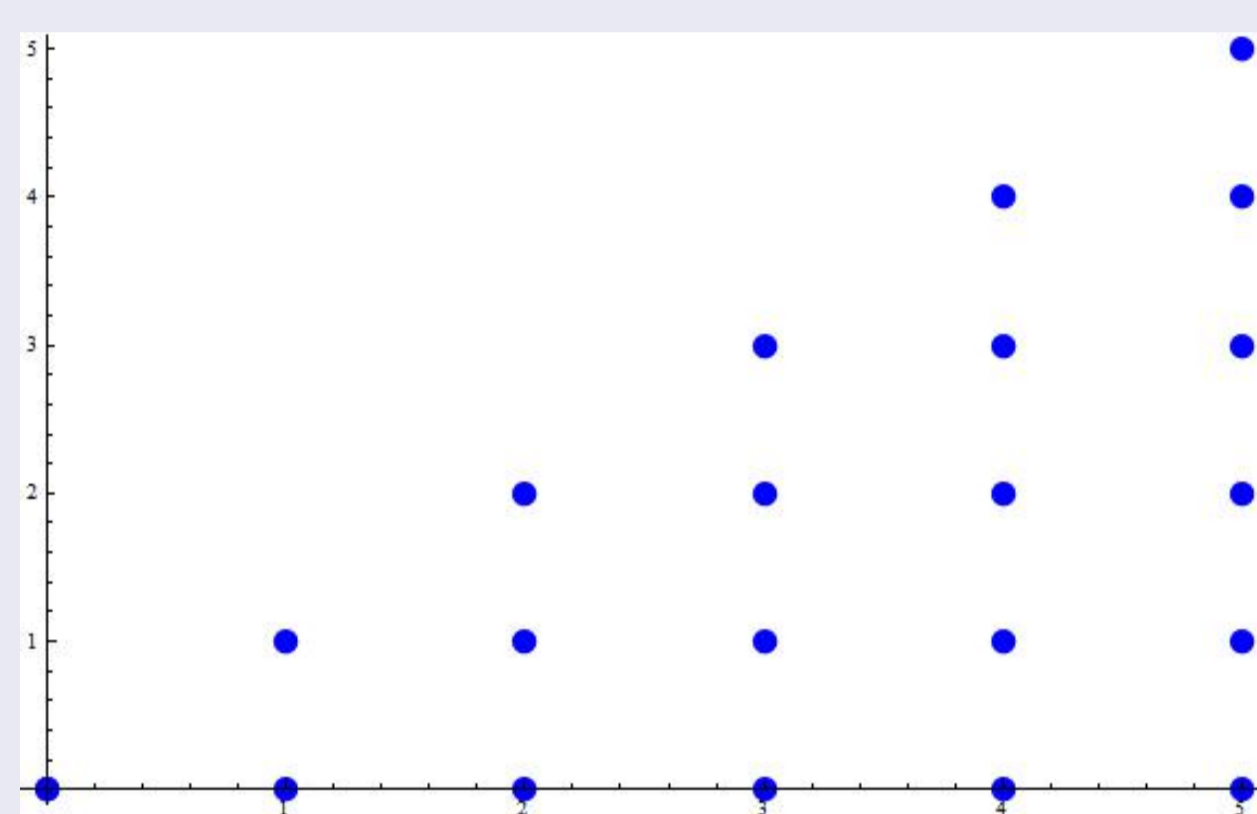# Visibility in Random Forests

Abigail Turner, Ananya Uppal, Suhan Liu, Amita Malik, Prof. Jayadev Athreya, Prof. Francesco Cellarosi

ILLINOIS — UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Illinois Geometry Lab

Overview

Imagine a tree at every point in the plane with integer coordinates while you stand at the origin.

Only a fraction of the trees in the infinite forest will be visible to you. Surprisingly enough, this fraction is $\frac{6}{\pi^2}$.

Which Trees Are Visible?

