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Zeros of normalized combinations of $\xi^{(k)}(s)$ on $\mathrm{Re}(s) = 1/2$

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A B S T R A C T

We consider functions of the form $F_{\vec{c},a,T}(s) = \sum_{j=0}^{M} \frac{c_j(-1)^j}{L^{2j}} \xi^{(a+2j)}(s)$, with $L = \log \frac{T}{2\pi}$ and c_j real constants satisfying a certain constraint. We show that as $T \to \infty$, the proportion of zeros of $F_{\vec{c},a,T}(s)$ on the critical line $\mathrm{Re}(s) = 1/2$ tends to 1.

1. Introduction

Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s = \sigma + it$, $\sigma > 1$ and $t \in \mathbb{R}$, denote the Riemann zeta-function. The analytic continuation of $\zeta(s)$ to a meromorphic function on the complex plane is achieved by the functional equation

$$\xi(s) = \xi(1-s),$$

where, the Riemann ξ-function is defined as

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

For $\sigma > 1$, the Euler product is

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$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

where the product is taken over all the primes p. This links the Riemann zeta-function to multiplicative number theory [18, §1 and §2]. It is well understood from the work of Riemann and von Mangoldt that the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ are located inside the critical strip $0 < \beta < 1$, see [18, §3]. From the fact that Γ has no zeros, and has simple poles at the trivial zeros of $\zeta(s)$, it follows that the zeros of ξ are the same as the non-trivial zeros of ζ. The Riemann hypothesis states that $\beta = \frac{1}{2}$.

Now let $N(T)$ denote the number of zeros of $\xi(s)$ in the rectangle $0 < \sigma < 1$ and $0 < t \le T$, each zero counted with multiplicity. It is well-known that

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \tag{1.1}$$

where

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right) \ll \log T,$$

as $T \to \infty$, see [18, §9]. Let us now define $N^{(0)}(T)$ to be the number of zeros of $\zeta(s)$ with $\beta = \frac{1}{2}$ on $0 < t \le T$, where each zero is counted with multiplicity. We further set

$$\kappa = \liminf_{T\to\infty} \frac{N^{(0)}(T)}{N(T)}.$$

In 1942, Selberg [15] showed that $\kappa > 0$, and later Levinson [10] showed that $\kappa > 0.34$. This was improved by Conrey [5] to $\kappa > 0.4088$. The history of these results and the current best bound can be found in [2,7,9,11,14]. In particular, the current best bound $\kappa > 0.4149$ is presented in [11].

For a positive integer k, let $\xi^{(k)}(s)$ denote the kth derivative of the Riemann ξ-function. The Riemann hypothesis implies that for any positive integer k, all the zeros of $\xi^{(k)}(s)$ lie on the critical line. Suppose, in analogy to the above, that $N_k(T)$ denotes the number of zeros $\beta + i\gamma$ of $\xi^{(k)}(s)$ in the rectangle $0 < \beta < 1$ and $0 < \gamma \le T$ and that $N_k^{(0)}(T)$ denotes the number of zeros of $\xi^{(k)}(s)$ with $\beta = \frac{1}{2}$ and $0 < \gamma \le T$. A result of Conrey [3] states that if T is positive and sufficiently large, $L = \log\frac{T}{2\pi}$ and $U = TL^{-10}$, then

$$\liminf_{T\to\infty} \kappa_k(T, U) = 1 + O(k^{-2}) \tag{1.2}$$

as $k \to \infty$ and where

$$\kappa_k(T, U) = \frac{N_k^{(0)}(T+U) - N_k^{(0)}(T)}{N_k(T+U) - N_k(T)}.$$

Moreover, in [4], following work from Anderson [1] and Heath-Brown [8], Conrey also established corresponding bounds for simple zeros. The coefficient of k^{-2} was computed in [3] for zeros with multiplicity and in [4] for simple zeros. It was remarked that the proportion of simple zeros was always a bit smaller than that of zeros with potential multiplicity. Nonetheless, from (1.2) as the order of the derivative of ξ increases, the proportion of zeros on the critical line increases to one. This strong result is due to Conrey [3].

Rezvyakova [12,13] computed the coefficients of k^{-2} in 2005 and her result holds uniformly for k in a certain range depending on T. In particular, Rezvyakova showed that the coefficient in front of k^{-2} could be taken to be $\frac{e^2+2}{16}$ for both simple as well as higher order zeros.

In the late 1990s, Selberg considered combinations of Dirichlet L-functions on the critical line. More specifically, let

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s},\quad \sigma>1,$$

be a Dirichlet L-function of modulus q, where χ denotes a primitive character. The functional equation of $L(s,\chi)$ is given by

$$\phi(s,\chi)=\varepsilon\pi^{-s/2}q^{s/2}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi)=\overline{\phi(1-\bar{s},\chi)},$$

where

$$\mathfrak{a}=\frac{1-\chi(-1)}{2}\quad\text{and}\quad|\varepsilon|=1,$$

see e.g. [6]. If we have n distinct even characters (a similar result holds for odd characters) and form the function

$$F(s)=\sum_{j=1}^{n}c_j\varepsilon_j q_j^{s/2}L(s,\chi_j) \tag{1.3}$$

for real $c_j\neq 0$, then

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)F(s)$$

is real for $s=\frac{1}{2}+it$. In a series of unpublished notes [16], Selberg proved a beautiful result on the zeros of $F(s)$. He derived a formula analogous to (1.1) for $F(s)$, and also showed that $N^{(0)}(T,F)>c(n)T\log T$ for $T>T_0(F)$, where $c(n)$ is a positive constant that depends only on n. Moreover, in those lectures, there is mention of the conjecture that almost all the zeros have real part equal to $\frac{1}{2}$.

