Geometry of the Apollonian Circle Packings

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Introduction

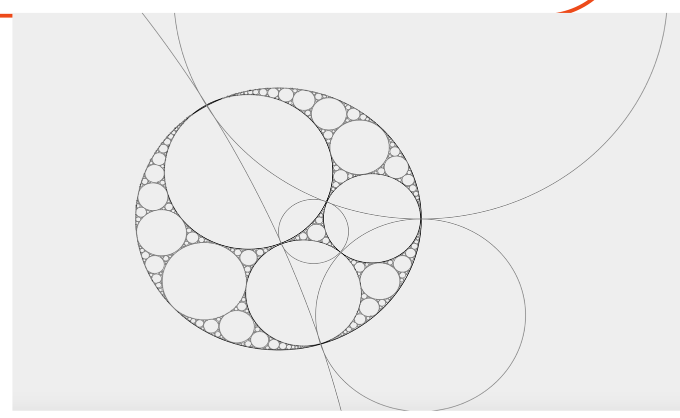
The fine structures of Apollonian gaskets are encoded by local spatial statistics. In this article, we report our empirical results on some of such statistics, namely, pair correlation, nearest neighbor spacing and electrostatic energy. We find that all these statistics, after proper normalization, exhibit some limiting behavior when the growing parameter T approaches infinity. We will make conjectures based on these numerical results.

These spatial statistics have been widely used in disciplines such as physics and biology. For instance, in microscopic physics, the Kirkwood-Buff Solution Theory links the pair cor- relation function of gas molecules, which encodes the microscopic details of the distribution of these molecules, to macroscopic thermodynamical properties of the gas such as pressure and potential energy. On macro level, cosmologists also use pair correlation to study the distribution of stars and galaxies.

Project Goals

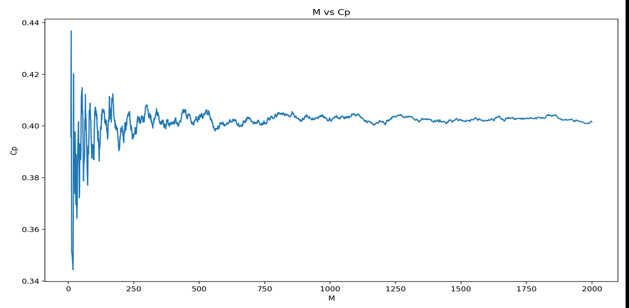
- Realize a continuous variation of Apollonian Gaskets on computer.
- Collect the information of centers and radius from circles. Study the distribution of centers.
- If time permits, we will also investigate the distribution of circles from other circle packings

Definition. *An Apollonian gasket, named after an ancient Greek mathematics Apollonius of Perga (200 BC), is a fractal set obtained by starting from three mutually tangent circles and iteratively inscribing new circles to curvilinear triangular gaps.*



Background

Theorem 1. [Oh-Shah]For any Apollonian gasket $\mathcal{P}$ placed in $\mathbb{R}^2$, there exists a finite Borel measure μ, such that for any region $\mathcal{R} \subset \mathbb{R}^2$ with piecewise analytic boundary, the cardinality of $\mathcal{P}_T(\mathcal{R})$, the set of circles from $\mathcal{P}_T$ lying in $\mathcal{R}$, satisfies



$$\lim_{T\to\infty} \frac{\mathcal{P}_T(\mathcal{R})}{T^\delta} = \mu(\mathcal{R})$$

Theorem above gives an satisfactory explanation on how circles are distributed in large scale. However, it provides little infomation on how circles distributed on a smaller scale. We can ask the question like

"If one sits at a center of a random circle from P, how many circles can one see within a distance of $\frac{10}{T}$?"

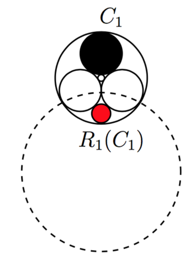
Quesion like this motivate us to do research in this area. With proof of **Conjecture 2** in the following report, we will be able to give an approximate answer to question above when T is large.

Methods and Results

Method:

First, we need to construct model for Apollonian Circle Packings.

- We are able to use Mobius Transformation to map the circles tocircles that will be easier to determine centers and radius.
- With inversion we are able to generate even more tangentcircles.



- We applied tree structure algorithm to the inversion of theoriginal circle and three generated circles. Our program willgenerate the data of radius and centers of all circles with radiuslarger than $\frac{1}{M}$.

Results:

Pair correlation function $F_T(s)$ for the set C_T is definited to be

$$F_T(s) = \frac{1}{2\#C_T} \sum_{p,q\in C_T; p\neq q} 1\{\frac{d(x,y)}{T} < s\}$$

where d(·, ·) is the Euclidean distance function, and 1 {A} is the indicator function which obtains the value 1 if the condition A is true and 0 otherwise.

