases}
\]

Here, \( d(n) \) denotes the divisor function,
\[
b(n) = \sum_{d|n} \mu^{2}(d) \log d, \quad c_{1}(n) = \sum_{d|n} \log \left( \frac{n}{d} \right) \mu^{2}(d) \quad \text{and} \quad c(n) = \sum_{d|n} \log \left( \frac{n}{d} \right) \mu^{2}(d) \log d. \tag{4.9}
\]

Therefore, for \( Re \, s > 1 \),
\[
\sum_{n=1}^{\infty} \frac{b(n)}{n^{s}} = \zeta(s) \left( \frac{\zeta(s)}{\zeta(2s)} \right)^{'} , \quad \sum_{n=1}^{\infty} \frac{c_{1}(n)}{n^{s}} = \zeta^{(1)}(s) \frac{\zeta(s)}{\zeta(2s)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c(n)}{n^{s}} = \zeta^{(j)}(s) \frac{\zeta(s)}{\zeta(2s)} \tag{4.10}
\]

Since \( h(s) \) is analytic for \( \sigma \geq 2 \) and \( h(s) \to 1 \) as \( \sigma \to \infty \), by the residue theorem,
\[
\int_{T/2}^{T} \log h(2 + iT) \, dt = \int_{T/2}^{\infty} \log h\left(\sigma + i\frac{T}{2}\right) \, d\sigma - \int_{\sigma_{0}}^{\infty} \log h(\sigma + iT) \, d\sigma. \tag{4.10}
\]

One has
\[
\log|h(s)| \leq \log(1 + |f(s)|^{2}) \leq |f(s)|^{2} \leq |f(\sigma)|^{2} \tag{4.11}
\]
and
\[
\log |h(s)| = Re(\log h(s)).
\]

Using this along with (4.10) and (4.11), we have
\[
\int_{T/2}^{T} \log|h(2 + iT)| \, dt = Re\left( \int_{T/2}^{T} \log h(2 + iT) \, dt \right)
\]
\[
\ll \int_{2}^{\infty} \log|h\left(\sigma + i\frac{T}{2}\right)| \, d\sigma + \int_{2}^{\infty} \log|h(\sigma + iT)| \, d\sigma
\]
\[
\ll \int_{2}^{\infty} |f(\sigma)|^{2} \, d\sigma. \tag{4.12}
\]

From (4.1), (4.6), (4.7) and (4.8), we have
\[
(f(s))^{2} = \sum_{n=2}^{\infty} \frac{a(n)}{n^{s}}
\]
for $\text{Re}(s) > 1$ and $a(n) \ll \varepsilon^{O(m)}(d(n))^3 \log^{2m}(n)$. Thus, from (4.12),

$$\int_{\sqrt{T}/2}^{T} \log |h(2 + it)| \, dt = \varepsilon^{O(m)}.$$

Thus, it remains to estimate only the first integral in (4.4), which is done by using the convexity theorem. From (4.1), we find that

$$I_1 := \int_{\sqrt{T}/2}^{T} \left| f \left( \frac{1}{2} - \frac{R}{\log T} + it \right) \right|^2 dt \ll \int_{\sqrt{T}/2}^{T} \left| M_{X} F_m \left( \frac{1}{2} - \frac{R}{\log T} + it \right) \right|^2 dt + T.$$

From Lemma 2.3 and integrating by parts, we have

$$\int_{\sqrt{T}/2}^{T} \left| M_{X} F_m \left( \frac{1}{2} - \frac{R}{\log T} + it \right) \right|^2 dt \sim \int_{\sqrt{T}/2}^{T} \left| M_{X} V \left( \frac{1}{2} - \frac{R}{\log T} + it \right) \right|^2 dt,$$

where

$$V(s) = \zeta(s) + \sum_{k=1}^{m} \left( \frac{m}{k} \right) 2^{k} \log^{k} \zeta(s).$$

Let $Q(x) = (1 + 2x)^m$, $R \ll 1$ be fixed and $P(x) = x$. Then, by Lemma 2.1 and (2.2),

$$\int_{\sqrt{T}/2}^{T} \left| M_{X} V \left( \frac{1}{2} - \frac{R}{\log T} + it \right) \right|^2 dt \sim cT,$$

where

$$c \ll \left( \max_{|s| \leq 1} (Q(x), Q'(x)) \right)^2 \ll 3^m.$$

Next, we compute the integral

$$I_2 := \int_{\sqrt{T}/2}^{T} |f(2 + it) - 1|^2 dt = \int_{\sqrt{T}/2}^{T} \left| M_{X} F_m(2 + it) - 1 \right|^2 dt.$$