To state our results, we need to introduce some further notations. For a fixed positive integer M, let us fix a vector $\vec{c}=(c_0,c_1,\cdots,c_M)$ such that $c_j\in\mathbb{R}$ for all j and define

$$c^*:=\sum_{j=0}^{M}\frac{(-1)^j c_j}{4^j}. \tag{1.4}$$

For all large numbers T, we set

$$L=\log\frac{T}{2\pi},\quad\text{and}\quad U=TL^{-10}.$$

Then for each positive integer a, we consider the function

$$F_{\vec{c},a,T}(s):=\sum_{j=0}^{M}\frac{c_j(-1)^j}{L^{2j}}\xi^{(a+2j)}(s). \tag{1.5}$$

The presence of L^j has the effect of balancing the size of $\xi^{(j)}(s)$ in $F_{\vec{c},a,T}(\mathrm{s})$, so that no one particular term dominates the entire combination. The constants appearing in the error terms may depend on the vector $\vec{c}$ throughout the paper.

Inspired by the result of Selberg and the techniques of Levinson and Conrey, our object of study in this note is the number of zeros of $F_{\vec{c},a,T}(s)$ on the critical line $\sigma=\frac{1}{2}$ with imaginary part between T and $T+U$. With this in mind, we define the counting functions $N_{\vec{c},a}(T)$ and $N_{\vec{c},a}^{(0)}(T)$ by

$$N_{\vec{c},a}(T) = \sum_{\substack{F_{\vec{c},a,T}(\rho)=0\\ 0<\operatorname{Im}\rho\le T}} 1, \quad \text{and} \quad N^{(0)}_{\vec{c},a}(T) = \sum_{\substack{F_{\vec{c},a,T}(\rho)=0\\ \operatorname{Re}\rho=1/2\\ 0<\operatorname{Im}\rho\le T}} 1.$$

Moreover, the proportion of zeros of $F_{\vec{c},a,T}(s)$ in the above rectangle on the critical line is given by the quotient

$$\kappa_{\vec{c},a,T} := \frac{N^{(0)}_{\vec{c},a}(T+U) - N^{(0)}_{\vec{c},a}(T)}{N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T)}. \tag{1.6}$$

Now we are ready to state our main result.

Theorem 1.1. *For any positive integer M, fix a vector $\vec{c} = (c_1, \cdots, c_M)$ with real components such that c^* as defined in (1.4) is nonzero. Also, for $F_{\vec{c},a,T}(s)$ defined in (1.5), let $\kappa_{\vec{c},a,T}$ be as in (1.6). Then*

$$\kappa_{\vec{c},a,T} \ge 1 - \frac{e^2+2}{16a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right), \tag{1.7}$$

as a and T tend to infinity such that

$$a \le \frac{1}{2}\frac{\log\log T}{\log\log\log T}.$$

Some comments are in order. The above result maintains the quality of the bounds and the uniformity achieved in [12,13]. The function $F_{\vec{c},a,T}(s)$ satisfies a functional equation given by

$$F_{\vec{c},a,T}(s) = (-1)^a F_{\vec{c},a,T}(1-s). \tag{1.8}$$

Moreover if all the zeros of $F_{\vec{c},a,T}(s)$ satisfy $\sigma_2 \le \operatorname{Re}(s) \le \sigma_1$ for some $\sigma_1, \sigma_2 \in \mathbb{R}$, then so do the zeros of all its higher order derivatives. This follows from the arguments developed in [12].

In particular, when the linear combination in (1.5) consists of a single term, then under the Riemann hypothesis, all derivatives of $\xi(s)$ will also have all their zeros on the critical line (see [3] and [12, Lemma 3]). The computations to follow show that the techniques from [3,10,12] can be applied in the same way to the function $F_{\vec{c},a,T}(s)$ defined in (1.5), and the proportion of zeros on the critical line tends to 1 in this case as well, even if, evidently, such functions $F_{\vec{c},a,T}(s)$ in general do not satisfy the Riemann hypothesis.

2. Zero free region, $N_{\vec{c},a}(T)$ and an inequality for $\kappa_{\vec{c},a,T}$

We first obtain a zero free region for our function $F_{\vec{c},a,T}(s)$ and then we find an asymptotic formula for the number of zeros $N_{\vec{c},a}(T)$. Observe that the zeros of $F_{\vec{c},a,T}(s)$ are the same as the zeros of the function

$$F_1(s) := \frac{i^a}{L^a}\sum_{j=0}^{M} \frac{c_j(-1)^j}{L^{2j}}\xi^{(a+2j)}(s)$$

and so we use the above function $F_1(s)$ to perform our computations. Let $s = \sigma + it$ with $\sigma > 1$ and $T \le t \le T+U$. For some $\beta_{\vec{c}} \gg a$, we now prove that

$$F_{\vec{c},a,T}(s) \ne 0 \quad \text{whenever} \quad \sigma_{\vec{c}} > \beta_{\vec{c}} \quad \text{or} \quad \sigma_{\vec{c}} < 1 - \beta_{\vec{c}}.$$

