ON THE PARITY OF BROKEN k-DIAMOND PARTITIONS

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Abstract. In this paper, we prove several new parity results for broken k-diamond partitions on certain types of arithmetic progressions. We also obtain bounds for the parity of broken k-diamond partitions and more general colored partitions.

1. Introduction

Ramanujan's beautiful work on congruences for the partition function has inspired many mathematicians to further explore this area. Andrews, Paule and Riese [2] introduced "partition diamonds" - a new variation of plane partitions studied by MacMahon in his book "Combinatory Analysis" [10]. For plane partitions, the non-negative integer parts a_i satisfy the relations

$$a_1 \geq a_2, a_1 \geq a_3, a_2 \geq a_4 \text{ and } a_3 \geq a_4. \tag{1.1}$$

Pictorially, one can represent this as a directed graph with the arrows describing the relation "$\geq$". The parts a_i are placed at the vertices of a square and the arrow pointing from a_i to a_j depicts the relation $a_i \geq a_j$. For instance, Figure 1 represents the relations given in

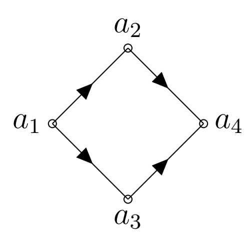

Figure 1

(1.1). For the generating function for such partitions, MacMahon considered the generating series

$$\sum_{a_1,a_2,a_3,a_4} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} = \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},$$

the sum being extended to all quadruples satisfying (1.1). Hence after substituting q to each x_j, the closed form of the generating function of the number $A(n)$ of such quadruples is given by

$$\sum_{n=0}^{\infty} A(n) q^n = \frac{1}{(1-q)(1-q^2)^2(1-q^3)} \quad (|q| < 1).$$

Instead of using squares as building blocks of the chain, Andrews and Paule [1] considered k-elongated diamonds of length m. A k- elongated diamond of length 1 and length m are

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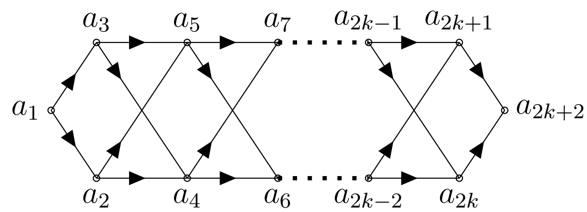

FIGURE 2

shown below in Figure 2 and Figure 3 respectively. A broken k-diamond [Figure 4] consists

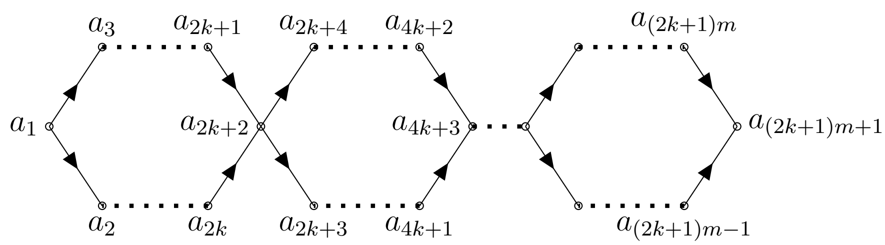

FIGURE 3

of two separated k-elongated diamond partitions, each of length m, where in one of them the source (vertex with no incoming arrows) is deleted. For an integer $k \geq 0$, the number

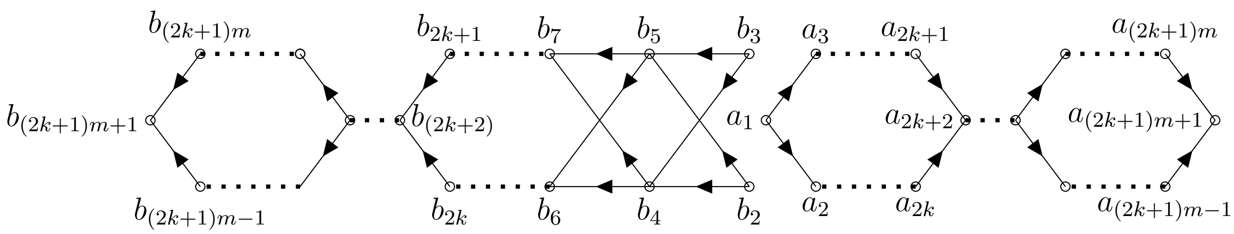

FIGURE 4

of broken k-diamond partitions of a non-negative integer n is denoted by $\Delta_k(n)$ and its generating function in [1] is given by

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}, \ |q|<1. \tag{1.2}$$

Throughout the paper, we assume $|q| < 1$. Andrews and Paule [1] made the following conjectures about congruences satisfied by the broken 2-diamond partitions. For all non-negative integers n,

$$\Delta_2(10n+2) \equiv 0 \pmod 2, \tag{1.3}$$

$$\Delta_2(25n+14) \equiv 0 \pmod 5, \tag{1.4}$$

and

$$\Delta_2(625n+314) \equiv 0 \pmod{5^2}. \tag{1.5}$$

Hirschhorn and Sellers [9] proved (1.3) and more, including the following congruences. For all non-negative integers n,

$$\Delta_1(4n+2) \equiv 0 \pmod 2,$$
$$\Delta_1(4n+3) \equiv 0 \pmod 2,$$

and

$$\Delta_2(10n+6) \equiv 0 \pmod 2.$$

Conjecture (1.4) was proved by Chan [6] who also gave a more general result involving higher powers of 5 in the modulus. Radu and Sellers [11] extended this list of congruences significantly. Their congruences concerned the parity of broken k-diamond partitions along arithmetic progressions of the form $an + b$, where $a = 4k + 2$.