- Looking at the above picture, it is evident that we want to say one tree is "behind" another one (i.e., not visible) if there exists another tree that is closer to the origin but has the same slope.
- So, the only trees that are visible to us are the ones that have integer coordinates that are co-prime.
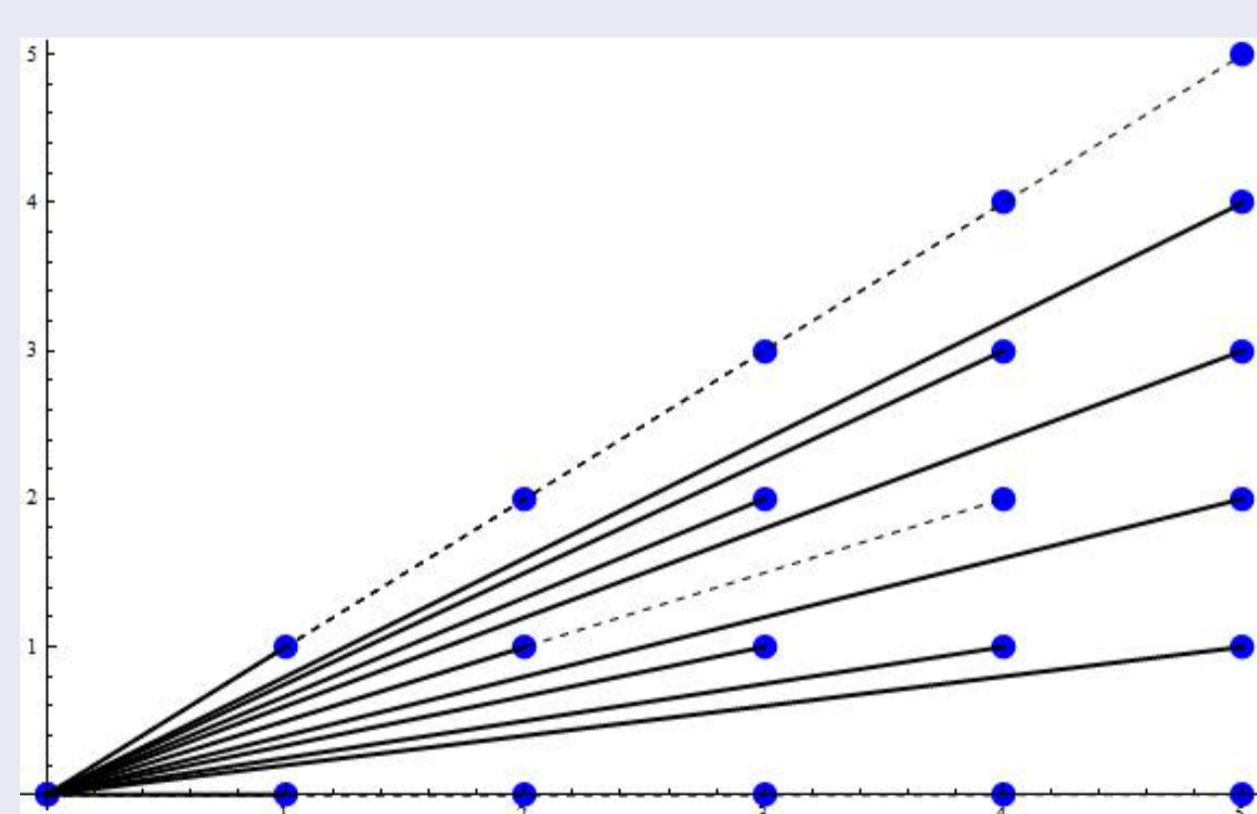
Given that these coordinates are coprime, we can identify each visible tree at coordinate (a, b) with its slope, $\frac{b}{a}$. The set of all the slopes of visible trees form the set of Farey fractions.

Farey Fractions

The Farey sequence of order n is the set F_n of fractions $\gamma = \frac{a}{q}$ with $1 \le q \le n$ in the interval $(0, 1]$, written in lowest terms and arranged in increasing order of magnitude. The elements of this sequence are the Farey fractions.