Conjecture 1 *There exists a positive, monotone, continuously differentiable function F on [0,∞), which is supported away from 0, such that*

$$\lim_{T\to\infty} F_{T,R}(s) = F(s)$$

where $R \subset \mathbb{R}^2$ with $\mu(R) > 0$, and $F_{T,R}$ is the pair correlation function for different T in a certain area R. **Moreover, F is independent of the Apollonian circle packing P.**

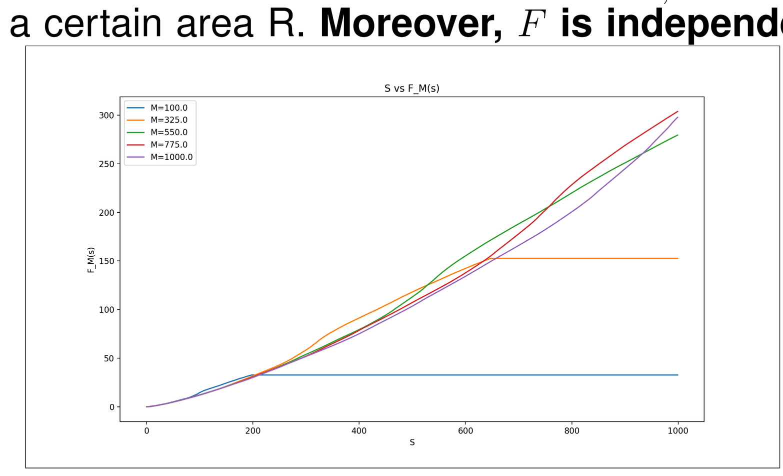


Figure 1: *Pair Correlation Graph*

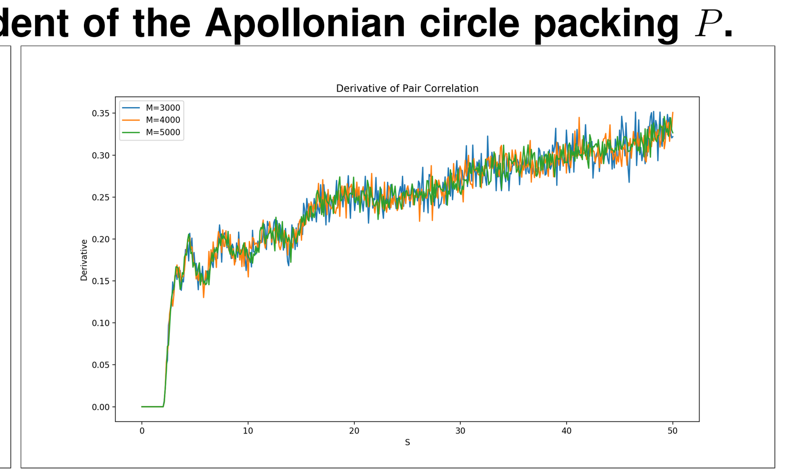


Figure 1: *Derivative Graph*

The electrostatic energy function $G(T)$ is defined by

$$G(T) = \frac{1}{T^{2\delta}} \sum_{p,q\in C_T; p\neq q} \frac{1}{d(p,q)}$$

Ourexperiment suggests that $G(T)$ converges to some positive constant when T get large.

Conjecture 2. *There exists a constant $b > 0$, such that*

$$\lim_{T\to\infty} G(T) = b$$

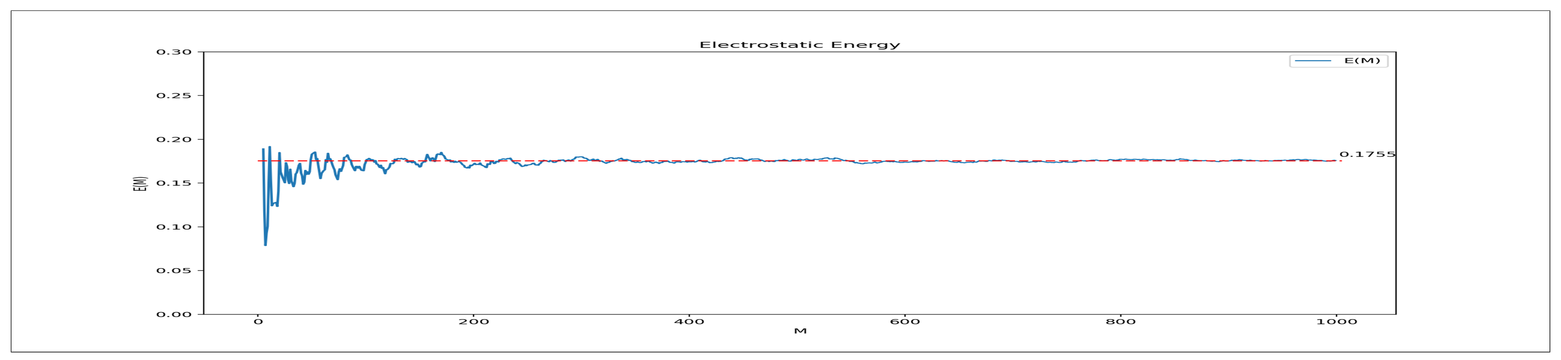


Figure 2: *The electrostatic energy*

Nearest spacing. For the set C_T and a point $x \in C_T$, we let $g_T(x)$to be the distance between x and a closest point in C_T to x. The nearest spacing function $H_T(s)$for the set C_T is then defined by the following:

$$H_T(s) = \frac{1}{\#C_T} \sum_{x\in C_T} 1\{\frac{g_T(x)}{T} < s\}$$

Conjecture 3. *There exists a positive, monotone, continuous function H on [0,∞), which is supported away from 0, such that*

$$\lim_{T\to\infty} H_T(s) = H(s)$$

for any $s \in [0, \infty)$.