In this paper, we discuss bounds on the parity of broken k-diamond partitions and prove congruences modulo 2, including some ones for arithmetic progressions of the form $M(2k+1)n + b$ for $M = 4$ and $M = 8$. Notice that the case $M = 2$ has been considered by Radu and Sellers [11].

We now state our main results where simultaneous congruences $\Delta_k(an+b_i) \equiv 0 \pmod 2$, $1 \le i \le l$ are denoted as $\Delta_k(an + b_1, b_2, \dots, b_l) \equiv 0 \pmod 2$.

Theorem 1.1. *For any non-negative integer n,*

$$\Delta_9(76n + 11, 15, 27, 39, 43, 47, 51, 59, 67) \equiv 0 \pmod 2, \tag{1.6}$$
$$\Delta_{15}(124n + 10, 26, 30, 38, 42, 50, 54, 58, 62, 78, 86, 94, 98, 102, 122) \equiv 0 \pmod 2. \tag{1.7}$$

Theorem 1.2. *For any non-negative integer n,*

$$\Delta_{12}(10n + 9) \equiv 0 \pmod 2.$$

Theorem 1.3. *For any non-negative integer k, and for N large enough,*

$$\#\{n \le N : \Delta_k(n) \text{ is odd}\} \ge N^{\frac{1}{2} - \frac{c}{\log\log N}},$$

for some positive real number c.

For $k, a, N \in \mathbb{N}$ and $b \in \mathbb{Z}_{\ge 0}$, define $\rho_k(a, b, N)$ to be the density of even values of $\Delta_k(an+b)$ for n up to N, that is,

$$\rho_k(a, b, N) := \frac{\#\{n \in \{0, ..., N-1\} : \Delta_k(an + b) \equiv 0 \pmod 2\}}{N}.$$

For example, from (1.6), $\rho_9(76, 11, N) = 1$ for each N.

A natural question that arises is whether more generally the limit $\lim\limits_{N\to\infty} \rho_k(a, b, N)$ exists for any choice of k, a and b.

Question: Is it true that for any $k, a \in \mathbb{N}$ and any non-negative integer b, the limit $\lim\limits_{N\to\infty} \rho_k(a, b, N)$ exists?

In favor of a positive answer, in Table 1, we present the values of $\rho_k(1, 0, N)$ for $0 \le k \le 10, a = 1, b = 0$, and some values of N up to $20,000$.

k	N: 1000	2000	5000	10000	20000
0	0.7220	0.7355	0.7476	0.7459	0.7498
1	0.7410	0.7455	0.7476	0.7488	0.7518
2	0.5770	0.5840	0.5864	0.5922	0.5927
3	0.7510	0.7720	0.7926	0.8076	0.8172
4	0.4960	0.5015	0.4998	0.5059	0.5030
5	0.7510	0.7640	0.7816	0.7900	0.7958
6	0.6190	0.6270	0.6066	0.6135	0.6174
7	0.6230	0.6450	0.6634	0.6773	0.6864
8	0.7280	0.7240	0.7280	0.7378	0.7441
9	0.6710	0.6825	0.7002	0.7120	0.7203
10	0.4870	0.4860	0.4944	0.4930	0.4999

TABLE 1. Values of $\rho_k(1,0,N)$

2. CONGRUENCES FOR $\Delta_k((8k+4)n+b)$

In this section, we consider the congruences for broken k-diamond partitions along arithmetic progressions $an+b$ with modulus $a=8k+4$. The arithmetic progressions in Theorem 1.1 are of this form for $k=9$ and $k=15$. To prove Theorem 1.1, we use the following result which relates congruences along $(8k+4)n+b$, for broken k-diamond partitions and $(2k+1)$-core partitions. For the definitions and more on t-core partitions, the reader is referred to [3].

Lemma 2.1. *For $n,b,k\in\mathbb{N}$,*

$$\Delta_k((8k+4)n+b)\equiv 0 \pmod 2 \text{ if and only if } a_{2k+1}((8k+4)n+b)\equiv 0 \pmod 2,$$

where $a_t(n)$ denotes the number of t-core partitions of n.