Considering the Gaps

Assume that only trees within a certain distance of the integer forest are visible to us, i.e. consider only a finite section of this infinite forest. Then,

- We want to look at the distribution of the angular gaps between these trees, i.e. the difference between their slopes.
- By symmetry, we need only consider the trees with slopes less than or equal to 1.
- Our project aims to generalize this problem to study the gap distribution of a randomly chosen subset of trees.
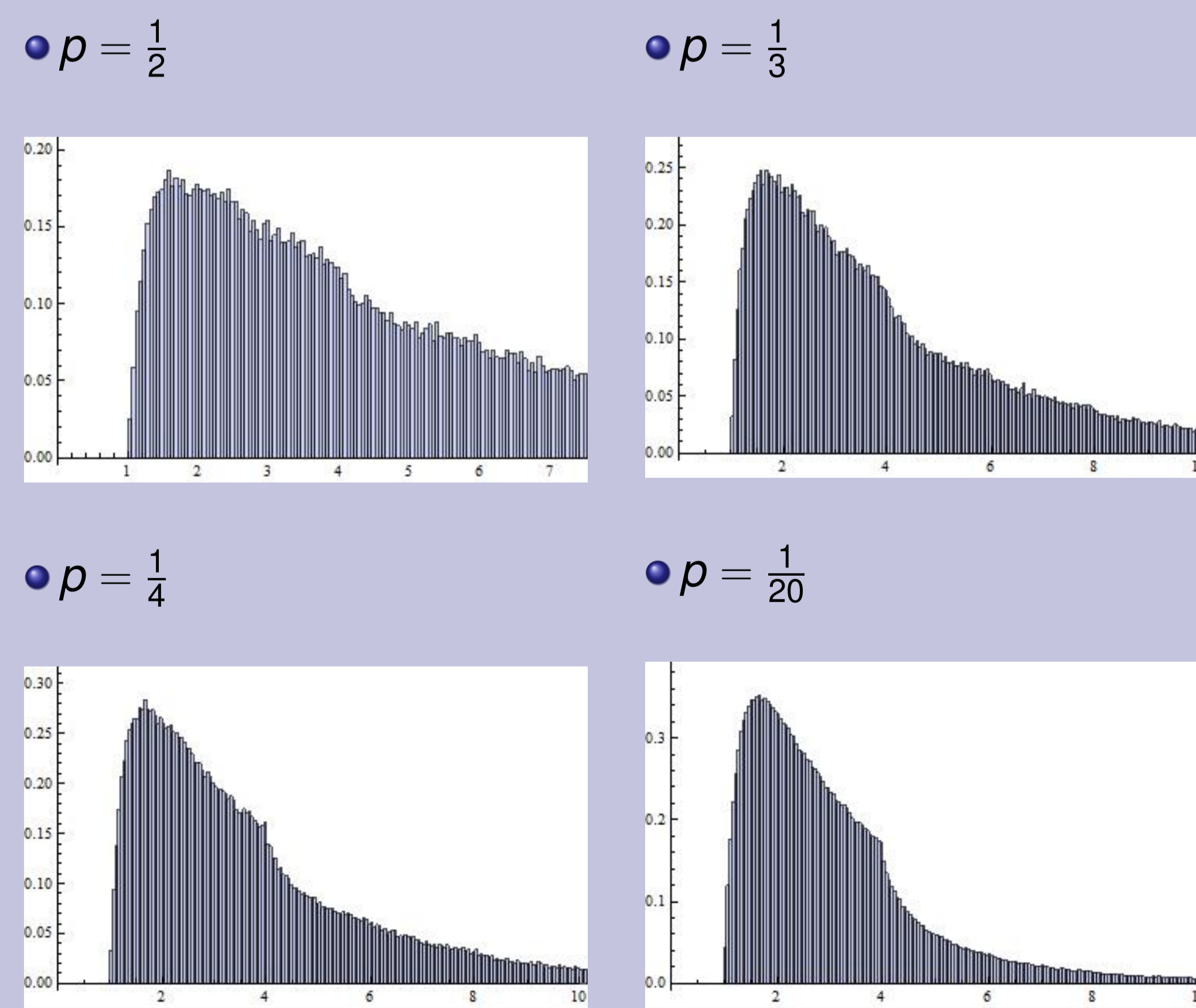- There are two cases as follows..

