k-MOMENTS OF DISTANCES BETWEEN CENTERS OF FORD CIRCLES

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Abstract. In this paper, we investigate a problem on the distribution of Ford circles, which concerns moments of distances between centers of these circles that lie above a given horizontal line.

1. Introduction

Introduced in 1938 by Lester R. Ford [8], a Ford circle is a circle tangent to the x-axis at a given point with rational coordinates $(p/q, 0)$ in reduced form, centered at $(p/q, 1/(2q^2))$. Any two Ford circles are either disjoint or tangent to each other. In the present paper, we study a question concerning the distribution of Ford circles.

For a fixed interval $I := [\alpha, \beta] \subseteq [0, 1]$ with rational end points and for each large positive integer Q, we consider the set $\mathcal{F}_{I,Q}$ consisting of Ford circles with centers lying between the vertical lines $x = \alpha$ and $x = \beta$ or possibly on the line $x = \beta$ but not below the line $y = \frac{1}{2Q^2}$. Note that these are the Ford circles that are tangent to the real axis at the rational points $(a/q, 0)$ with a/q in the interval $I = [\alpha, \beta]$ and $q \leq Q$. Let $N_I(Q)$ denote the number of elements in $\mathcal{F}_{I,Q}$. The circles $C_{Q,1}, C_{Q,2}, \cdots, C_{Q,N_I(Q)}$ in $\mathcal{F}_{I,Q}$ are arranged in such a way that any two consecutive circles are tangent to each other. For each j in $\{1, 2, \cdots, N_I(Q)\}$, denote the center of $C_{Q,j}$ by $O_{Q,j}$ and the radius of $C_{Q,j}$ by $r_{Q,j}$.

For any positive integer k, consider the k-moment

$$\mathcal{M}_{k,I}(Q) := \frac{1}{|I|} \sum_{j=1}^{N_I(Q)-1} \left(D(O_{Q,j}, O_{Q,j+1})\right)^k, \tag{1.1}$$

where $D(O_{Q,j}, O_{Q,j+1})$ denotes the Euclidean distance between the centers $O_{Q,j}$ and $O_{Q,j+1}$. For all large X, we consider the average

$$\mathcal{A}_{k,I}(X) := \frac{1}{X} \int_X^{2X} \mathcal{M}_{k,I}(Q)\, dY, \tag{1.2}$$

where here and in what follows, Y denotes a real variable and the positive integer Q is a function of Y; more precisely, Q is the integer part of Y. Although, as Q increases, the sequence of individual distances $D(Q_{Q,j}, O_{Q,j+1})$ changes wildly as more and more circles of various sizes appear between any two given circles, the k-averages $\mathcal{A}_{k,I}(X)$ satisfy nice asymptotic formulas.

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Theorem 1.1. *Fix an interval* $I := [\alpha, \beta] \subseteq [0,1]$ *with* $\alpha, \beta \in \mathbb{Q}$*, and let* $\mathcal{A}_{k,I}(X)$ *be defined as in* (1.2)*. Then, for* $k = 1$*,*

$$\mathcal{A}_{1,I}(X) = \frac{6}{\pi^2} \log 4X + B_1(I) + \mathrm{O}\left(\frac{\log^2 X}{X}\right), \tag{1.3}$$

where $B_1(I)$ *is a constant depending only on the interval* I*.*

We remark that when $I = [0,1]$, the constant $B_1(I)$ is given by

$$B_1([0,1]) = \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)},$$

where $\zeta(s)$ denotes the Riemann zeta function and γ denotes Euler's constant. In this case, we also obtain a better bound for the error term in (1.3), namely,

$$\mathcal{A}_{1,[0,1]}(X) = \frac{6}{\pi^2} \log 4X + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} + \mathrm{O}\left(\frac{1}{X e^{c_0 (\log X)^{3/5} (\log\log X)^{-1/5}}}\right), \tag{1.4}$$

where $c_0 > 0$ is an absolute constant.

Theorem 1.2. *For* I *as in Theorem 1.1 and* $k = 2$ *in* (1.2)*,*

$$\mathcal{A}_{2,I}(X) = B_2(I) + \frac{3}{\pi^2}\frac{\log X}{X^2} + \frac{D_2(I)}{X^2} + \mathrm{O}_\epsilon\left(X^{-21/10+\epsilon}\right),$$

where $B_2(I)$ *and* $D_2(I)$ *are constants depending only on the interval* I*.*

In particular, for $I = [0,1]$, one has

$$B_2([0,1]) = \frac{\zeta(3)}{2\zeta(4)} \text{ and } D_2([0,1]) = \frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right).$$

In this case also, we obtain a better bound for the error term,

$$\mathcal{A}_{2,[0,1]}(X) = \frac{\zeta(3)}{2\zeta(4)} + \frac{3}{\pi^2}\frac{\log X}{X^2} + \frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right)\frac{1}{X^2} + \mathrm{O}_\epsilon\left(\frac{\log^{5/3} X (\log\log X)^{1+\epsilon}}{X^3}\right). \tag{1.5}$$

Theorem 1.3. *For* I *as in Theorem 1.1 and* $k \geq 3$ *in* (1.2)*,*

$$\mathcal{A}_{k,I}(X) = B_k(I) + \mathrm{O}_k\left(\frac{1}{X^2}\right),$$

where $B_k(I)$ *is a constant depending on* k *and the interval* I*.*

In particular, for the full interval $I = [0,1]$, one has

$$B_k([0,1]) = \frac{\zeta(2k-1)}{2^{k-1}\zeta(2k)}.$$

In this case, we obtain a second order term and a better bound for the error term,

$$\mathcal{A}_{k,[0,1]}(X) = \frac{\zeta(2k-1)}{2^{k-1}\zeta(2k)} + \frac{k\zeta(2k-3)}{2^k\zeta(2k-2)}\frac{1}{X^2} + \mathrm{O}_k\left(\frac{1}{X^3}\right). \tag{1.6}$$

It would be interesting to investigate similar questions for Apollonian circle packings.