Proof. This follows immediately from Corollary 1.2 in [11]. □

Therefore, in order to obtain the congruences in Theorem 1.1, it suffices to prove the following congruences for $(2k+1)$-core partitions:

$$a_{19}(76n+11,15,27,39,43,47,51,59,67)\equiv 0 \pmod 2, \tag{2.1}$$

$$a_{31}(124n+10,26,30,38,42,50,54,58,62,78,86,94,98,102,122)\equiv 0 \pmod 2. \tag{2.2}$$

We begin by introducing some notations and definitions to be used in Lemma 2.3, which plays a key role in proving these congruences.

Let $\Gamma := SL_2(\mathbb{Z})$, and for a positive integer N, let

$$\Gamma_0(N) := \left\{\begin{pmatrix} x & y \\ z & w\end{pmatrix}\in\Gamma : N|z\right\} \text{ and } \Gamma_\infty := \left\{\begin{pmatrix} 1 & h \\ 0 & 1\end{pmatrix} : h\in\mathbb{Z}\right\}.$$

For $M\in\mathbb{N}$, let

$$R(M) := \{\mathbf{r}=(r_{\delta_1}, r_{\delta_2}, ..., r_{\delta_D}) : r_{\delta_i}\in\mathbb{Z}, 1\le i\le D\},$$

where δ_i runs over the set of positive divisors of M and D is the number of such divisors. Next, for $\mathbf{r} \in R(M)$, we define

$$\sum_{n=0}^{\infty} c_{\mathbf{r}}(n) q^n := \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta}. \tag{2.3}$$

For $m, M \in \mathbb{N}, \mathbf{r} \in R(M), t \in \mathbb{Z}_m, \kappa := \gcd(m^2 - 1, 24)$ and $\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma$, let

$$\mathcal{A}_{m,\mathbf{r}}(\gamma) := \min_{\lambda \in \mathbb{Z}_m} \sum_{\delta|M} r_\delta \frac{(\gcd(\delta x + \delta\kappa\lambda z, mz))^2}{24\delta m}, \tag{2.4}$$

$$\mathcal{B}_{\mathbf{r}}(\gamma) := \sum_{\delta|M} r_\delta \frac{(\gcd(\delta, z))^2}{24\delta}. \tag{2.5}$$

For $\mathbf{r} \in R(M)$, let $\pi(M, \mathbf{r})$ be the tuple of non-negative integers (s, j) such that $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$ where j is odd. For such m, M, N, t and $\mathbf{r} \in R(M)$, we define the set Ω consisting of elements of the form $(m, M, N, t, \mathbf{r})$ with the conditions:

(1) $\kappa N \sum_{\delta|M} r_\delta \frac{mN}{\delta} \equiv 0 \pmod{24}$ and $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod 8$;

(2) either 4 divides κN and 8 divides Ns, or 2 divides s and 8 divides $N(1-j)$;

(3) $\dfrac{24m}{\gcd(-24\kappa t - \kappa \sum_{\delta|M} \delta r_\delta, 24m)}$ divides N;

(4) $p|m \Rightarrow p|N$ for every prime p and $\delta|M \Rightarrow \delta|mN$ for every δ with $r_\delta \neq 0$.

For a positive integer m, let $\mathbb{Z}_m$ be the set of residue classes modulo m identified to the (ordered) set $\{0, 1, \ldots, m-1\}$ and let $\mathbb{Z}^*{}_m$ be the set of residues coprime to m. Let $\mathbb{S}_m$ be the set of squares in $\mathbb{Z}^*{}_m$. For $\mathbf{r} \in R(M)$, define the function

$$\odot : \mathbb{S}_{24m} \times \mathbb{Z}_m \to \mathbb{Z}_m,$$

whose image is uniquely determined by the relation

$$[s] \odot t \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod m.$$

For $t \in \mathbb{Z}_m$, let $P_{m,\mathbf{r}}(t)$ denote the set

$$P_{m,\mathbf{r}}(t) := \{[s] \odot t : [s] \in \mathbb{S}_{24m}\}.$$

Now, we consider the generating function for t-core partitions $a_t(n)$ given by

$$\sum_{n=0}^{\infty} a_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{1 - q^n}.$$

Observe that the above generating function for $a_t(n)$ for $t = 19$ and $t = 31$ coincides with that of $c_{\mathbf{r}}(n)$ in (2.3) for $\mathbf{r} = (-1, 19)$ and $\mathbf{r} = (-1, 31)$, respectively. Recall that $M = 19 = m$ in the first case and $M = 31 = m$ in the second one. Thus, proving congruences (2.1) and (2.2) is equivalent to proving the following result.

Theorem 2.2. *For any non-negative integer n,*

$$c_{\mathbf{r}}(76n+11,15,27,39,43,47,51,59,67)\equiv 0 \pmod 2 \text{ for } \mathbf{r}=(-1,19);$$
$$c_{\mathbf{r}}(124n+10,26,30,38,42,50,54,58,62,78,86,94,98,102,122)\equiv 0 \pmod 2$$
$$\text{for } \mathbf{r}=(-1,31).$$

We provide a proof of this theorem using the result below. With the help of this result, it suffices to check the above congruences up to only a finite number of terms.