Visible Forest

- In the first case we consider removing trees from only the set of visible trees. This is equivalent to considering a subset of the Farey fractions.
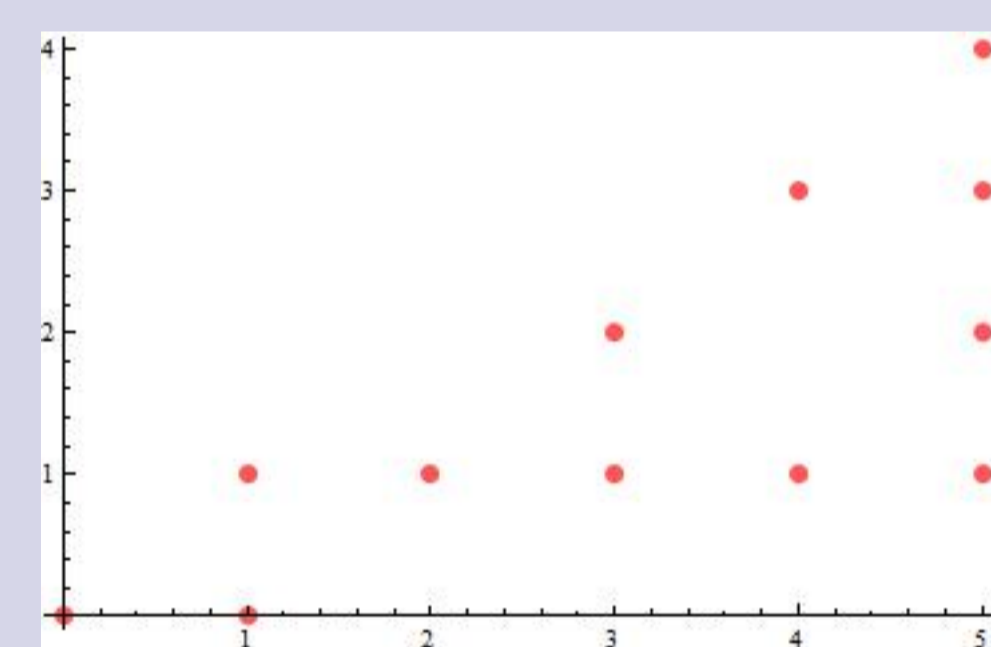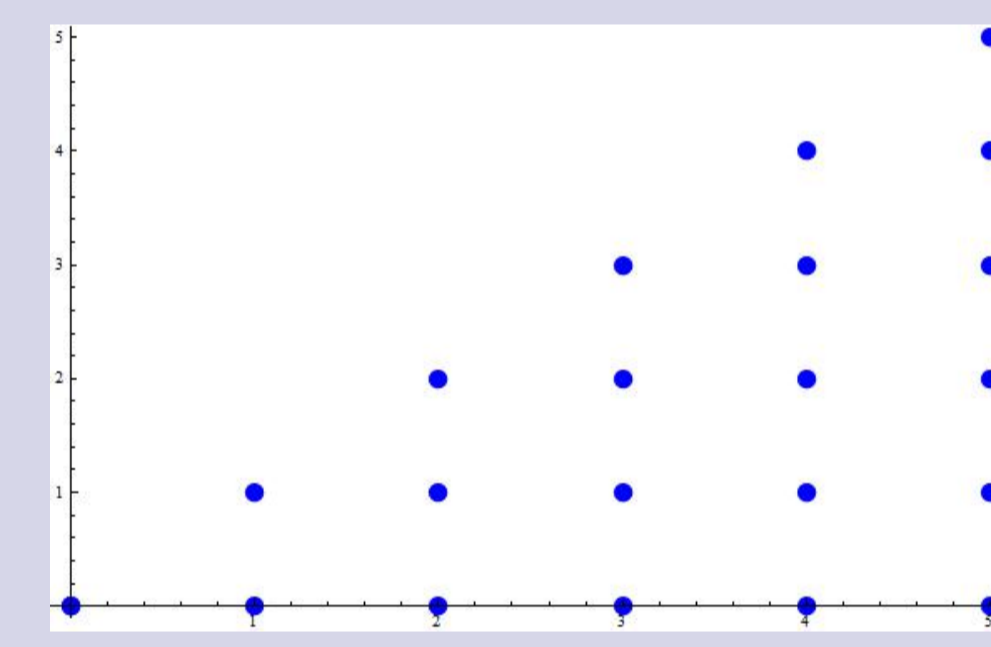

- The following graphs show the distribution of gaps after removing each tree in the visible forest with probability p.