2. General Setup

In this section, we fix a positive integer k and express the k-th moment $\mathcal{M}_{k,I}$ in terms of the Euler-phi function and the Mobius function. Next, we rewrite $\mathcal{A}_{k,I}$ as an integral involving the Riemann zeta function, and then shift the path of integration based on the Vinogradov-Korobov zero free region. To proceed, we first review some facts about Farey fractions. Given a positive integer Q, by a Farey fraction of order Q, we mean a rational number in reduced form in the interval $[0,1]$ with denominator at most Q. We denote by F_Q the sequence of Farey fractions of order Q, arranged in order of increasing size. Two Farey fractions $a/b < c/d$ in F_Q are neighbours if and only if $bc-ad=1$ and $b+d>Q$. The Farey sequence F_Q is in bijection with the set of Ford circles tangent to the real line at points in the interval $[0,1]$ and radius at least $\frac{1}{2Q^2}$. Therefore, the distance between the centers $O_{Q,j}$ and $O_{Q,j+1}$ of two consecutive Ford circles $C_{Q,j}$ and $C_{Q,j+1}$ is given by

$$D(Q_{Q,j}, O_{Q,j+1}) = \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}, \tag{2.1}$$

where a_j/q_j and a_{j+1}/q_{j+1} are neighbours in F_Q and correspond to a pair of Ford circles tangent to the x-axis at $x = a_j/q_j$ and $x = a_{j+1}/q_{j+1}$ respectively. Thus, $N_I(Q)$ is same as the number of Farey fractions of order Q inside the interval $I = [\alpha, \beta]$. For two neighbouring Farey fractions $a'/q' < a/q$ of order Q in the interval I, we note that $a' \equiv -\bar{q} \pmod{q'}$, since $q'a - qa' = 1$. The notation $\bar{x} \pmod{n}$ is used for the multiplicative inverse of $x \pmod{n}$ in the interval $[1,n]$ for positive integers x and n with $\gcd(x,n)=1$. Thus, the conditions $a'/q' \in (\alpha, \beta]$ and $a/q \in (\alpha, \beta]$ are equivalent to

$$\bar{q} \in [q' - q'\beta, q' - q'\alpha), \text{ and } \bar{q'} \in (q\alpha, q\beta], \tag{2.2}$$

respectively. Questions on the distribution of Farey fractions have been studied extensively, see for example [1], [2], [4], [9] and the references therein.

For a fixed k and the interval I, from (1.1) and (2.1), one has

$$\begin{aligned}
|I|\mathcal{M}_{k,I}(Q) &= \sum_{j=1}^{N_I(Q)-1} \left(\frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2} \right)^k \\
&= \frac{1}{2^{k-1}} \sum_{j=2}^{N_I(Q)} \frac{1}{q_j^{2k}} + \frac{1}{2^k q_1^{2k}} - \frac{1}{2^k q_{N_I(Q)}^{2k}} + \frac{1}{2^k} \sum_{j=1}^{N_I(Q)-1} \sum_{i=1}^{k-1} \binom{k}{i} \left(\frac{1}{q_j^i q_{j+1}^{k-i}} \right)^2 \\
&= \frac{1}{2^{k-1}} \sum_{1 \le q \le Q} \frac{1}{q^{2k}} \sum_{\substack{\alpha q < a \le \beta q \\ (a,q)=1}} 1 + \frac{1}{2^k} \sum_{i=1}^{k-1} \binom{k}{i} \sum_{j=1}^{N_I(Q)-1} \frac{1}{q_j^{2i} q_{j+1}^{2k-2i}} + \left(\frac{1}{2^k q_1^{2k}} - \frac{1}{2^k q_{N_I(Q)}^{2k}} \right) \\
&=: S_k + S_k' + R_k(I),
\end{aligned} \tag{2.3}$$

where in the last inequality, without loss of generality, we assume that for large Q, the end-points of the interval $I = [\alpha, \beta]$ are Farey fractions of order Q since $\alpha, \beta \in \mathbb{Q}$. This implies

$R_k(I)$ in (2.3) is a constant depending only on k and end-points of the interval I. Therefore,

$$\begin{aligned}|I|\mathcal{A}_{k,I} &= \frac{1}{X}\int_X^{2X} \left(S_k + S_k' + R_k(I)\right)\, dY\\ &= \frac{1}{X}\int_1^{2X} S_k\, dY - \frac{1}{X}\int_1^{X} S_k\, dY + \frac{1}{X}\int_X^{2X} S_k'\, dY + R_k(I). \end{aligned} \tag{2.4}$$