Lemma 2.3. [11, Lemma 1.8] *Let l be a positive integer and $(m,M,N,t,\mathbf{r})\in\Omega$. Let $\mathbf{u}\in R(N)$ and $\{\gamma_1,\gamma_2,...,\gamma_{n_0}\}\subseteq\Gamma$ be a complete set of representatives of the double cosets $\Gamma_0(N)\setminus\Gamma/\Gamma_\infty$. Assume that $\mathcal{A}_{m,\mathbf{r}}(\gamma_i)+\mathcal{B}_{\mathbf{u}}(\gamma_i)\geq 0$ for all $i\in\{1,2,...,n_0\}$. By $t_{\min}$, denote the minimum value of $t'\in P_{m,\mathbf{r}}(t)$, and let*

$$\nu:=\frac{1}{24}[\Gamma:\Gamma_0(N)]\left(\sum_{\delta|N}u_\delta+\sum_{\delta|M}r_\delta\right)-\frac{1}{24}\sum_{\delta|N}\delta u_\delta-\frac{1}{24m}\sum_{\delta|M}\delta r_\delta-\frac{t_{\min}}{m}. \tag{2.6}$$

If

$$\forall\ (n,t')\in\{0,\dots,\lfloor\nu+1\rfloor\}\times P_{m,\mathbf{r}}(t),\ \sum_{n=0}^{\infty}c_{\mathbf{r}}(mn+t')q^n\equiv 0 \pmod l,$$

then, for all non-negative integers n, the congruence

$$c_{\mathbf{r}}(mn+t')\equiv 0 \pmod l$$

holds for all $t'\in P_{m,\mathbf{r}}(t)$.

In the above lemma, we set $m=4p, M=p$ and $\mathbf{r}=(-1,p)$, where p is a prime, and for these values, we compute the sets $P_{m,\mathbf{r}}(t)$ using the following result.

Lemma 2.4. *Let $p\geq 5$ be a prime. Let $\mathbf{r}=(-1,p)\in R(p)$. Then*

$$P_{4p,\mathbf{r}}(t)=\left\{t'\in\mathbb{Z}_{4p}\,\middle|\,\left(\frac{24t'-1}{p}\right)=\left(\frac{24t-1}{p}\right)\ \text{and } t'\equiv t \pmod 4\right\},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Proof. For $\mathbf{r}=(-1,p), m=4p, M=p$, we have $\sum_{\delta|M}\delta r_\delta=p^2-1$ and since $\gcd(p,6)=1$, we have $24|p^2-1$. Therefore,

$$P_{4p,\mathbf{r}}(t)=\left\{t'\in\mathbb{Z}_{4p}\,|\,t'\equiv ts+(s-1)\frac{p^2-1}{24} \pmod{4p},\ [s]\in\mathbb{S}_{96p}\right\}.$$

As $[s]\in\mathbb{S}_{96p}$, one has $s\equiv 1 \pmod 4$ and by the Chinese remainder theorem, one has

$$P_{4p,\mathbf{r}}(t)=\left\{t'\,|\,t'\equiv ts+(s-1)\frac{p^2-1}{24} \pmod p,\ t'\equiv t \pmod 4,\ [s]\in\mathbb{S}_{96p}\right\}.$$

Notice that

$$t'\equiv ts+(s-1)\frac{p^2-1}{24} \pmod p \Leftrightarrow 24t'-1\equiv s(24t-1) \pmod p.$$

Since the equality $\left(\frac{24t'-1}{p}\right) = \left(\frac{24t-1}{p}\right)$ (respectively the congruence $t' \equiv t \pmod 4$) means that $24t-1$ and $24t'-1$ are simultaneously square or non-square modulo p (respectively modulo 96), one arrives to

$$P_{4p,\mathbf{r}}(t) = \left\{ t' \in \mathbb{Z}_{4p} \,\middle|\, \left(\frac{24t'-1}{p}\right) = \left(\frac{24t-1}{p}\right) \text{ and } t' \equiv t \pmod 4 \right\}.$$

□

Notice that the special case where p divides $24t-1$ corresponds to $P_{4p,\mathbf{r}}(t) = \{t\}$. We now proceed to prove Theorem 2.2.

Proof of Theorem 2.2. As mentioned before, we set $m = N = 4p, M = p$ and $\mathbf{r} = (-1, p)$ in Lemma 2.3. In order to use this lemma, we first show that $(4p, p, 4p, t, \mathbf{r})$ belongs to Ω and then compute the sets $P_{m,\mathbf{r}}(t)$ for $p = 19, t = 11$ in the first case and for $p = 31, t = 10$ in the other one.