From equation (20) of [12], we have

$$F_1(s) = H(s)\sum_{j=0}^{M} \frac{c_j i^{a+2j}}{L^{a+2j}} \left(\frac{1}{2}\log\frac{s}{2\pi}\right)^{a+2j} (1 + R_{a+2j,\vec{c}}(s)), \tag{2.1}$$

where $H(s) = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$, and the remainder term $R_{a+2j,\vec{c}}$ is given by

$$R_{a+2j,\vec{c}}(s) = \zeta(s) - 1 + \sum_{l=1}^{a+2j} \binom{a+2j}{l} \left(\frac{1}{2}\log\frac{s}{2\pi}\right)^{-l} \zeta^{(l)}(s)\left(1 + O\left(\frac{1}{\log^2 t}\right)\right) + O\left(\frac{1}{\log^2 t}\right).$$

By equation (23) of [12], $|R_{a+2j,\vec{c}}| < 1/2$ for $\beta_{\vec{c}} \gg a$. Using this in (2.1), we conclude that

$$F_1(s) \neq 0 \quad \text{for} \quad \operatorname{Re}(s) > \beta_{\vec{c}},$$

which in turn implies from our earlier discussion about the zeros of $F_1(s)$ and $F_{\vec{c},a,T}$ being the same that

$$F_{\vec{c},a,T}(s) \neq 0 \quad \text{for} \quad \operatorname{Re}(s) > \beta_{\vec{c}}. \tag{2.2}$$

The functional equation (1.8) yields,

$$F_{\vec{c},a,T}(s) = (-1)^a F_{\vec{c},a,T}(1-s) \neq 0 \quad \text{for} \quad \operatorname{Re}(s) < 1 - \beta_{\vec{c}}. \tag{2.3}$$

This completes the argument for $F_{\vec{c},a,T}(s) \neq 0$ for $\operatorname{Re}(s) > \beta_{\vec{c}}$ or when $\operatorname{Re}(s) < 1 - \beta_{\vec{c}}$.

In a standard way and using the arguments above, one can compute $N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T)$, the number of zeros with imaginary part between T and $T+U$ with $U = T/L^{10}$, as claimed in the lemma below for which we omit the details. Note that it is enough to count the zeros in the rectangle with vertices as $\beta_{\vec{c}} + iT$, $1 - \beta_{\vec{c}} + iT$, $\beta_{\vec{c}} + i(T+U)$, and $1 - \beta_{\vec{c}} + i(T+iU)$. More precisely, one has the following.

Lemma 2.1. *For $T \gg 0$, and $0 < U \leq T$, we have*

$$N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T) = \left(\frac{T+U}{2\pi}\log\frac{T+U}{2\pi} - \frac{T+U}{2\pi}\right) - \left(\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}\right) + O_{\vec{c}}\left(a\log T\right).$$

This implies

$$N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T) = \frac{UL}{\pi} + O_{\vec{c}}\left(aU\right).$$

We now state an inequality satisfied by $\kappa_{\vec{c},a,T}$ involving zeros of a certain arithmetic function $V(s)$ which can be proved using Lemma 2.1 and arguments from [12].

Lemma 2.2. *With the notations as above, one has*

$$\kappa_{\vec{c},a,T} \geq 1 - \frac{4\pi}{UL}N + O_{\vec{c}}\left(\frac{a}{U}\right),$$

where N is the number of zeros of $V(s)$ inside the rectangular contour $\mathcal{R} = [1/2 + iT, \mathcal{B} + iT, \mathcal{B} + i(T+U), 1/2 + i(T+U)]$ of the function $V(s)$, for some $\mathcal{B}$ such that $\mathcal{B} > \beta_c$ defined as

$$V(s) = \frac{2^a}{H(s)}\sum_{j=0}^{M}\frac{c_j i^{2j}}{L^{a+2j}}\left(\sum_{m=0}^{a+2j}\binom{a+2j}{m}H^{(m)}(s)\int_{0\swarrow 1}\frac{z^{-s}e^{\pi i z^2}}{2i\sin(\pi z)}(-\log z)^{a+2j-m}\left(1 - \frac{\log z}{L}\right)\,dz\right.$$
$$\left. + \sum_{m=0}^{a+2j}(-1)^m\binom{a+2j}{m}H^{(m)}(1-s)\int_{0\searrow 1}\frac{z^{s-1}e^{-\pi i z^2}}{2i\sin(\pi z)}(\log z)^{a+2j-m}\frac{\log z}{L}\right)\,dz, \tag{2.4}$$

where, as before $H(s)$ is given by

$$H(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

The notation $\int_{0\searrow 1}$ denotes an integral along a line directed from the upper right to lower left which is inclined at an angle of $\pi/4$ to the real axis and intersects it between 0 and 1, see [17] and [18, §2.10].

3. An upper bound for N

In this section we prove the following lemma which gives an upper bound on N, the number of zeros of $V(s)$, defined in (2.4), inside the contour.