Consider,

$$\begin{aligned}\frac{1}{X}\int_1^{X} S_k\, dY &= \frac{1}{2^{k-1}X}\int_1^X \sum_{1\le q\le Q}\frac{1}{q^{2k}}\sum_{\substack{\alpha q<a\le\beta q\\ (a,q)=1}} 1\, dY\\
&= \frac{1}{2^{k-1}X}\int_1^X \sum_{q\le Q}\frac{1}{q^{2k}}\sum_{\alpha q<a\le\beta q} 1\sum_{d|(a,q)}\mu(d)\, dY\\
&= \frac{1}{2^{k-1}X}\int_1^X \sum_{q\le Q}\frac{1}{q^{2k}}\sum_{d|q}\mu(d)\sum_{\frac{\alpha q}{d}<l\le\frac{\beta q}{d}} 1\, dY\\
&= \frac{1}{2^{k-1}X}\int_1^X \sum_{q\le Q}\frac{1}{q^{2k}}\left(\sum_{d|q}\mu(d)\frac{(\beta-\alpha)q}{d} - \sum_{d|q}\mu(d)\left(\left\{\frac{\beta q}{d}\right\}-\left\{\frac{\alpha q}{d}\right\}\right)\right) dY\\
&= \frac{1}{2^{k-1}X}\int_1^X \left(|I|\sum_{q\le Q}\frac{\phi(q)}{q^{2k}} - \sum_{d\le Q}\frac{\mu(d)}{d^{2k}}\sum_{m\le\frac{Q}{d}}\left(\frac{\{\beta m\}-\{\alpha m\}}{m^{2k}}\right)\right) dY\\
&= \frac{|I|}{2^{k-1}}\sum_{1\le q\le X}\frac{\phi(q)}{q^{2k}}\left(1-\frac{q}{X}\right) - \frac{1}{2^{k-1}X}\int_1^X \sum_{\substack{d,m\ge1\\ dm\le Y}}\frac{\mu(d)}{d^{2k}m^{2k}}\left(\{\beta m\}-\{\alpha m\}\right) dY\\
&= \frac{|I|}{2^{k-1}}\sum_{1\le q\le X}\frac{\phi(q)}{q^{2k}}\left(1-\frac{q}{X}\right) - \frac{1}{2^{k-1}}\sum_{1\le d\le X}\frac{\mu(d)}{d^{2k}}\sum_{1\le m\le\frac{X}{d}}\frac{\{\beta m\}-\{\alpha m\}}{m^{2k}}\\
&\quad + \frac{1}{2^{k-1}X}\sum_{1\le d\le X}\frac{\mu(d)}{d^{2k-1}}\sum_{1\le m\le\frac{X}{d}}\frac{\{\beta m\}-\{\alpha m\}}{m^{2k-1}}\\
&=: S_{k,1} - S_{k,2} + S_{k,3}. \end{aligned}\tag{2.5}$$

We first estimate the sums $S_{k,2}$ and $S_{k,3}$. Since for $x\ge1$, and $a\ge2$,

$$\sum_{n\ge x}\frac{1}{n^a} = \mathrm{O}\left(x^{1-a}\right),$$

one has

$$\begin{aligned} S_{k,2} &= \frac{1}{2^{k-1}} \sum_{1\le d\le X} \frac{\mu(d)}{d^{2k}} \left(\sum_{m\ge 1} \frac{\{\beta m\}-\{\alpha m\}}{m^{2k}} - \sum_{m>X/d} \frac{\{\beta m\}-\{\alpha m\}}{m^{2k}} \right) \\ &= \frac{C_{2k,I}}{2^{k-1}} \sum_{1\le d\le X} \frac{\mu(d)}{d^{2k}} + \mathrm{O}\left(\sum_{1\le d\le X} \frac{|\mu(d)|}{d^{2k}} \sum_{m>X/d} \frac{|\{\beta m\}-\{\alpha m\}|}{m^{2k}} \right) \\ &= \frac{C_{2k,I}}{2^{k-1}\zeta(2k)} + \mathrm{O}\left(\frac{\log X}{X^{2k-1}}\right), \end{aligned} \tag{2.6}$$

where for a natural number j, we denote

$$C_{j,I} := \sum_{m\ge 1} \frac{\{\beta m\}-\{\alpha m\}}{m^j}.$$

Similarly for $k \ge 2$,

$$\begin{aligned} S_{k,3} &= \frac{1}{2^{k-1}X} \sum_{1\le d\le X} \frac{\mu(d)}{d^{2k-1}} \left(\sum_{m\ge 1} \frac{\{\beta m\}-\{\alpha m\}}{m^{2k-1}} - \sum_{m>X/d} \frac{\{\beta m\}-\{\alpha m\}}{m^{2k-1}} \right) \\ &= \frac{C_{2k-1,I}}{2^{k-1}} \frac{1}{X} \sum_{1\le d\le X} \frac{\mu(d)}{d^{2k-1}} + \mathrm{O}\left(\frac{1}{X} \sum_{1\le d\le X} \frac{|\mu(d)|}{d^{2k-1}} \sum_{m>X/d} \frac{|\{\beta m\}-\{\alpha m\}|}{m^{2k-1}} \right) \\ &= \frac{C_{2k-1,I}}{2^{k-1}\zeta(2k-1)} \frac{1}{X} + \mathrm{O}\left(\frac{\log X}{X^{2k-1}}\right). \end{aligned} \tag{2.7}$$