Case I: $p = 19, t = 11$. Here, $\mathbf{r} = (-1, 19) \in R(19)$ and $\kappa = \gcd(76^2 - 1, 24) = 3$. Also $\prod_{\delta|M} \delta^{|r_\delta|} = 19^{19}$, therefore $s = 0$ and $j = 19^{19}$. Thus, conditions (1) to (4) stated earlier are easily verified, and we conclude that $(76, 19, 76, 11, (-1, 19)) \in \Omega$.
Next, using Lemma 2.4, we compute the set $P_{4p,\mathbf{r}}(t)$. Since,

$$\left(\frac{24 \times t - 1}{p}\right) = \left(\frac{263}{19}\right) = 1,$$

the set $P_{4p,\mathbf{r}}(t)$, consisting of $t' \in \mathbb{Z}_{4p}$ for which $\left(\frac{24t'-1}{19}\right) = 1$ and $t' \equiv 11 \pmod 4$, is given by $\{11, 15, 27, 39, 43, 47, 51, 59, 67\}$. Hence for $\mathbf{r} = (-1, 19)$,

$$P_{76,\mathbf{r}}(11) = \{11, 15, 27, 39, 43, 47, 51, 59, 67\}.$$

Case II: $p = 31, t = 10$. In this case, we find $\mathbf{r} = (-1, 31) \in R(31), \kappa = 3, s = 0$ and $j = 31^{31}$. It can be verified that conditions (1) to (4) hold here as well. This shows that $(124, 31, 124, 10, (-1, 31)) \in \Omega$. To compute the set $P_{4p,\mathbf{r}}(t)$, since

$$\left(\frac{24 \times t - 1}{p}\right) = \left(\frac{239}{31}\right) = -1,$$

we need to find all those $t' \in \mathbb{Z}$ such that $0 \le t' \le 4p - 1$, $\left(\frac{24t'-1}{31}\right) = -1$ and $t' \equiv 10 \pmod 4$. This means $t' \in \{10, 26, 30, 38, 42, 50, 54, 58, 62, 78, 86, 94, 98, 102, 122\}$. Therefore, for $\mathbf{r} = (-1, 31)$,

$$P_{124,\mathbf{r}}(10) = \{10, 26, 30, 38, 42, 50, 54, 58, 62, 78, 86, 94, 98, 102, 122\}.$$

Next, we check the assumption in Lemma 2.3 that for prime p and $\mathbf{u} \in R(N)$, the inequality $\mathcal{A}_{m,\mathbf{r}}(\gamma) + \mathcal{B}_{\mathbf{u}}(\gamma) \ge 0$ holds for all $\gamma \in \Gamma$. Choose $\mathbf{u}$ to be the zero tuple so that $\mathcal{B}_{\mathbf{u}}(\gamma) = 0$ for all $\gamma \in \Gamma$. Therefore, it suffices to show $\mathcal{A}_{m,\mathbf{r}}(\gamma) \ge 0$ for all $\gamma \in \Gamma$. Let $\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma$. From the definition of $\mathcal{A}_{m,\mathbf{r}}(\gamma)$ in (2.4) and the fact that x and z are coprime, since $wx - yz = 1$, we have

$$\begin{aligned}\mathcal{A}_{4p,\mathbf{r}}(\gamma) &= \frac{1}{24}\min_{\lambda\in\mathbb{Z}_{4p}}\left(-\frac{(\gcd(x+\kappa\lambda z,4pz))^2}{4p}+p\frac{(\gcd(px+p\kappa\lambda z,4pz))^2}{4p^2}\right)\\ &= \frac{1}{24}\min_{\lambda\in\mathbb{Z}_{4p}}\frac{1}{24}\left(-\frac{(\gcd(x+\kappa\lambda z,4p))^2}{4p}+p\frac{(\gcd(x+\kappa\lambda z,4))^2}{4}\right).\end{aligned}$$

Let

$$F(\gamma,p,\lambda) := \frac{-(\gcd(x+\kappa\lambda z,4p))^2}{4p}+\frac{p\,(\gcd(x+\kappa\lambda z,4))^2}{4}.$$

Now we consider all possibilities for $\gcd(x+\kappa\lambda z,4p)$, and this yields the following implications:

$$\begin{aligned}
\gcd(x+\kappa\lambda z,4p)=1 &\Rightarrow F(\gamma,p,\lambda)=\frac{-1}{4p}+\frac{p}{4}\geq 0,\\
\gcd(x+\kappa\lambda z,4p)=2 &\Rightarrow F(\gamma,p,\lambda)=\frac{-4}{4p}+\frac{4p}{4}\geq 0,\\
\gcd(x+\kappa\lambda z,4p)=4 &\Rightarrow F(\gamma,p,\lambda)=\frac{-16}{4p}+\frac{16p}{4}\geq 0,\\
\gcd(x+\kappa\lambda z,4p)=p &\Rightarrow F(\gamma,p,\lambda)=\frac{-p^2}{4p}+\frac{p}{4}=0,\\
\gcd(x+\kappa\lambda z,4p)=2p &\Rightarrow F(\gamma,p,\lambda)=\frac{-4p^2}{4p}+\frac{4p}{4}=0,\\
\gcd(x+\kappa\lambda z,4p)=4p &\Rightarrow F(\gamma,p,\lambda)=\frac{-16p^2}{4p}+\frac{16p}{4}=0.
\end{aligned}$$

This proves that $\mathcal{A}_{4p,\mathbf{r}}(\gamma)\geq 0$ for each $\gamma\in\Gamma$.