Lemma 3.1. *Let N be as in the previous lemma. Then the following inequality holds*

$$N \leq \frac{UL}{2\pi}\log\left(\frac{I}{U}\right) + O_{\vec{c}}(aU).$$

Here

$$I = \frac{1}{|c^*|}\int_T^{T+U} |\psi B(\sigma_a + it) + \chi^* \psi D(\sigma_a + it)|dt + O\left(\frac{U}{L^{9/2}}\right),$$

with

$$\chi^*(t) = e^{1+i\left(\frac{\pi}{4} - t\log\left(\frac{t}{2\pi e}\right)\right)}, \quad \sigma_a = 1/2 - 1/L,$$

$$B(s) = \sum_{j=0}^{M} C_j B_j(s); \quad D(s) = \sum_{j=0}^{M} C_j D_j(s)$$

$$C_j := \frac{c_j(-1)^j}{4^j}, \quad B_j(s) := \sum_{n\leq\sqrt{\frac{T}{2\pi}}} \left(1 + \frac{\pi i}{2L} - \frac{2\log n}{L}\right)^{a+2j}\left(1 - \frac{\log n}{L}\right)n^{-s},$$

$$D_j(s) := \sum_{n\leq\sqrt{\frac{T}{2\pi}}} \frac{\log n}{L}\left(\frac{2\log n}{L} + \frac{\pi i}{2L} - 1\right)^{a+2j} n^{s-1},$$

and

$$\psi(s) = \sum_{n\leq y}\frac{a(n)}{n^s} \quad \text{with} \quad a(n) = \frac{\mu(n)}{n^{1/L}}h\left(\frac{\log y/n}{\log y}\right),$$

and h is some polynomial satisfying $h(0) = 0$ as well as $h(1) = 1$ and here $y = T^{1/2}L^{-20}$.

Proof. Notice that N, the number of zeros of $V(s)$, as in (2.4), inside the contour $\mathcal{R}$, is less than the number of zeros of $V(s)\psi(s)$ therein where $\psi(s)$ is a mollifying function which on average approximates the behavior of inverse of the function $F_{\vec{c},a,T}(s)$. This mollifier is defined in the following way

$$\psi(s) = \sum_{n\leq y}\frac{a(n)}{n^s}$$

where

$$a(n) = \frac{\mu(n)}{n^{1/L}} h\left(\frac{\log y/n}{\log y}\right)$$

and h is a polynomial satisfying $h(0) = 0$ and $h(1) = 1$ to be chosen later. Therefore to bound N, we bound the number of zeros of $\frac{1}{c*}V(s)\psi(s)$. For this we shall apply Littlewood's lemma [10] to $\frac{1}{c*}V(s)\psi(s)$ on the rectangular contour $\Omega = [\sigma_a + iT, \sigma_1 + iT, \sigma_1 + i(T+U), \sigma_a + i(T+U)]$, where $\sigma_1 = \log L/\log 2$, $\sigma_a = 1/2 - 1/L$. Littlewood's lemma gives

$$2\pi i \sum_{\rho=\beta+i\gamma} (\beta - \sigma_a) = -\oint_{\Omega} \log\left(\frac{1}{c^*}\psi(s)V(s)\right) ds, \tag{3.1}$$

where the summation is performed over all the zeros ρ of $V(s)\psi(s)$ inside Ω and on its upper side. Using estimates from [12, §3] we get approximations for our integral in (3.1) along the right and horizontal sides of the contour Ω. In doing so, we note that

$$\left|\frac{1}{c^*}V(s)\psi(s) - 1\right| = O_{\vec{c}}\left(\frac{a}{L}\right)$$

on the right side of the contour. This implies that the change in argument of $\dfrac{1}{c^*}V(s)\psi(s)$ is bounded by π in absolute value on this side. Now the number of zeros of the product $V(s)\psi(s)$ in a larger domain Ω is greater than or equal to the number of zeros of $V(s)$ in a smaller domain $\mathcal{R}$, the imaginary part of the left hand side of (3.1) is at least $\dfrac{N}{L}$. Putting all these facts together, we conclude

$$N \le \frac{L}{2\pi} \int_T^{T+U} \log\left|\frac{1}{c^*}\psi(\sigma_a + it)V(\sigma_a + it)\right| dt + O_{\vec{c}}(aU).$$

Finally using Jensen's inequality, we get an expression for the number of zeros in terms of an integral as

$$N \le \frac{UL}{2\pi} \log\left(\frac{I}{U}\right) + O_{\vec{c}}(aU), \tag{3.2}$$

where

$$I = \frac{1}{|c^*|} \int_T^{T+U} |\psi(\sigma_a + it)V(\sigma_a + it)| dt. \tag{3.3}$$

As in [12], we first write $V(\sigma_a + it)$ as

$$V(\sigma_a + it) = \tilde{B}(\sigma_a + it) + \chi(\sigma_a + it)\tilde{D}(\sigma_a + it),$$

where

$$\tilde{B}(\sigma_a + it) = B(\sigma_a + it) + \sum_{j=0}^{M} \frac{c_j i^{a+2j}}{2^{a+2j}} \Bigg(\sum_{n \le \sqrt{\frac{T}{2\pi}}} \Bigg(\left(\frac{\log(t/2\pi)}{L} + \frac{\pi i}{2L} - \frac{2\log n}{L}\right)^{a+2j} - \left(1 + \frac{\pi i}{2L} - \frac{2\log n}{L}\right)^{a+2j} \Bigg)$$

$$\times\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\Bigg)$$
$$+\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\Bigg(\sum_{\sqrt{\frac{T}{2\pi}}\le n\le\sqrt{\frac{t}{2\pi}}}\left(\frac{\log(t/2\pi)}{L}+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\Bigg)$$
$$+O(aT^{-3/4}), \tag{3.4}$$

with

$$B(\sigma_a+it)=\sum_{j=0}^{M}C_j\cdot B_j:=\sum_{j=0}^{M}\frac{c_j i^{2j}}{2^{2j}}\cdot\Bigg(\sum_{n\le\sqrt{\frac{T}{2\pi}}}\left(1+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\Bigg),$$

and $\tilde{D}(\sigma_a+it)$ is the sum

$$\sum_{j=0}^{M}\frac{c_j i^{2j}2^a}{L^{a+2j}}\sum_{n\le\sqrt{\frac{t}{2\pi}}}\left(\log n-\frac{1}{2}\log\left(\frac{1-\sigma_a-it}{2\pi}\right)+O\left(\frac{1}{|1-t|}\right)\right)^{a+2j}\left(\frac{\log n}{L}\right)n^{\sigma_a+it-1}.$$