From (3.5) and (4.6),

$$I_2 \ll m \int_{\sqrt{T}/2}^{T} \sum_{n \gg 1} \frac{a_X(n)}{n^{2 + \frac{1}{2}}} \, dt + \sum_{j=1}^{m} \left( \frac{m}{j} \right)^2 \frac{j^{2i}}{\log^{2j} T} \int_{\sqrt{T}/2}^{T} \sum_{n \gg 1} \frac{b_{j,i}(n)}{n^{2 + \frac{1}{2}}} \, dt.$$

Employing Lemma 2.7, (4.7), (4.8), and (4.9), we have

$$I_2 = \varepsilon^{O(m)} \frac{T}{\log^2 T}.$$

From an easy modification of the classical convexity theorem (see [34, p. 233]), one can deduce that

$$\int_{\sqrt{T}/2}^{T} |f(\sigma_0 + it)|^2 dt = \varepsilon^{O(m)} T \log^{1 - 2m} T$$

uniformly for $\frac{1}{2} - \frac{R}{\log T} \leq \sigma_0 \leq 2$. From (4.11) and (4.13), we find that

$$\int_{\sqrt{T}/2}^{T} \log \left| h(\sigma_0 + it) \right| \, dt = \varepsilon^{O(m)} T \log^{1 - 2m} T.$$
Combining (4.4), (4.5), (4.12), (4.14), and the inequality
\[
\int_{\sigma_0}^{\sigma} \psi(\sigma, \frac{T}{2}, \sigma) \, d\sigma \geq \int_{\sigma_0}^{\sigma} N_m(\sigma, T) \, d\sigma - \int_{\sigma_0}^{\sigma} N_m(\sigma, \frac{T}{2}) \, d\sigma,
\]
which follows from (4.2), we obtain
\[
\int_{\sigma_0}^{\sigma} N_m(\sigma, T) \, d\sigma - \int_{\sigma_0}^{\sigma} N_m(\sigma, \frac{T}{2}) \, d\sigma = e^{O(m)} T \log \frac{1 - \delta}{\delta},
\]
uniformly for \(\frac{1}{2} \leq \sigma_0 \leq 1\). Now, we replace \(T\) by \(T/2^n\), \(n \geq 0\), in the above estimate, and sum over \(n\) for \(0 \leq n \leq \infty\) to complete the proof of Theorem 1.5.

5 Uniform distribution and Discrepancy Bounds: Proofs of Theorems 1.1 and 1.2

5.1 Proof of Theorem 1.1

We start with the identity
\[
\sum_{0 \leq y_n \leq T} x^{y_n} = \sum_{0 \leq y_n \leq T} x^{\beta_n - 1/2} + \sum_{0 \leq y_n \leq T} (x^{y_n} - x^{\beta_n - 1/2}),
\]
which holds for any \(x\). Let \(x = e^{2\pi n}\), where \(\alpha > 0\) is any fixed real number. From (1.10), it can be shown that the non-trivial zeros of \(\xi^m(s)\) are symmetric with respect to the line \(\sigma = 1/2\). Therefore,
\[
\sum_{0 \leq y_n \leq T} (x^{y_n} - x^{\beta_n - 1/2}) \ll \sum_{0 \leq y_n \leq T} \left|1 - x^{\beta_n - 1/2}\right| \ll \sqrt{x} \log x \sum_{0 \leq y_n \leq T} \sum_{\beta_n > 1/2} (\beta_m - 1/2) = \sqrt{x} \log x \int_1^{1/2} N_m(\sigma, T) \, d\sigma,
\]
where in the penultimate step, we use the mean value theorem. Combining this with Theorem 1.5, we find that
\[
\sum_{0 \leq y_n \leq T} (x^{y_n} - x^{\beta_n - 1/2}) \ll e^{O(m)} \sqrt{x} T \log x.
\]
Let \(T\) be large enough such that
\[
\log 2x \leq (\log T)^{1-\epsilon}.
\]
for some \(\epsilon > 0\). For \(x > 1\), from Theorem 1.3, we have
\[
\sum_{0 \leq y_n \leq T} x^{\beta_n - 1/2} \ll \frac{T \log x}{\sqrt{x}} + \sqrt{x} \log (2xT) \log x + \frac{T}{\log x} \frac{(2\log 2x)^{mK+1}}{(\log T)^{mk}}
\]
\[
+ \sqrt{x} \log x(T e^{O(1)} + \log T) \log T
\]
\[
+ \sqrt{x} \log x(2 \log 2x)^{mK+1} \left(\frac{\log T}{\delta(\log T)^{\delta}}\right)^{mK}.
\]
Hence, from Lemma 2.10 and estimates (5.1), (5.2) and (5.3), we have
\[
\frac{1}{N_m(T)} \sum_{0 \leq y_n \leq T} x^{y_n} = o(1)
\]
as \(T \to \infty\) and uniformly for \(m \leq \mathcal{L}\). A similar result also holds for \(0 < x < 1\). In this case, we first use (1.11) on the left side of (5.3), and then apply Theorem 1.3.