In a similar fashion one estimates $S_{k,3}$ for $k = 1$,

$$\begin{aligned} |S_{1,3}| &= \left| \frac{1}{X} \sum_{1\le d\le X} \frac{\mu(d)}{d} \sum_{m<X/d} \frac{\{\beta m\}-\{\alpha m\}}{m} \right| \\ &\le \frac{1}{X} \sum_{1\le d\le X} \frac{|\mu(d)|}{d} \sum_{m<X/d} \left| \frac{\{\beta m\}-\{\alpha m\}}{m} \right| = \mathrm{O}\left(\frac{\log^2 X}{X}\right). \end{aligned} \tag{2.8}$$

Next we consider the sum

$$S_{k,1} = \frac{|I|}{2^{k-1}} \sum_{1\le q\le X} \frac{\phi(q)}{q^{2k}} \left(1 - \frac{q}{X}\right).$$

The arithmetic function $\phi(n)n^{-2k}$ is multiplicative and its Dirichlet series is given by

$$\sum_{n=1}^{\infty} \frac{\phi(n)n^{-2k}}{n^s} = \frac{\zeta(s+2k-1)}{\zeta(s+2k)},$$

which converges for $\Re(s) > 2-2k$. For a complex number s, we write $s = \sigma + it$. By Perron's formula ([12, page 130]), for $c > 0$,

$$\frac{1}{X} \sum_{q\le X} \frac{\phi(q)}{q^{2k}} (X - q) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{X^s \zeta(s+2k-1)}{s(s+1)\zeta(s+2k)}\, ds. \tag{2.9}$$

Fix $T, U > 0$ such that $2 \leq T \leq X$, $X^2 \leq U \leq X^{2k}$ and $c = \frac{a}{\log X}$ for some absolute constant $a > 0$. Let

$$d = -2k + 1 - \frac{A}{(\log 2T)^{2/3}(\log\log 2T)^{1/3}},$$

where A will be a suitably chosen absolute constant. In order to evaluate the above integral, we modify the path of integration from $c - iU$ to $c + iU$ along the line segments $l_j, 1 \leq j \leq 9$ described below.

We let l_1 be the half line from $c + iU$ to $c + i\infty$, l_2 be the line segment from $-2k + 1 + iU$ to $c + iU$, l_3 be the line segment from $-2k + 1 + iT$ to $-2k + 1 + iU$, l_4 be the line segment from $d + iT$ to $-2k + 1 + iT$, l_5 be the line segment from $d - iT$ to $d + iT$, l_6 be the line segment from $-2k + 1 - iT$ to $d - iT$, l_7 be the line segment from $-2k + 1 - iU$ to $-2k + 1 - iT$, l_8 be the line segment with endpoints $-2k + 1 - iU$ and $c - iU$, and lastly let l_9 be the half line from $c - i\infty$ to $c - iU$. The main contribution on the right side of (2.9) comes from the residues at the poles of the function

$$f_k(s) := \frac{X^s \zeta(s + 2k - 1)}{s(s + 1)\zeta(s + 2k)},$$

encountered when we modified the path of integration. By the residue theorem,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f_k(s)\, ds = \sum \mathrm{Res}(f_k(s)) + \sum_{m=1}^{9} J_m, \tag{2.10}$$

where J_i is the integral of $f_k(s)$ along l_i. Here the sum $\sum \mathrm{Res}(f_k(s))$ is taken over all the poles of $f_k(s)$ inside the region bounded by segments $l_2, l_3, \ldots, l_8$ and the vertical segment joining $c - iU$ and $c + iU$. To estimate the integrals $J_1, \ldots, J_9$, we use standard bounds for $\zeta(s)$ and $\frac{1}{\zeta(s)}$ ([13, page 47]),

$$\zeta(\sigma + it) = \begin{cases} \mathrm{O}\left(t^{\sigma - \frac{1}{2}} \log t\right), & -1 \leq \sigma \leq 0, \\ \mathrm{O}\left(t^{\frac{1-\sigma}{2}} \log t\right), & 0 \leq \sigma \leq 1, \\ \mathrm{O}\left(\log t\right), & 1 \leq \sigma \leq 2, \\ \mathrm{O}\left(1\right), & \sigma \geq 2, \end{cases}$$

and

$$\frac{1}{\zeta(\sigma + it)} = \begin{cases} \mathrm{O}\left(\log t\right), & 1 \leq \sigma \leq 2, \\ \mathrm{O}\left(1\right), & \sigma \geq 2. \end{cases}$$

We also use the Vinogradov-Koroborov zero free region ([14], [11]),

$$\sigma \geq 1 - B(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}},$$

where

$$\frac{1}{\zeta(s)} = \mathrm{O}\left((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}\right),$$

and B is an absolute constant.

3. First Moment

In this section we provide a proof of Theorem 1.1. For the first moment, we set $k = 1$ in Section 2.