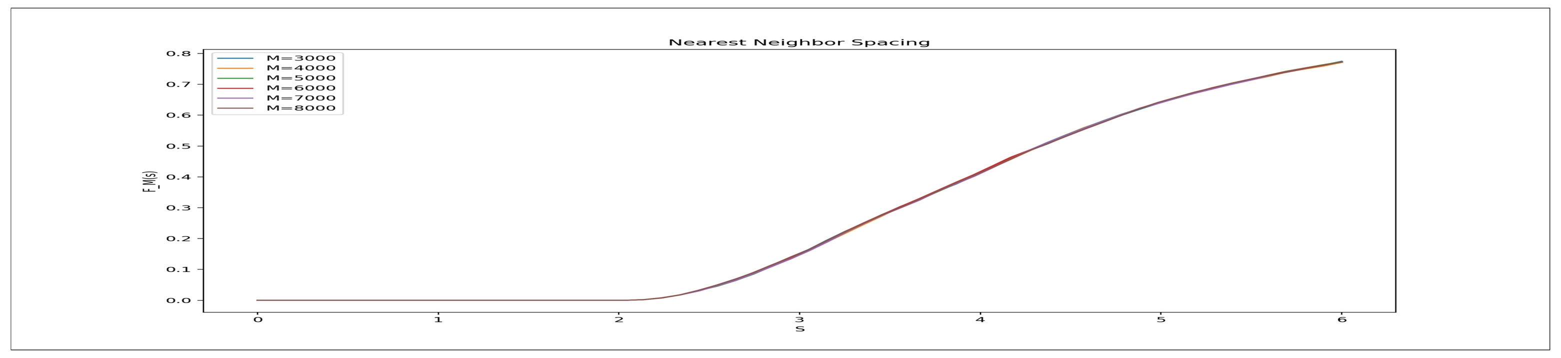


Figure 3: *Nearest spacing*

Future Directions

Further research of this topic could be carried out in numerous areas in the physical sciences and Mathematics:

- Analysis of different groups of Circle Packings (e.g Bianchi groups, Schmidt arrangements);
- Modeling the distribution of stars in Spiral Galaxies using fractal groups: How are the stars in the Spiral Galaxies distributed?
- Applications to various forms of fine-scale structures: Biology and Physics.

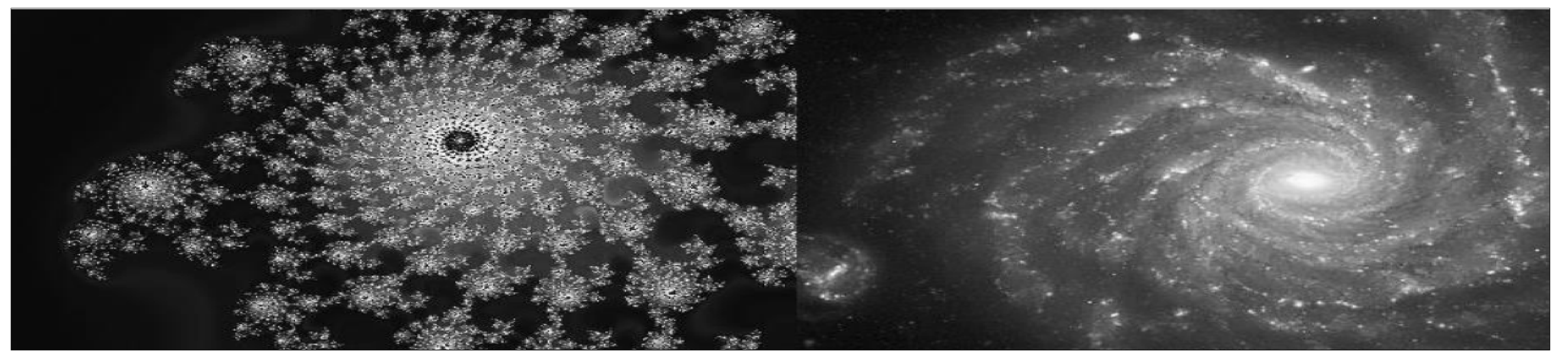


Figure 4: *Fractal Structure*

References

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[2] L. Gardi; The Mandelbrot set and the fractal nature of light, the Universe, and everything . Proc. SPIE 8832, The Nature of Light: What are Photons? V, 883210 (October 1, 2013); doi:10.1117/12.2023739.

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