Invoking the Weyl criterion, Lemma 2.5, we conclude that the sequence \((a y_m)\) is uniformly distributed modulo one. This completes the proof of Theorem 1.1.
5.2 Proof of Theorem 1.2

Case (i). Assuming the Riemann hypothesis. Let \( \rho_m = \frac{1}{2} + i\gamma_m \). Also, note that \( \Lambda(n_x)\delta_x, T \ll T \log x \). Then, from Theorem 1.3 and Lemma 2.6, we have

\[
D_m^*(\alpha; T) \ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \left| \sum_{0 \leq y \leq T} x^{iky_m} \right|
\]

\[
= \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \left| \sum_{0 \leq y \leq T} (x^k)^{\rho_m - 1/2} \right|
\]

\[
\ll \frac{1}{M+1} + \frac{1}{T \log T} \sum_{k=1}^{M} \left\{ T \log x \frac{x^{k/2}}{k} + x^{k/2} k \log(2x) \log x \right\}
\]

\[
+ \frac{T}{x^{k/2}} \frac{(2k \log 2x)^{mk+1}}{k(\log T)^{mk+1}} + x^{k/2} \log x (2k \log 2x)^{mk+1} \left( \frac{\log \log T}{\delta(\log T)^{\delta}} \right)^{mk}
\]

\[
+ \frac{x^{k/2}}{k} \left( \log T \right)^{\delta} e^{(\log T)^{\delta}} + x^{k/2} \left( \log T \right)^{\frac{\delta + 1}{\log T}} \right\}.
\]

(5.4)

Now, we set

\[
\delta = 1 - \frac{\log 10}{\log \log T}, \quad M + 1 = \left\lfloor \frac{\log T}{2 \log 2x} \right\rfloor \quad \text{and} \quad K = \left\lfloor \frac{4 \log T}{\log \log T} \right\rfloor.
\]

Therefore, we obtain

\[
D_m^*(\alpha; T) \leq \frac{c_1}{T} \log x + \exp \left( \frac{c_2 \log T}{\log \log T} \right) \log x
\]

where \( c_1 \) and \( c_2 \) are absolute constants, and this holds uniformly for \( 0 \leq m \leq L \). This completes the proof of Theorem 1.2.

Case (ii). Unconditional bound. From (5.1), we have

\[
D_m^*(\alpha; T) \ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \left| \sum_{0 \leq y \leq T} x^{iky_m} \right|
\]

\[
\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \left( \sum_{0 \leq y \leq T} (x^k)^{\rho_m - 1/2} \right)
\]

\[
\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \left( \sum_{0 \leq y \leq T} (x^k)^{\rho_m - 1/2} \right)
\]

(5.5)

From (5.4) and (5.2), we have

\[
D_m^*(\alpha; T) \ll \frac{1}{M+1} + \frac{1}{T \log T} \left\{ T \log x \frac{x^{k/2}}{k} + x^{k/2} k \log(2x) \log x \right\}
\]

\[
+ \frac{T}{x^{k/2}} \frac{(2k \log 2x)^{mk+1}}{k(\log T)^{mk+1}} + x^{k/2} \log x (2k \log 2x)^{mk+1} \left( \frac{\log \log T}{\delta(\log T)^{\delta}} \right)^{mk}
\]

\[
+ \frac{x^{k/2}}{k} \left( \log T \right)^{\delta} e^{(\log T)^{\delta}} + x^{k/2} \left( \log T \right)^{\frac{\delta + 1}{\log T}} \right\}.
\]

Now, we set

\[
M + 1 = \left\lfloor \frac{\log T}{2 \log 2x} \right\rfloor.
\]

Hence, we deduce that

\[
D_m^*(\alpha; T) \leq \frac{a_1}{T} \log x + \frac{e^{a_2 \log x}}{\sqrt{\log T}},
\]

where \( a_1 \) and \( a_2 \) are absolute constants, and this holds uniformly for \( 0 \leq m \leq L \).
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References