Proof of Theorem 1.1. From (2.3), we note that

$$|I|\mathcal{M}_{1,I} = \sum_{q\le Q}\frac{1}{q^2}\sum_{\substack{\alpha q<a\le\beta q\\(a,q)=1}} 1 + \left(-\frac{1}{2q_1^2}+\frac{1}{2q_{N_I(Q)}^2}\right) = S_1 + R_1(I).$$

Therefore, as in (2.4) we have

$$|I|\mathcal{A}_{1,I} = \frac{1}{X}\int_1^{2X} S_1\ dY - \frac{1}{X}\int_1^{X} S_1\ dY + R_1(I). \tag{3.1}$$

Now from (2.5),

$$\frac{1}{X}\int_1^{X} S_1\ dY = S_{1,1} - S_{1,2} + S_{1,3}. \tag{3.2}$$

The sums $S_{1,2}$ and $S_{1,3}$ have already been estimated in (2.6) and (2.8) respectively. In order to estimate $S_{1,1}$, we bound the integrals J_m in (2.10) as follows. One has

$$\begin{aligned}|J_1|, |J_9| &= \mathrm{O}\left(\int_U^\infty \frac{|X^{c+it}||\zeta(c+1+it)|}{|c+it||c+1+it||\zeta(c+2+it)|}\ dt\right)\\ &= \mathrm{O}\left(X^c\int_U^\infty \frac{\log t}{t^2}\ dt\right) = \mathrm{O}\left(\frac{\log U}{U}\right).\end{aligned}$$

And,

$$\begin{aligned}|J_2|, |J_8| &= \mathrm{O}\left(\int_{-1}^{c} \frac{|X^{\sigma+iU}||\zeta(1+\sigma+iU)|}{|\sigma+iU||\sigma+1+iU||\zeta(2+\sigma+iU)|}\ d\sigma\right)\\ &= \mathrm{O}\left(\frac{(\log U)^2}{U^2}\int_{-1}^{0}\left(\frac{X}{\sqrt{U}}\right)^\sigma d\sigma + \frac{\log U}{U^2}\int_0^c X^\sigma\ d\sigma\right) = \mathrm{O}\left(\frac{\log^2 U}{U^2}\right).\end{aligned}$$

Next,

$$\begin{aligned}|J_3|, |J_7| &= \mathrm{O}\left(\int_T^U \frac{|X^{-1+it}||\zeta(it)|}{|-1+it||it||\zeta(1+it)|}\ dt\right) = \mathrm{O}\left(X^{-1}\int_T^U \frac{\log t}{t^{3/2}}\ dt\right)\\ &= \mathrm{O}\left(\frac{\log^2 T}{X\sqrt{T}}\right).\end{aligned}$$

Also,

$$\begin{aligned}|J_4|, |J_6| &= \mathrm{O}\left(\int_d^{-1} \frac{|X^{\sigma+iT}||\zeta(1+\sigma+iT)|}{|\sigma+1+iT||\sigma+iT||\zeta(\sigma+2+iT)|}\ d\sigma\right)\\ &= \mathrm{O}\left(\frac{(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}{T^2}\int_d^{-1} X^\sigma|\zeta(1+\sigma+iT)|\ d\sigma\right)\\ &= \mathrm{O}\left(\frac{(\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}}{T^2}\int_d^{-1}\left(\frac{X}{\sqrt{T}}\right)^\sigma d\sigma\right) = \mathrm{O}\left(\frac{(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}{XT^{3/2}}\right).\end{aligned}$$

Lastly,

$$|J_5| = \mathrm{O}\left(\int_{-T}^{T} \frac{|X^{d+it}||\zeta(1+d+it)|}{|d+it||d+1+it||\zeta(d+2+it)|}\, dt\right)$$
$$= \mathrm{O}\left(X^d \int_{-T}^{T} \frac{t^{-1/2-d}(\log(2+|t|))^{5/3}(\log\log(3+|t|))^{1/3}}{1+t^2}\, dt\right) = \mathrm{O}\left(X^d\right).$$

Collecting all the above estimates and setting $U = X^2$ and $T = \exp\left(\frac{c_1(\log X)^{3/5}}{(\log\log X)^{1/5}}\right)$, one obtains

$$S_{1,1} = |I|\,\mathrm{Res}(f_1(s)) + \mathrm{O}\left(\frac{1}{Xe^{c_0(\log X)^{3/5}(\log\log X)^{-1/5}}}\right), \tag{3.3}$$

where c_0 and c_1 are suitable positive absolute constants. Here, in the prescribed region, $f_1(s)$ has only one pole at $s = 0$ of order two with residue

$$\mathrm{Res}(f_1(s)) = \frac{\log X}{\zeta(2)} + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}.$$

From (2.6), (2.8), (3.1), (3.2), (3.3) and above,

$$\mathcal{A}_{1,I}(X) = \frac{6}{\pi^2}\log 4X + \frac{\gamma-1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} - \frac{C_{2,I}}{|I|\zeta(2)} + \frac{R_1(I)}{|I|} + \mathrm{O}\left(\frac{\log^2 X}{X}\right).$$

This concludes the proof of Theorem 1.1. □

Remark. In the case of the full interval $I = [0, 1]$, we observe that $R_1([0, 1]) = 0$ in (2.3) and $S_{1,2} = 0 = S_{1,3}$ in (2.5). Therefore, $S_{1,1}$ in (3.2) is the only term which contributes to the average $\mathcal{A}_{1,[0,1]}$ in (3.1) and we obtain

$$\mathcal{A}_{1,[0,1]}(X) = \frac{6}{\pi^2}\log 4X + \frac{\gamma-1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} + \mathrm{O}\left(\frac{1}{Xe^{c_0(\log X)^{3/5}(\log\log X)^{-1/5}}}\right),$$

as claimed in (1.4).