Lastly, using the fact that $[\Gamma:\Gamma_0(N)] = N\prod_{p|N}\left(1+\frac{1}{p}\right)$, we calculate ν in (2.6) for the two cases. For $p=19, t=11, \mathbf{r}=(-1,19), \mathbf{u}=(0,0,...,0)$ and $t_{\min}=11$,

$$\nu = 90-\frac{26}{76} \text{ and hence } \lfloor\nu\rfloor = 89.$$

For $p=31, t=10, \mathbf{r}=(-1,31), \mathbf{u}=(0,0,...,0)$ and $t_{\min}=10$,

$$\nu = 240-\frac{50}{124}, \text{ therefore } \lfloor\nu\rfloor = 239.$$

Taking $l=2$ in Lemma 2.3, we see that

- if $c_{\mathbf{r}}(76n+t')\equiv 0 \pmod 2$ for all $0\leq n\leq 90$ and $t'\in P_{76,\mathbf{r}}(11)$, $\mathbf{r}=(-1,19)$, then $c_{\mathbf{r}}(76n+t')\equiv 0 \pmod 2$ for all $n\geq 0$ and $t'\in P_{76,\mathbf{r}}(11)$.
- if $c_{\mathbf{r}}(124n+t')\equiv 0 \pmod 2$ for all $0\leq n\leq 240$ and $t'\in P_{124,\mathbf{r}}(10)$, $\mathbf{r}=(-1,31)$, then $c_{\mathbf{r}}(124n+t')\equiv 0 \pmod 2$ for all $n\geq 0$ and $t'\in P_{124,\mathbf{r}}(10)$.

Using Mathematica, we verify the calculations $c_{\mathbf{r}}(76n+t')\equiv 0 \pmod 2$ for $0\leq n\leq 90$ and $c_{\mathbf{r}}(124n+t')\equiv 0 \pmod 2$ for $0\leq n\leq 240$, and thus conclude that

$$\begin{aligned}&c_{\mathbf{r}}(76n+t')\equiv 0 \pmod 2 \text{ for all } n\geq 0 \text{ and } t'\in P_{76,\mathbf{r}}(11),\\ \text{and } &c_{\mathbf{r}}(124n+t')\equiv 0 \pmod 2 \text{ for all } n\geq 0 \text{ and } t'\in P_{124,\mathbf{r}}(10).\end{aligned}$$

This completes the proof of Theorem 2.2. □

Remark. In a similar fashion, one can prove congruences for broken k-diamond partitions along arithmetic progressions $(16k+8)n+b$ by taking m in Lemma 2.3 and Lemma 2.4 to be $8p$ instead of $4p$. The proof works in a similar manner and one obtains, for instance, the congruences

$$\Delta_3(56n+2,15,20,28,29,31,34,39,42,44,45,47,53)\equiv 0 \pmod 2 \quad \text{for all integers } n\geq 0.$$

3. Some congruences for broken 12-diamond partitions

Radu and Sellers [11] considered broken $(2k+1)$-diamond partitions, where $2k+1$ is a prime and $2\leq k\leq 11$. They obtained parity results for arithmetic progressions $(4k+2)n+b$. In fact, for each such k, there are exactly k many arithmetic progressions for which $\Delta_k((4k+2)n+b)\equiv 0 \pmod 2$. We investigate such congruences for $k=12$. In this case, we find only five congruences, namely

$$\Delta_{12}(50n+9,19,29,39,49)\equiv 0 \pmod 2,$$

for all non-negative integers n. This is possibly due to the fact that 50 is not a prime number. Notice that this result is equivalent to Theorem 1.2.

Proof of Theorem 1.2. We make use of the following result of Ramanujan, [8]:

$$\begin{aligned}\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\{&R(q^5)^4+qR(q^5)^3+2q^2R(q^5)^2+3q^3R(q^5)+5q^4\\ &-3q^5R(q^5)^{-1}+2q^6R(q^5)^{-2}-q^7R(q^5)^{-3}+q^8R(q^5)^{-4}\},\end{aligned} \tag{3.1}$$

where $R(q)=\prod_{n\geq 1}\frac{(1-q^{5n-3})(1-q^{5n-2})}{(1-q^{5n-4})(1-q^{5n-1})}$, and as usual $(z;q)_\infty := \prod_{l=0}^{\infty}(1-zq^l)$. From (1.2) and (3.1), we have

$$\begin{aligned}\sum_{n\geq 0}\Delta_{12}(n)q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty^2(-q^{25};q^{25})_\infty}\\ &\equiv \frac{1}{(q;q)_\infty(q^{25};q^{25})_\infty} \pmod 2\\ &\equiv \frac{(q^{25};q^{25})_\infty^4}{(q^5;q^5)_\infty^6}\{R(q^5)^4+qR(q^5)^3+q^3R(q^5)+q^4\\ &\quad +q^5R(q^5)^{-1}+q^7R(q^5)^{-3}+q^8R(q^5)^{-4}\} \pmod 2.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n\geq 0}\Delta_{12}(5n+4)q^n \pmod 2\\ &\equiv \frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^6} \pmod 2 \equiv \prod_{n\geq 1}\frac{(1-q^{5n})^4}{(1-q^n)^6} \pmod 2\\ &\equiv 1/\prod_{n\geq 1}(1-q^{5n-4})^6(1-q^{5n-3})^6(1-q^{5n-2})^6(1-q^{5n-1})^6(1-q^{5n})^2 \pmod 2\\ &\equiv 1/\prod_{n\geq 1}(1-q^{10n-8})^3(1-q^{10n-6})^3(1-q^{10n-4})^3(1-q^{10n-2})^3(1-q^{10n}) \pmod 2.\end{aligned}$$