Here

$$\chi(s)=\frac{H(1-s)}{H(s)}.$$

Let

$$D(\sigma_a+it)=\sum_{j=0}^{M}C_j\cdot D_j:=\sum_{n\le\sqrt{\frac{T}{2\pi}}}\frac{c_j i^{2j}}{2^{2j}}\cdot\frac{\log n}{L}\left(\frac{2\log n}{L}+\frac{\pi i}{2L}-1\right)^{a+2j}n^{\sigma_a+it-1}$$

Using the above notations in (3.3) we obtain

$$I=\frac{1}{|c^*|}\int_T^{T+U}|\psi(\sigma_a+it)V(\sigma_a+it)|\,dt$$
$$=\frac{1}{|c^*|}\int_T^{T+U}|\psi B(\sigma_a+it)+\chi^*\psi D(\sigma_a+it)|dt+O\left(\frac{U}{L^{9/2}}\right),$$

where

$$\chi^*(t)=e^{1+i(\frac{\pi}{4}-t\log(\frac{t}{2\pi e}))}.$$

Thus using the above inequality and (3.2), the number of zeros is bounded by

$$N\le\frac{UL}{2\pi}\log\left(\frac{I}{U}\right)+O_{\vec{c}}(aU),$$

which proves the lemma. □

4. Proportion of zeros on the critical line

We now apply the results of the previous two sections to obtain the proportion of zeros of $F_{\vec{c},a,T}(s)$ on $\mathrm{Re}(s) = \frac{1}{2}$. As we shall see this proportion tends to 1, at a speed independent of the vector $\vec{c}$. We start by denoting

$$\frac{J}{U} := \frac{1}{|c^*|^2} \sum_{j,l=0}^{M} C_j C_l A_{j,l}, \tag{4.1}$$

where

$$A_{j,l} = \frac{1}{U} \int_{T}^{T+U} (B_j + \chi^* D_j)\psi(\sigma_a + it)(\overline{B}_l + \overline{\chi}^* \overline{D_l \psi}(\sigma_a + it))dt. \tag{4.2}$$

As in [3] and [12], using the Cauchy–Schwarz equality and Lemma 3.1, we conclude that

$$N \leq \frac{UL}{4\pi} \log\left(\frac{J}{U}\right) + O_{\vec{c}}(aU). \tag{4.3}$$

Let

$$\phi_k(x) := (1-x)\left(1 - 2x + \frac{\pi i}{2L}\right)^{a+2k} \quad \text{for } 0 \leq k \leq M.$$

Using simplifications as in [12] we express the integral in (4.2) as the sum

$$\begin{aligned} A_{j,l} = \sum_{n_3,n_4 \leq y} \frac{a(n_3)\overline{a(n_4)}}{(n_3 n_4)^{2\sigma_a}} {m^*}^{2\sigma_a} \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \Bigg(\sum_{v=0}^{n} \binom{n}{v} {\phi_j}^{(v)}\left(\frac{1}{L}\log\left(\frac{xn_4}{m^*}\right)\right) \\ \times \overline{{\phi_l}^{(n-v)}\left(\frac{1}{L}\log\left(\frac{xn_3}{m^*}\right)\right)} x^{-2a} \Bigg)\Bigg|_1^{\frac{T}{2\pi}\frac{{m^*}^2}{n_3 n_4}} + O_{\vec{c}}\left(\frac{a}{L}\right), \end{aligned} \tag{4.4}$$

with $m^* = \gcd(n_3, n_4)$. Let

$$H_a = \int_0^1 h^2(x)dx \quad \text{and} \quad H'_a = \int_0^1 {h'}^2(x)dx,$$

where the polynomial $h(x)$ satisfying $h(0) = 0$ and $h(1) = 1$ is chosen as

$$h(x) = \sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c_r} x^r, \tag{4.5}$$

such that $\tilde{c_r} \in \mathbb{R}$ and $\tilde{a} = a - X \leq a \leq a + 2M \leq a + X = \tilde{b}$ for some constant $X = X(M)$. Using the Taylor series expansion of $\phi_j^{(v)}(x)$ and $\phi_l^{(n-v)}(x)$ at $x = 1$ and at $x = 0$, and generalizations of Lemmas 18 and 20 from [12], (4.4) becomes

$$
\begin{aligned}
A_{j,l} &= \frac{1}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\left(e^2(\phi_j(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=1} - (\phi_j(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\right)\left(\frac{H_a}{2}+2H_a'\right)\\
&+\frac{1}{2}\sum_{n=0}^{j+l+2}\left(e^2(\phi_j(x)\overline{\phi_l{}'(x)}+\phi_j{}'(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=1} - (\phi_j(x)\overline{\phi_l{}'(x)}+\phi_j{}'(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\right)\frac{H_a}{2}\\
&+\frac{1}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\left(e^2(\phi_j{}'(x)\overline{\phi_l'(x)})^{(n)}\Big|_{x=1} - (\phi_j{}'(x)\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=0}\right)\frac{H_a}{2}\\
&+\sum_{r,m=\tilde{a}}^{\tilde{b}}\tilde{c}_r\tilde{c}_m\mathcal{U} + O((2a)^{2a+4}L^{-1}\log^5 L),
\end{aligned}
$$

where $\mathcal{U}$ is defined by

$$
\begin{aligned}
\mathcal{U} &:= \frac{e^2}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\times\\
&\times\left((\phi_j(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1} + (\phi_j(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=1}\frac{r}{r+m} + (\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}\frac{m}{r+m}\right)\\
&+\frac{1}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\\
&\times\left((\phi_j(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=0} + (\phi_j(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=0}\frac{r}{r+m} + (\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\frac{m}{r+m}\right).
\end{aligned}
\tag{4.6}
$$