4. Higher Moments

In this section, we prove Theorem 1.2 and Theorem 1.3. We first estimate the integral $\frac{1}{X}\int_1^X S_k\, dY$ for $k \geq 2$ in (2.4). From (2.5),

$$\frac{1}{X}\int_1^X S_k\, dY = S_{k,1} - S_{k,2} + S_{k,3}.$$

Estimates for $S_{k,2}$ and $S_{k,3}$ for $k \geq 2$ have already been obtained in (2.6) and (2.7). For $k \geq 2$, estimates for

$$S_{k,1} = \frac{|I|}{2^{k-1}X}\sum_{q\leq X}\frac{\phi(q)}{q^{2k}}(X-q)$$

can be obtained as before where we set $U = X^{2k}$. In this case, the corresponding function $f_k(s)$ has poles at $s = 0$, $s = -1$ and $s = 2-2k$ in the region described before. All these poles are simple and the sum of the residues of $f_k(s)$ at these poles is given by

$$\sum \mathrm{Res}(f_k(s)) = \frac{\zeta(2k-1)}{\zeta(2k)} - \frac{\zeta(2k-2)}{\zeta(2k-1)}\frac{1}{X} + \frac{1}{(2k-3)(2k-2)\zeta(2)}\frac{1}{X^{2k-2}}.$$

One can estimate the line integrals J_m of the function $f_1(s)$ along l_i for $1 \le i \le 9$ in (2.10) as before. In this case one has

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{X^s\zeta(s+2k-1)}{s(s+1)\zeta(s+2k)}\,ds = \frac{\zeta(2k-1)}{\zeta(2k)} - \frac{\zeta(2k-2)}{\zeta(2k-1)}\frac{1}{X} + \frac{1}{(2k-3)(2k-2)\zeta(2)X^{2k-2}} + \mathrm{O}\left(\frac{1}{X^{2k-1}e^{c_0(\log X)^{3/5}(\log\log X)^{-1/5}}}\right).$$

Therefore, from (2.9) and the above equation, we obtain

$$S_{k,1} = \frac{|I|\zeta(2k-1)}{2^{k-1}\zeta(2k)} - \frac{|I|\zeta(2k-2)}{2^{k-1}\zeta(2k-1)}\frac{1}{X} + \frac{|I|}{2^{k-1}(2k-3)(2k-2)\zeta(2)X^{2k-2}} + \mathrm{O}\left(\frac{1}{X^{2k-1}e^{c_0(\log X)^{3/5}(\log\log X)^{-1/5}}}\right).$$

From (2.5), (2.6), (2.7) and above, we derive

$$\frac{1}{X}\int_X^{2X} S_k\,dY = \frac{|I|\zeta(2k-1)}{2^{k-1}\zeta(2k)} - \frac{C_{2k,I}}{2^{k-1}\zeta(2k)} + \frac{|I|(1-2^{2k-3})}{2^{3k-4}(2k-3)(2k-2)\zeta(2)X^{2k-2}} + \mathrm{O}\left(\frac{\log X}{X^{2k-1}}\right). \tag{4.1}$$

In order to prove Theorem 1.2 and Theorem 1.3, it remains to estimate the remaining integral

$$\frac{1}{X}\int_X^{2X} S_k'\,dY$$

for $k \ge 2$ in (2.4).

Proof of Theorem 1.2. For $k = 2$ in (2.3), we have

$$\begin{aligned} |I|\mathcal{M}_{2,I}(Q) &= \sum_{j=1}^{N_I(Q)-1}\left(\frac{1}{2q_j^2}+\frac{1}{2q_{j+1}^2}\right)^2 \\ &= \frac{1}{2}\sum_{q\le Q}\frac{1}{q^4}\sum_{\substack{\alpha q<a\le\beta q\\(a,q)=1}} 1 + \frac{1}{2}\sum_{j=1}^{N_I(Q)-1}\left(\frac{1}{q_jq_{j+1}}\right)^2 + \left(-\frac{1}{4q_1^4}+\frac{1}{4q_{N_I(Q)}^4}\right) \\ &= S_2 + S_2' + R_2(I). \end{aligned}$$

Therefore,

$$|I|\mathcal{A}_{2,I} = \frac{1}{X}\int_X^{2X} S_2\,dY + \frac{1}{X}\int_X^{2X} S_2'\,dY + R_2(I). \tag{4.2}$$

From [3, Theorem 2], we obtain

$$S_2' = \frac{|I|}{2}S_0(Q) + \frac{C_{2,I}}{Q^2} + \mathrm{O}_\epsilon\left(Q^{-21/10+\epsilon}\right),$$

where

$$S_0(Q) = \frac{12}{\pi^2 Q^2}\left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\right) + \mathrm{O}_\epsilon\left(\frac{\log^{5/3} Q(\log\log Q)^{1+\epsilon}}{Q^3}\right).$$

We remark in passing that the saving in the exponent above (from -2 to $-21/10$) was obtained by employing Weil type estimates ([7], [10], [15]) for Kloosterman sums.