Gap Distributions

- $p = \frac{1}{2}$
- $p = \frac{1}{3}$

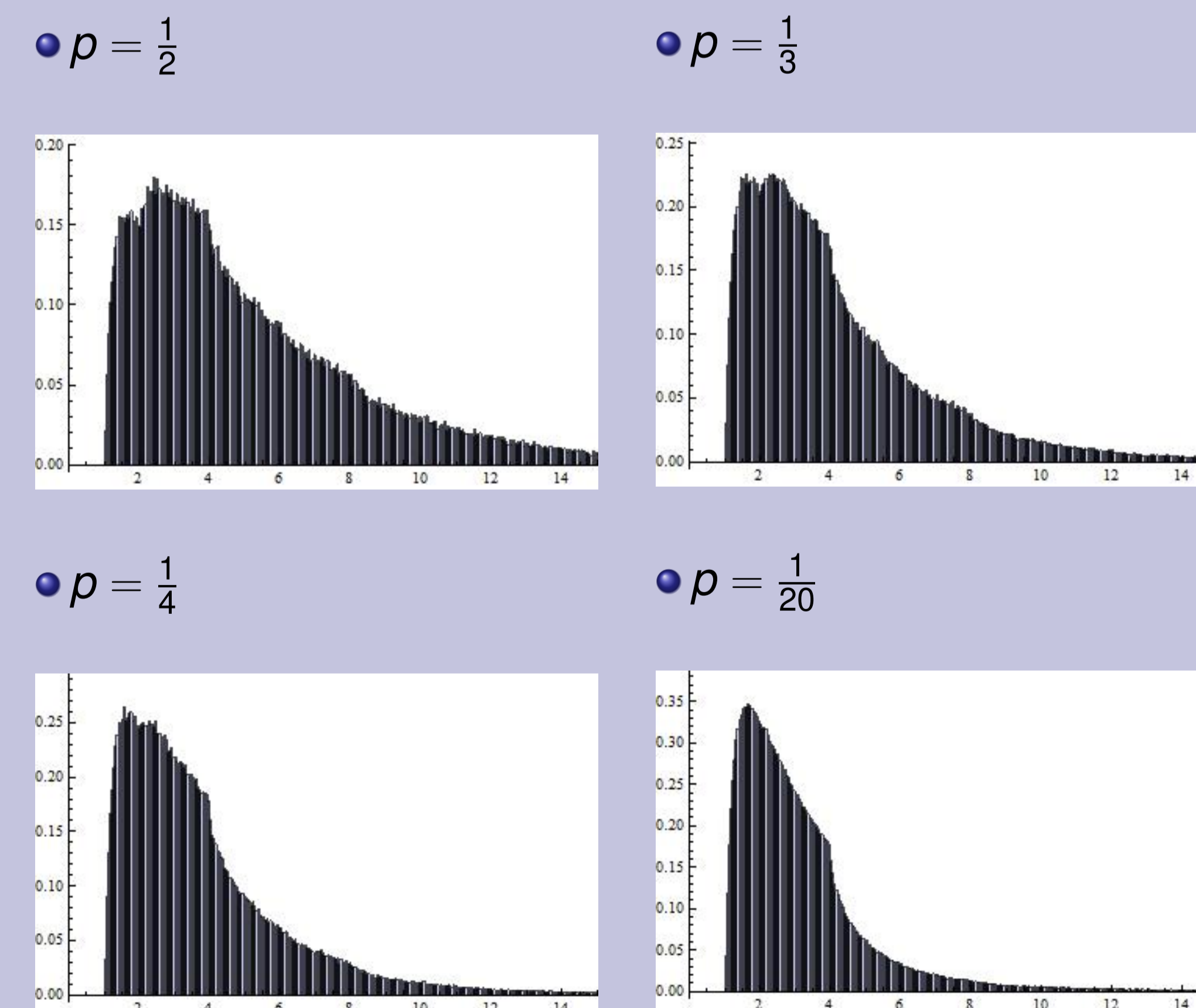- $p = \frac{1}{4}$
- $p = \frac{1}{20}$

Whole Forest

- In the second case we consider removing trees from the whole forest (i.e. even trees that are not visible to us) and consider the gaps between the visible trees.
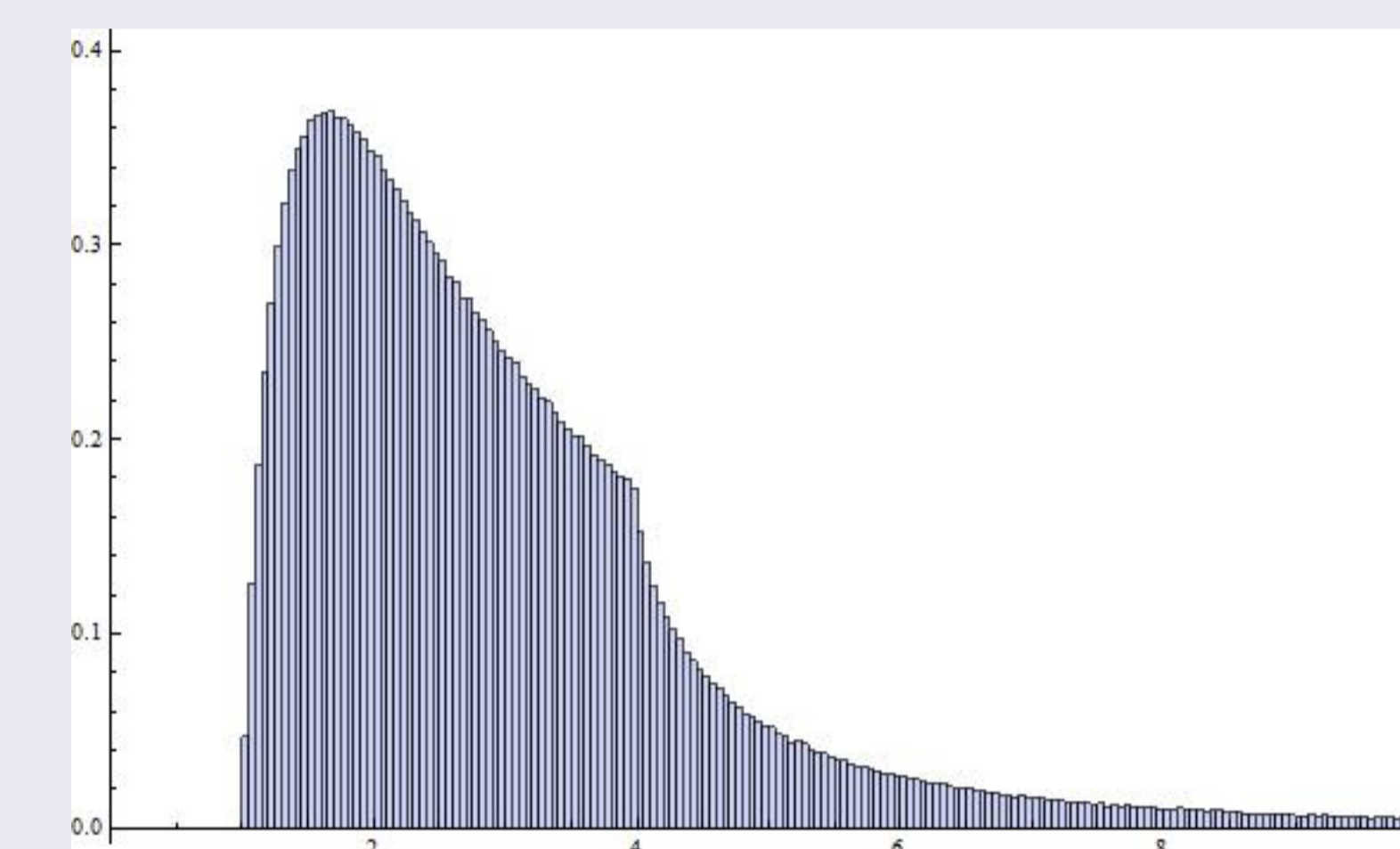

- The following graphs show the distribution of gaps after removing each tree with probability p in the entire forest.

Gap Distributions

- $p = \frac{1}{2}$
- $p = \frac{1}{3}$

- $p = \frac{1}{4}$
- $p = \frac{1}{20}$

An Important Distribution

The Hall's distribution, shown below, describes the distribution of gaps between the visible trees i.e. the differences between the Farey fractions.

In both cases to the left, notice that this means that as $p \to 0$ (i.e. you kill fewer and fewer trees) the distribution of the gaps between visible trees tends to the Hall's distribution above.

Future Directions

- In the future, one could try to relate the distributions we have observed to other known distributions.
- Another important direction could be to look for some underlying theoretical relationships that connect these different gap distributions.

References

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