Lemma 28 from [12] and some simplifications yield

$$
A_{j,l} = \frac{1}{2}\left(\left(4H_a' - H_a\right)\Phi_a + H_a\left(\Phi_a' - 1\right) + \sum_{r,m=\tilde{a}}^{\tilde{b}} 2\tilde{c}_r\tilde{c}_m\mathcal{U}\right) + O_{\vec{c}}\left(\frac{(2a)^{2a+4}\log^5 L}{L}\right), \tag{4.7}
$$

where

$$
\Phi_a = \int_0^1 e^{2x}\phi^2(x)dx, \quad \text{and} \quad \Phi_a' = \int_0^1 e^{2x}\phi'^{\,2}(x)dx,
$$

where

$$
\phi(x) = \sum_{j=0}^{M}(1-x)(1-2x)^{a+2j}.
$$

We write a formula for Φ_a using repeated integration by parts and obtain,

$$
\begin{aligned}
&\int_0^1 e^{2x}(1-x)^2(1-2x)^{2a+2j+2l}\,dx\\
&= \frac{1}{2(2a+2j+2l)} - \frac{1}{2(2a+2j+2l)^2} + \frac{(e^2+1)}{4(2a+2j+2l)^3} + O\left(a^{-4}\right).
\end{aligned}
\tag{4.8}
$$

For Φ_a', we have

$$\Phi'_a = \frac{2(a+2j)(a+2l)}{2a+2j+2l} + \frac{2(a+2j)(a+2l)}{(2a+2j+2l)^2} + 1 + \left(\frac{(a+2j)(a+2l)}{(2a+2j+2l)^2} + 1 + \frac{(a+2j)(a+2l)(e^2-1)}{(2a+2j+2l)^2}\right)\frac{1}{2a+2j+2l} + \left(\frac{-2(3e^2+1)(a+2j)(a+2l)}{(2a+2j+2l)^2} + e^2 - 1 + \frac{4(a+2j)(a+2l)}{(2a+2j+2l)^2}\right)\frac{1}{2(2a+2j+2l)^2} + O(a^{-3}). \tag{4.9}$$

We now expand the terms involving r and m in (4.7), and substitute (4.8), (4.9) in (4.7) to obtain

$$\begin{aligned} A_{j,l} = \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m \mathcal{U} &+ \frac{(rm+(a+2j)(a+2l))}{(2a+2j+2l)(r+m)} + \frac{(rm-(a+2j)(a+2l))}{(2a+2j+2l)(r+m)}\left(\frac{1}{r+m} - \frac{1}{2a+2j+2l}\right) \\ &+ \frac{rm}{(2a+2j+2l)(r+m)^3} - \frac{rm}{(2a+2j+2l)^2(r+m)^2} + \frac{(e^2+1)rm}{2(2a+2j+2l)^3(r+m)} \\ &- \frac{1}{4(2a+2j+2l)(r+m)} + \frac{(a+2j)(a+2l)}{(2a+2j+2l)(r+m)^3} - \left(\frac{2jl}{(2a+2j+2l)^2}+1\right)\frac{1}{2(r+m)^2} \\ &+ \left(\frac{2(a+2j)(a+2l)}{(2a+2j+2l)^2} + 1 + \frac{(a+2j)(a+2l)(e^2-1)}{(2a+2j+2l)^2}\right)\frac{1}{2(2a+2j+2l)(r+m)} \\ &+ \left.\frac{1}{2(r+m)^2}\right) + O_{\vec{c}}\left(\frac{(2a)^{2a+4}\log^5 L}{L}\right). \end{aligned} \tag{4.10}$$

We compute the term corresponding to $\mathcal{U}$ in $A_{j,l}$ separately below.

$$\begin{aligned} \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m\mathcal{U} = {} & \frac{e^2}{2}\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c}_r^2 \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n}\left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{1}{2}(\phi_j(x)\overline{\phi_l}'(x))^{(n)} + ({\phi_j}'(x)\overline{\phi_l}(x))^{(n)}\right)\Big|_{x=1} \\ & + \frac{e^2}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\sum_{\substack{m=\tilde{a}\\ m\neq r}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m \sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n} \\ & \times\left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{r}{r+m}(\phi_j(x)\overline{\phi_l}'(x))^{(n)} + \frac{m}{r+m}({\phi_j}'(x)\overline{\phi_l}(x))^{(n)}\right)\Big|_{x=1} \\ & + \frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c}_r^2 \sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n} \\ & \times\left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{1}{2}((\phi_j(x)\overline{\phi_l}'(x))^{(n)} + ({\phi_j}'(x)\overline{\phi_l}(x))^{(n)}\right)\Big|_{x=0} \\ & + \frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\sum_{\substack{m=\tilde{a}\\ m\neq r}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m \sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n} \\ & \times\left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{r}{r+m}(\phi_j(x)\overline{\phi_l}'(x))^{(n)} + \frac{m}{r+m}({\phi_j}'(x)\overline{\phi_l}(x))^{(n)}\right)\Big|_{x=0}. \end{aligned} \tag{4.11}$$