Next, we have

$$\frac{1}{|I|X}\int_X^{2X} S_2'\, dY = \frac{3}{\pi^2}\frac{\log X}{X^2} + \left(\frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{3}{2} - \log 2\right) + \frac{C_{2,I}}{2|I|}\right)\frac{1}{X^2} + \mathrm{O}_\epsilon\left(X^{-21/10+\epsilon}\right).$$

Combining (4.1), (4.2) and above, we conclude that

$$\begin{aligned}\mathcal{A}_{2,I} = \frac{|I|\zeta(3) - C_{4,I}}{2|I|\zeta(4)} + \frac{R_2(I)}{|I|} + \frac{3}{\pi^2}\frac{\log X}{X^2} + \left(\frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right) + \frac{C_{2,I}}{2|I|}\right)\frac{1}{X^2} \\ + \mathrm{O}_\epsilon\left(X^{-21/10+\epsilon}\right).\end{aligned}$$

This completes the proof of Theorem 1.2. □

Remark. For the full interval $I = [0,1]$, observe that $R_2(I) = 0$, and

$$S_2' = \frac{6}{\pi^2 Q^2}\left(\log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\right) + \mathrm{O}_\epsilon\left(\frac{\log^{5/3} Q(\log\log Q)^{1+\epsilon}}{Q^3}\right).$$

This along with (4.1) and (4.2) proves (1.5),

$$\begin{aligned}\mathcal{A}_{2,[0,1]}(X) = \frac{\zeta(3)}{2\zeta(4)} - \frac{1}{2} + \frac{3}{\pi^2}\frac{\log X}{X^2} + \frac{3}{\pi^2}\left(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right)\frac{1}{X^2} \\ + \mathrm{O}_\epsilon\left(\frac{\log^{5/3} X(\log\log X)^{1+\epsilon}}{X^3}\right).\end{aligned}$$

Proof of Theorem 1.3. For $k \geq 3$,

$$\begin{aligned}|I|\mathcal{M}_{k,I}(Q) &= \sum_{j=1}^{N_I(Q)-1}\left(\frac{1}{2{q_j}^2} + \frac{1}{2{q_{j+1}}^2}\right)^k \\ &= \sum_{q\leq Q}\frac{1}{q^{2k}}\sum_{\substack{\alpha q<a\leq\beta q\\(a,q)=1}}\frac{|I|}{2^{k-1}} + \frac{1}{2^k}\sum_{i=1}^{k-1}\binom{k}{i}\sum_{j=1}^{N_I(Q)-1}\left(\frac{1}{q_j^i q_{j+1}^{k-i}}\right)^2 - \left(\frac{1}{2^k q_1^{2k}} + \frac{1}{2^k q_{N_I(Q)}^{2k}}\right) \\ &= S_k + S_k' + R_k(I).\end{aligned}$$

Therefore,

$$|I|\mathcal{A}_{k,I} = \frac{1}{X}\int_X^{2X} S_k\, dY + \frac{1}{X}\int_X^{2X} S_k'\, dY + R_k(I). \tag{4.3}$$

For each $1 \leq i \leq k-1$, consider the sum

$$S_{k,i} := \sum_{j=1}^{N_I(Q)-1}\frac{1}{q_j^{2i}q_{j+1}^{2k-2i}}.$$

For any positive integer m, let $\mathcal{L}_m$ denote the set

$$\mathcal{L}_m := \left\{ \begin{array}{l} l \in \mathbb{N} \colon l > m, Q - m < l \leq Q, \gcd(m, l) = 1, \\ \qquad\quad \bar{l} \pmod{m} \in (m\alpha, m\beta], \bar{m} \pmod{l} \in [l - l\beta, l - l\alpha) \end{array} \right\}.$$

Employing (2.2), we have

$$S_{k,i} = \sum_{1 \leq r \leq Q} \sum_{q \in \mathcal{L}_r} \frac{1}{q^{2i} r^{2k-2i}} + \sum_{1 \leq q \leq Q} \sum_{r \in \mathcal{L}_q} \frac{1}{q^{2i} r^{2k-2i}}.$$

As noted earlier in Section 2, when q and r are denominators of neighbouring Farey fractions in F_Q, then $r + q > Q$. Therefore, $q > r$ implies $q > Q/2$ and for $r > q$, we have $r > Q/2$. Also,

$$\sum_{q \in \mathcal{L}_r} 1 = \mathrm{O}\left(\phi(r)\right) \text{ and } \sum_{r \in \mathcal{L}_q} 1 = \mathrm{O}\left(\phi(q)\right).$$

Using the above relations and the fact that for $x \geq 2$ and $a \geq 3$,

$$\sum_{1 \leq n \leq x} \frac{\phi(n)}{n^a} = \frac{\zeta(a-1)}{\zeta(a)} + \mathrm{O}\left(x^{2-a}\right), \tag{4.4}$$

we obtain

$$\begin{aligned} S_{k,i} &\leq \left(\frac{2}{Q}\right)^{2i} \sum_{r \leq Q} \frac{1}{r^{2k-2i}} \sum_{q \in \mathcal{L}_r} 1 + \left(\frac{2}{Q}\right)^{2k-2i} \sum_{q \leq Q} \frac{1}{q^{2i}} \sum_{r \in \mathcal{L}_q} 1 \\ &= \mathrm{O}\left(\frac{1}{Q^{2i}} \sum_{r \leq Q} \frac{1}{r^{2k-2i}} \phi(r)\right) + \mathrm{O}\left(\frac{1}{Q^{2k-2i}} \sum_{q \leq Q} \frac{1}{q^{2i}} \phi(q)\right) \\ &= \mathrm{O}\left(\frac{\log Q}{Q^{2i}}\right) + \mathrm{O}\left(\frac{\log Q}{Q^{2k-2i}}\right). \end{aligned}$$

Here on the far right side, the first $\log Q$ may be replaced by 1 unless $i = k - 1$, and the second $\log Q$ may be replaced by 1 unless $i = 1$. Hence,

$$\frac{1}{X} \int_X^{2X} S'_k \, dY = \frac{1}{X} \int_X^{2X} \frac{1}{2^k} \sum_{i=1}^{k-1} \binom{k}{i} S_{k,i} \, dY = \mathrm{O}\left(\frac{1}{X^2}\right).$$