Since the last expression is an even function of q, we conclude that

$$\Delta_{12}(5(2n+1)+4) = \Delta_{12}(10n+9) \equiv 0 \pmod 2 \text{ for all integers } n \geq 0. \qquad \square$$

4. A congruence for broken 3-diamond partitions

In this section, we prove a result for broken 3-diamond partitions along the arithmetic progression $8n+7$, which is not of the form $(4k+2)n+b$.

Theorem 4.1. *For $n \geq 0$, $\Delta_3(8n+7) \equiv 0 \pmod 2$.*

Proof. Let $\mathrm{asc}_t(n)$ denote the number of self-conjugate t-core partitions of n, [3]. Garvan, Kim and Stanton [7, Equations (7.1a) and (7.1b)] give the generating function for $\mathrm{asc}_t(n)$ as

$$\sum_{n=0}^{\infty} \mathrm{asc}_t(n) q^n = (-q;q^2)_\infty (q^{2t};q^{2t})_\infty^{t/2}, \text{ if } t \text{ is even,}$$

$$\sum_{n=0}^{\infty} \mathrm{asc}_t(n) q^n = \frac{(-q;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-1)/2}}{(-q^t;q^{2t})_\infty}, \text{ if } t \text{ is odd.}$$

In particular,

$$\sum_{n=0}^{\infty} \mathrm{asc}_7(n) q^n = \frac{(-q;q^2)_\infty (q^{14};q^{14})_\infty^{3}}{(-q^7;q^{14})_\infty}.$$

Using (1.2) and above, we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \Delta_3(n) q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty^2 (-q^7;q^7)_\infty} \equiv \frac{(-q;q^2)_\infty}{(-q^7;q^7)_\infty} \pmod 2 \\
&\equiv \frac{(-q;q^2)_\infty}{(q^{14};q^{14})_\infty (-q^7;q^{14})_\infty} \pmod 2 \\
&\equiv \frac{(-q;q^2)_\infty (q^{14};q^{14})_\infty^3}{(q^{14};q^{14})_\infty^4 (-q^7;q^{14})_\infty} \pmod 2 \\
&\equiv \frac{1}{(q^{14};q^{14})_\infty^4} \sum_{n\geq 0} \mathrm{asc}_7(n) q^n \pmod 2. \qquad (4.1)
\end{aligned}$$

Baruah and Sarmah [3, Theorem 3.1] show that $\mathrm{asc}_7(8n+7) = 0$ for all $n \geq 0$, which along with (4.1) implies that $\Delta_3(8n+7) \equiv 0 \pmod 2$ for all integers $n \geq 0$. $\square$

5. Counts for odd values of $\Delta_k(n)$

In this section, we give a proof of Theorem 1.3, which provides a lower bound for the number of odd values of $\Delta_k(n)$ for n not exceeding N, where N is any large fixed positive integer. We employ the methods developed in [4] and [5].

Proof of Theorem 1.3. We investigate the parity of $\Delta_k(n)$, which is the same as the parity of the coefficient of q^n in the formal power series,

$$F(q) := \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3 (1-q^{(4k+2)n})}.$$

By reducing the coefficients of $F(q)$ modulo 2, we see that

$$\begin{aligned} F(q) &\equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(2k+1)n})^2} \pmod 2 \\ &\equiv \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{(2k+1)n})} \pmod 2. \end{aligned}$$

Let

$$G(q) := \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{(2k+1)n})}. \tag{5.1}$$

Therefore, the parity of $\Delta_k(n)$ is the same as that of the coefficient of q^n in $G(q)$, and we have to prove that the desired lower bound holds for $G(q)$. Passing to the formal logarithmic derivative and then multiplying the resultant by q, equation (5.1) leads to

$$\begin{aligned} \frac{qG'(q)}{G(q)} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nq^{nm} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(2k+1)nq^{(2k+1)nm} \\ &= \sum_{h=1}^{\infty} q^h \sum_{n|h} n + (2k+1)\sum_{h=1}^{\infty} q^{(2k+1)h} \sum_{n|h} n \\ &= \sum_{h=1}^{\infty} \sigma(h)q^h + (2k+1)\sum_{h=1}^{\infty} \sigma(h)q^{(2k+1)h}. \end{aligned}$$