Using Lemmas 1 and 29 from [12], we combine first and third terms together, and pair the second and fourth terms to yield

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m\mathcal{U} = \frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c}_r^2 + \frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\sum_{\substack{m=\tilde{a}\\ m\neq r}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m + O_{\vec{c}}\left(\frac{a}{L}\right)$$

$$= \frac{1}{2}\left(\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c}_r\right)^2 + O_{\vec{c}}\left(\frac{a}{L}\right) = \frac{1}{2} + O_{\vec{c}}\left(\frac{a}{L}\right). \tag{4.12}$$

Next, we focus on the sums not involving $\mathcal{U}$ occurring in (4.10). Let $r = a+u$, $m = a+v$ for $-X \leq u, v \leq X$. Note that

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m \frac{(rm-(2j+a)(2l+a))}{(2j+2l+2a)(r+m)}\left(\frac{1}{r+m} - \frac{1}{2j+2l+2a}\right)$$

$$= \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m\left(\frac{(a+u)(a+v)-(2j+a)(2l+a)}{(2j+2l+2a)(2a+u+v)}\left(\frac{1}{2a+u+v} - \frac{1}{2a+2j+2l}\right)\right)$$

$$= \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m\left(\frac{a^2(1+\frac{u}{a})(1+\frac{v}{a}) - a^2(1+\frac{2j}{a})(1+\frac{2l}{a})}{4a^2(1+\frac{j+l}{a})(1+\frac{u+v}{2a})}\right)\frac{1}{2a}\left(\frac{1}{1+\frac{u+v}{2a}} - \frac{1}{1+\frac{j+l}{a}}\right)$$

$$= O_{\vec{c}}\left(\frac{1}{a^3}\right), \tag{4.13}$$

by expanding with the use of geometric series. Similarly

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m\Bigg(\frac{rm}{(2j+2l+2a)(r+m)^3} - \frac{rm}{(2j+2l+2a)^2(r+m)^2} + \frac{(e^2+1)rm}{(2j+2l+2a)^3 2(r+m)}$$

$$- \frac{1}{4(2j+2l+2a)(r+m)} + \frac{(2j+a)(2l+a)}{(2j+2l+2a)(r+m)^3} - \left(\frac{2(2j+a)(2l+a)}{(2j+2l+2a)^2}+1\right)\frac{1}{2(r+m)^2}$$

$$+ \left(\frac{2(2j+a)(2l+a)}{(2j+2l+2a)^2} + 1 + \frac{(2j+a)(2l+a)(e^2-1)}{(2j+2l+2a)^2}\right)\frac{1}{2(2j+2l+2a)(r+m)}$$

$$+ \frac{1}{2(r+m)^2}\Bigg) + O_{\vec{c}}\left(\frac{1}{a^3}\right) = \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r\tilde{c}_m\left(\frac{e^2+2}{16}\right)\frac{1}{a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right)$$

$$= \frac{e^2+2}{16}\frac{1}{a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.14}$$

Note that we have employed $h(1) = 1$ in (4.10), (4.12), (4.13) and (4.14). As before substituting $r = a+u$, $m = a+v$ for $-X \leq u, v \leq X$.

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \frac{\tilde{c}_r\tilde{c}_m(rm+(2j+a)(2l+a))}{(2j+2l+2a)(r+m)} = \sum_{-X\leq u,v\leq X} \tilde{c}_{a+u}\tilde{c}_{a+v}\frac{2a^2+a(u+v+2j+2l)+(uv+4jl)}{4a^2(1+\frac{2j+2l}{2a})(1+\frac{u+v}{2a})}.$$

Expanding the denominators using power series, we see that the above expression reduces to

$$\sum_{-X\leq u,v\leq X} \tilde{c}_{a+u}\tilde{c}_{a+v}\left(\frac{1}{2} + \left(\frac{-(u+v)(2j+2l)}{4} + \frac{uv+4jl}{4}\right)\frac{1}{a^2}\right) + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

Simplifying this term by term, we obtain

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \frac{\tilde{c}_r\tilde{c}_m(rm+(2j+a)(2l+a))}{(2j+2l+2a)(r+m)} = \frac{1}{2}\sum_{-X\le u,v\le X}\tilde{c}_{a+u}\tilde{c}_{a+v} + \frac{jl}{a^2}\sum_{-X\le u,v\le X}\tilde{c}_{a+u}\tilde{c}_{a+v}$$
$$-\frac{(j+l)}{2a^2}\sum_{-X\le u,v\le X}\tilde{c}_{a+u}\tilde{c}_{a+v}(u+v)$$
$$+\frac{1}{4a^2}\sum_{-X\le u,v\le X}\tilde{c}_{a+u}\tilde{c}_{a+v}uv + O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.15}$$

Now let

$$\mathcal{S} = \sum_{-X\le t\le X} t\tilde{c}_{a+t}. \tag{4.16}$$

Since $h(1) = 1$ equation (4.15) becomes

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \frac{\tilde{c}_r\tilde{c}_m(rm+(2j+a)(2l+a))}{(2j+2l+2a)(r+m)} = \frac{1}{2} + (4jl - 2(j+l)\mathcal{S} + \mathcal{S}^2)\frac{1}{4a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.17}$$