This combined with (4.1) and (4.3) yields

$$\mathcal{A}_{k,I} = \frac{|I|\zeta(2k-1) - C_{2k,I}}{|I| 2^{k-1} \zeta(2k)} + \frac{R_k(I)}{|I|} + \mathrm{O}_k\left(\frac{1}{X^2}\right) \quad \text{for } k \geq 3,$$

which completes the proof of Theorem 1.3. ∎

Remark. In the case of the full interval $I = [0,1]$, note that $R_k([0,1]) = 0$, and

$$\begin{aligned} S_{k,i} &= \sum_{\substack{1\le q,r\le Q\\ \gcd(q,r)=1,\\ q+r>Q}} \frac{1}{q^{2i} r^{2k-2i}} \\ &= \sum_{\substack{1\le q,r\le Q\\ \gcd(q,r)=1,\\ r<Q/2,\\ q+r>Q}} \frac{1}{q^{2i} r^{2k-2i}} + \sum_{\substack{1\le q,r\le Q\\ \gcd(q,r)=1,\\ q<Q/2,\\ q+r>Q}} \frac{1}{q^{2i} r^{2k-2i}} + \sum_{\substack{1\le q,r\le Q\\ \gcd(q,r)=1,\\ q,r\ge Q/2}} \frac{1}{q^{2i} r^{2k-2i}} \\ &=: \Sigma_{1,i} + \Sigma_{2,i} + \Sigma_{3,i}. \end{aligned}$$

First we estimate the sum $\Sigma_{1,i}$ for $1 \le i \le k-2$. Note that in this case $r < Q/2$, therefore $q > Q/2$ since $r + q > Q$. Also, $\dfrac{1}{q} = \dfrac{1}{Q}\left(1 + \mathrm{O}\left(\dfrac{Q-q}{Q}\right)\right)$ gives,

$$\begin{aligned} \Sigma_{1,i} &= \sum_{1\le r<Q/2} \frac{1}{r^{2k-2i}} \left(\sum_{\substack{\gcd(q,r)=1\\ Q-r<q\le Q}} \frac{1}{Q^{2i}} + \mathrm{O}\left(\frac{Q-q}{Q^{2i+1}}\right)\right) \\ &= \sum_{1\le r<Q/2} \frac{1}{r^{2k-2i}} \sum_{\substack{\gcd(q,r)=1\\ Q-r<q\le Q}} \frac{1}{Q^{2i}} + \mathrm{O}\left(\frac{1}{Q^{2i+1}} \sum_{1\le r<Q/2} \frac{1}{r^{2k-2i-1}} \sum_{\substack{\gcd(q,r)=1\\ Q-r<q\le Q}} 1 \right) \\ &= \frac{1}{Q^{2i}} \sum_{1\le r<Q/2} \frac{\phi(r)}{r^{2k-2i}} + \mathrm{O}\left(\frac{1}{Q^{2i+1}} \sum_{1\le r<Q/2} \frac{\phi(r)}{r^{2k-2i-1}} \right). \end{aligned}$$

Using (4.4), for $1 \le i \le k-2$,

$$\Sigma_{1,i} = \frac{\zeta(2k-2i-1)}{\zeta(2k-2i)} \frac{1}{Q^{2i}} + \mathrm{O}\left(\frac{1}{Q^{2i+1}}\right).$$

Using

$$\sum_{n\le x} \frac{\phi(n)}{n^2} = \frac{\log x}{\zeta(2)} + \mathrm{O}(1), \ \text{ and } \sum_{n\le x} \frac{\phi(n)}{n} = \mathrm{O}(x),$$

we have, for $i = k-1$,

$$\Sigma_{1,k-1} = \frac{1}{\zeta(2)} \frac{\log(Q/2)}{Q^{2k-2}} + \mathrm{O}\left(\frac{1}{Q^{2k-2}}\right).$$

Similarly, for the sum $\Sigma_{2,i}$ for $2 \le i \le k-1$,

$$\Sigma_{2,i} = \frac{\zeta(2i-1)}{\zeta(2i)} \frac{1}{Q^{2k-2i}} + \mathrm{O}\left(\frac{1}{Q^{2k-2i+1}}\right),$$

and

$$\Sigma_{2,1} = \frac{1}{\zeta(2)} \frac{\log(Q/2)}{Q^{2k-2}} + \mathrm{O}\left(\frac{1}{Q^{2k-2}}\right).$$

Lastly, for $1 \le i \le k-1$,

$$\Sigma_{3,i} = \mathrm{O}\left(\frac{1}{Q^{2k-2}}\right).$$

Therefore,

$$S_k' = \frac{k\zeta(2k-3)}{2^{k-1}\zeta(2k-2)}\frac{1}{Q^2} + \mathrm{O}\left(\frac{1}{Q^3}\right).$$

This combined with (4.3) gives (1.6),

$$\mathcal{A}_{k,[0,1]} = \frac{\zeta(2k-1)}{2^{k-1}\zeta(2k)} - \frac{1}{2^{k-1}} + \frac{k\zeta(2k-3)}{2^k\zeta(2k-2)}\frac{1}{X^2} + \mathrm{O}_k\left(\frac{1}{X^3}\right).$$

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