Now we work in the ring $\mathbb{F}_2[[q]]$ and set $H(q) := \sum_{h=1}^{\infty} \sigma(h)q^h + \sum_{h=1}^{\infty} \sigma(h)q^{(2k+1)h}$. Since, $\sigma(h)$ is odd if and only if each odd prime factor has an even exponent in the prime factorization of h, in other words, $h = 2^r(2n+1)^2$ for some integers $n, r \geq 0$, $H(q)$ has the form

$$\begin{aligned} H(q) &= \sum_{n,r\geq 0} q^{2^r(2n+1)^2} + \sum_{n,r\geq 0} q^{(2k+1)2^r(2n+1)^2} \\ &= \sum_{n=1}^{\infty} q^{n^2} + \sum_{n=1}^{\infty} q^{2n^2} + \sum_{n=1}^{\infty} q^{(2k+1)n^2} + \sum_{n=1}^{\infty} q^{2(2k+1)n^2}. \end{aligned} \tag{5.2}$$

By reducing modulo 2, $G(q)$ takes the form $G(q) = 1 + q^{n_1} + q^{n_2} + \cdots$ in $\mathbb{F}_2[[q]]$. Now, from $qG'(q) = G(q)H(q)$ and (5.2), we derive

$$qG'(q) + (q^{n_1} + q^{n_2} + \cdots)H(q) = \sum_{n=1}^{\infty} q^{n^2} + \sum_{n=1}^{\infty} q^{2n^2} + \sum_{n=1}^{\infty} q^{(2k+1)n^2} + \sum_{n=1}^{\infty} q^{2(2k+1)n^2}. \tag{5.3}$$

We now derive a lower bound for $\#\{j : n_j \leq N\}$.

Case I. If at least half of the $\lfloor\sqrt{N}\rfloor$ terms of the form q^{n^2} for $n^2 \leq N$ on the left side of (5.3) are canceled by terms from the series $qG'(q)$, then $G'(q)$ has at least $\lfloor\sqrt{N}/2\rfloor$ terms up to q^N. Hence, $G(q)$ has at least $\lfloor\sqrt{N}/2\rfloor$ terms up to q^N and we obtain the desired lower bound.

Case II. Assume that less than half of the terms of the form q^{n^2} for $n^2 \leq N$ are canceled by terms from $qG'(q)$. This implies that at least $\lfloor\sqrt{N}/2\rfloor$ such terms are left to be canceled by terms from the series $(q^{n_1} + q^{n_2} + \cdots)H(q)$. To see how many terms of the form q^{m^2}

for $m^2 \leq N$ may appear in a series of the form $q^{n_j}H(q)$ for a fixed n_j, we consider four diophantine equations in positive integers n and m, namely,

$$n_j + n^2 = m^2, \tag{5.4}$$

$$n_j + 2n^2 = m^2, \tag{5.5}$$

$$n_j + (2k+1)n^2 = m^2, \tag{5.6}$$

$$n_j + 2(2k+1)n^2 = m^2. \tag{5.7}$$

Using arguments from [4], we find bounds (from above) for the number of solutions of these equations. Equation (5.4) has at most $N^{\frac{2c_1}{\log\log N}}$ solutions for some constant $c_1 > 0$. The number of solutions of equation (5.5) is bounded by $c_2 \log N$, for some $c_2 > 0$. In order to bound the number of solutions of equations (5.6) and (5.7), we work in $\mathbb{Q}(\sqrt{2k+1})$, and find the number of solutions to be at most $N^{\frac{c_3}{\log\log N}}$, for some fixed $c_3 > 2\log 2$. Similarly for equation (5.7), the number of solutions is bounded by $c_4 \log N$. Therefore, the number of solutions of (5.4), (5.5), (5.6) and (5.7) is at most $N^{\frac{c}{\log\log N}}$ for some positive number $c > 0$. Thus, we arrive at the desired bound,

$$\#\{n \leq N : \Delta_k(n) \text{ is odd}\} \geq N^{\frac{1}{2} - \frac{c}{\log\log N}}.$$

□

Remark. For $l, m \in \mathbb{N}$, let $B_1, B_2, \ldots, B_l; D_1, D_2, \ldots, D_m$ be distinct positive integers. Let $C(n)$ denote the number of colored partitions of n in $l+m$ colors with the following conditions:

(1) the parts appearing in the partitions are multiples of B_j's and D_i's,
(2) the parts which appear as multiples of B_j's are distinct.

Then, the associated generating function is given by

$$\sum_{n=0}^{\infty} C(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^{B_1 n})(1+q^{B_2 n})\ldots(1+q^{B_l n})}{(1-q^{D_1 n})(1-q^{D_2 n})\ldots(1-q^{D_m n})}.$$

Note that for $l = 1, m = 2$, $B_1 = 2k+1, D_1 = 1, D_2 = 4k+2$, one obtains the generating function for the broken k-diamond partitions modulo 2. Using similar arguments as above, one concludes, for all N large enough,

$$\#\{n \leq N : C(n) \text{ is odd}\} \geq N^{\frac{1}{2} - \frac{c}{\log\log N}} ,$$

for some positive real number c.

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