Collecting the simplified expressions obtained in (4.12), (4.13), (4.14) and (4.17) and substituting them in the expression for $A_{j,l}$ as in (4.2), we arrive at

$$A_{j,l} = 1 + (4jl - 2(j+l)\mathcal{S} + \mathcal{S}^2)\frac{1}{4a^2} + \left(\frac{e^2+2}{16}\right)\frac{1}{a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

Also recall from (4.1) that

$$\frac{J}{U} = \frac{1}{|c^*|^2}\sum_{j,l=0}^{M} C_jC_lA_{j,l}.$$

On substituting $A_{j,l}$ here and using the definition of c^*, we obtain

$$\frac{J}{U} = 1 + \frac{e^2+2}{16a^2} + \left(4\sum_{j,l=0}^{M} C_jC_ljl - 2\sum_{j,l=0}^{M} C_jC_l(j+l)\mathcal{S} + |c^*|^2\mathcal{S}^2\right)\frac{1}{|c^*|^2 4a^2}$$
$$+ O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.18}$$

Here we would like to point out to the reader that although one has flexibility in choosing $\mathcal{S}$, the expression inside the parenthesis on the right side of (4.18) cannot be decreased below zero. For example, it is 1/4 when one considers $F_{\vec{c},a,T}(s) = \xi^{(a)}(s)$ and chooses $h(x)$ in (4.5) to be $\frac{x^{a-1}}{2} + \frac{x^a}{2}$, in which case

$$\frac{J}{U} = 1 + \frac{e^2+3}{16a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

For $F_{\vec{c},a,T}(s) = \xi^{(a)}(s) + \xi^{(a+2)}(s)/L^2$ and $h(x) = \frac{x^{a-2}}{4} + \frac{x^a}{2} + \frac{x^{a+1}}{4}$, one obtains $\mathcal{S} = -\frac{1}{4}$, $C_0 = 1$, $C_1 = 1/4$, $c^* = 5/4$ and the coefficient attached to $\frac{1}{a^2}$ in $\frac{J}{U}$ in (4.18) becomes $\frac{e^2+2}{16} + \frac{169}{1600}$.

The minimum of the expression inside the parenthesis on the right side of (4.18) is actually zero, and is attained at

$$\mathcal{S} = \frac{1}{|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l). \tag{4.19}$$

Let us also remark that our mollifier does allow one to arrange for such a condition to hold: there exist coefficients $\tilde{c}_{a+t}$ such that the minimum is attained and such that $\tilde{c}_{a+t}$ also satisfies

$$\sum_{-X \leq t \leq X} \tilde{c_{a+t}} = 1. \tag{4.20}$$

For example, if one chooses constants $\tilde{c}_{a+t} = 0$ for $-X \leq t \leq -2$, $\tilde{c}_a = 1$, $\tilde{c}_{a+t} = 0$ for $2 \leq t \leq X$, and

$$\tilde{c}_{a+1} = \frac{1}{2|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l), \quad \tilde{c}_{a-1} = \frac{-1}{2|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l),$$

then

$$\begin{aligned} \mathcal{S} &= \sum_{t=-X}^{X} t\tilde{c}_{a+t} = -\tilde{c}_{a-1} + \tilde{c}_{a+1} = -\frac{-1}{2|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l) + \frac{1}{2|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l) \\ &= \frac{1}{|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l). \end{aligned}$$

Therefore, in this example, with the choice of coefficients $\tilde{c}_j$ as above, the minimum is attained and (4.20) also holds true.

Finally, we compute the expression for J/U in (4.18) by substituting the minimum value of $\mathcal{S}$ from (4.19) in (4.18) and arrive at

$$\begin{aligned} \frac{J}{U} &= 1 + \frac{e^2+2}{16a^2} + \frac{1}{a^2|c^*|^2} \sum_{j,l=0}^{M} C_j C_l jl - \frac{1}{4a^2}\left(\frac{1}{|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l)\right)^2 \\ &\quad + O_{\vec{c}}\left(\frac{1}{a^3}\right). \end{aligned}$$

The two sums involving C_j, C_l cancel each other since

$$\begin{aligned} \left(\frac{1}{|c^*|^2} \sum_{j,l=0}^{M} C_j C_l (j+l)\right)^2 &= \left(\frac{1}{|c^*|^2} \sum_{j,l=0}^{M} jC_j C_l + \frac{1}{|c^*|^2} \sum_{j,l=0}^{M} lC_j C_l\right)^2 \\ &= \frac{4}{|c^*|^4}\left(\sum_{j,l=0}^{M} jC_j C_l\right)^2 = \frac{4}{|c^*|^4}\left(\sum_{j=0}^{M} jC_j\right)^2 \left(\sum_{l=0}^{M} C_l\right)^2 \\ &= \frac{4(-1)^a}{|c^*|^4}\left(\sum_{j=0}^{M} jC_j\right)^2 |c^*|^2 = \frac{4(-1)^a}{|c^*|^2}\left(\sum_{j=0}^{M} jC_j\right)^2 \\ &= \frac{1}{|c^*|^2} \sum_{j,l=0}^{M} 4C_j C_l jl. \end{aligned}$$

Consequently, this yields

$$\frac{J}{U} = 1 + \frac{e^2+2}{16a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

Finally, by putting (4.3), Lemmas 2.2 and 3.1 together, we complete the proof of Theorem 1.1.

Remark 4.1. Using the same techniques, one can get a similar result on the proportion of simple zeros of $F_{\vec{c},a,T}(s)$ on the critical line.

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