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Partitions into kth powers of terms in an arithmetic progression

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Abstract G. H. Hardy and S. Ramanujan established an asymptotic formula for the number of unrestricted partitions of a positive integer, and claimed a similar asymptotic formula for the number of partitions into perfect kth powers, which was later proved by E. M. Wright. Recently, R. C. Vaughan provided a simpler asymptotic formula in the case $k = 2$. In this paper, we consider partitions into parts from a specific set $A_k(a_0, b_0) := \{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\}$, for fixed positive integers k, a_0, and b_0. We give an asymptotic formula for the number of such partitions, thus generalizing the results of Wright and Vaughan. Moreover, we prove that the number of such partitions is even (odd) infinitely often, which generalizes O. Kolberg's theorem for the ordinary partition function $p(n)$.

Keywords Partitions · Parity · Arithmetic progression · Asymptotics · Hardy–Littlewood circle method

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1 Introduction

For any non-negative integer n, and $A \subseteq \mathbb{N}$, let $p_A(n)$ denote the number of partitions of n with parts in the set A. Note that for $A = \mathbb{N}$, the quantity $p_A(n)$ counts the number of unrestricted partitions of n, and is usually denoted by $p(n)$. This function has been studied extensively. However, it was not known that $p(n)$ takes even (odd) values infinitely often even until 1959, when Kolberg [8] established these facts. Other proofs of Kolberg's theorem were later found by Newman [9], and by Fabrykowski and Subbarao [5]. It is conjectured that $p(n)$ is even (odd) approximately half the time. Even though many results have been proved in this direction, for example, Ono [16], Nicolas et al. [11], and Ahlgren [1], the best known results are far from the estimates expected.

For fixed positive integers k, a_0, and b_0, define the subset $A_k(a_0, b_0)$ of positive integers by $A_k(a_0, b_0) := \{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\}$. Denote by $p_{A_k(a_0,b_0)}(n)$ the number of partitions of n where the parts are taken from the subset $A_k(a_0, b_0)$. Hardy and Ramanujan [7] initiated the study of $p(n)$ from an analytic point of view. They proved an asymptotic formula for $p(n)$, as n approaches infinity, and stated (without proof) a similar result for $p_{A_k(1,1)}(n)$, the number of partitions of n into perfect kth powers, for any $k \geq 2$. Later a proof was supplied for the case $k \geq 2$ by Wright [23] in 1934. His proof uses the ideas of Hardy and Ramanujan for the case $k = 1$, but relies heavily on a transformation for the generating function of $p_{A_k(1,1)}(n)$ involving generalized Bessel functions. In the case $k = 2$, a simpler aymptotic formula has recently been given by Vaughan [21], and has been generalized for any integer $k \geq 2$ by Gafni [6]. For asymptotics of some other restricted partitions, the reader is referred to [12] and [13].

In this paper, we focus our attention on the more general function $p_{A_k(a_0,b_0)}(n)$. In Sect. 2, we work in the ring of formal power series in one variable over the field of two elements $\mathbb{Z}/2\mathbb{Z}$. Using elementary differential equations and algebraic tools such as Hensel's lemma, we develop a new method to prove that this partition function assumes even (odd) values for infinitely many positive integers. This generalizes the result of Kolberg [8] for the ordinary partition function $p(n)$. In fact, our method works in more generality and can be applied to certain other restricted partition functions, including plane partitions for which there are no congruence results known, to obtain corresponding parity results. Yee along with the first and third authors [3] obtained a similar result in the case of $k = 1$ using more advanced tools such as the Prime Number Theorem for arithmetic progressions and properties of Dirichlet L-functions.

In the later sections, we provide an asymptotic expansion for $p_{A_k(a_0,b_0)}(n)$, as n approaches infinity. This extends results of Hardy and Ramanujan [7], Vaughan [21], and Wright [23]. Our proof is based on the Hardy–Littlewood circle method. A fine analysis and modification of results pertaining to exponential sums help us overcome the complications posed by the general arithmetic progression $a_0 \pmod{b_0}$ when $b_0 > 1$. Moreover, following similar arguments as in the proof of Theorem 1.2, one can also obtain asymptotics for the difference of the number of such partitions of two consecutive positive integers as they grow large.

In the last section, we mention possible future directions for this research.

Now we state the main results of this paper. We reiterate definitions from the second paragraph above. For fixed positive integers k, a_0, and b_0, define $A_k(a_0, b_0) \subseteq \mathbb{N}$ by

$$A_k(a_0, b_0) := \left\{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\right\}. \tag{1.1}$$

Also, let

$$p_{A_k(a_0,b_0)}(n) := \#\{\text{partitions of } n \text{ into parts from } A_k(a_0, b_0)\}. \tag{1.2}$$

The first result is about the parity of $p_{A_k(a_0,b_0)}(n)$.

Theorem 1.1 *Let k, a_0, and b_0 be fixed positive integers satisfying $a_0 \leq b_0$, and $(a_0, b_0) = 1$. Let $A_k(a_0, b_0)$ and $p_{A_k(a_0,b_0)}(n)$ be defined as in (1.1) and (1.2), respectively. Then, there are infinitely many positive integers n such that $p_{A_k(a_0,b_0)}(n)$ is even, and there are infinitely many positive integers m for which $p_{A_k(a_0,b_0)}(m)$ is odd.*

In the next result, we show that

$$p_{A_k(a_0,b_0)}(n) \sim \mathcal{B} \exp\left(\mathcal{M} n^{\frac{1}{k+1}}\right) n^{-\frac{b_0+b_0k+2a_0k}{2b_0(k+1)}},$$

where $\mathcal{B}$ and $\mathcal{M}$ are constants depending on the parameters a_0, b_0 and $k \geq 2$.

Theorem 1.2 *Fix positive integers k, a_0, and b_0 with $k \geq 2$, $a_0 \leq b_0$, and $(a_0, b_0) = 1$, let $A_k(a_0, b_0)$ and $p_{A_k(a_0,b_0)}(n)$ be defined as in (1.1) and (1.2), respectively. Set $\beta_0 = a_0/b_0$, and let $\zeta(s)$ and $\zeta(s, \beta_0)$ denote the Riemann zeta function and the Hurwitz zeta function, respectively. Let M be a fixed positive integer with*

$$M \leq \frac{1}{2016k^2}\left(\frac{1}{b_0k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)\right)^{-\frac{k}{k+1}} n^{\frac{1}{k+1}}. \tag{1.3}$$

Then, for any positive integer J, there exist constants $\mu_1, \ldots, \mu_{J-1}$ such that as $n \to \infty$,

$$\begin{aligned} p_{A_k(a_0,b_0)}(n) &= \frac{\exp\left(\frac{k+1}{b_0k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{\frac{1}{k}} + \zeta(0,\beta_0)(1-\log b_0^k) + k\zeta'(0,\beta_0)\right)}{2\sqrt{\pi}\sqrt{Y}X^{1-\zeta(0,\beta_0)}} \\ &\quad\times \exp\left(\sum_{m=1}^{M-1}\frac{b_0^{2mk}}{(2m)!}(1-2m)\zeta(1-2m)\zeta(-2mk,\beta_0)X^{-2m}\right) \\ &\quad\times\left(1+\sum_{j=1}^{J-1}\frac{\mu_j}{Y^j} + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right) + \mathrm{O}_{k,a_0,b_0}\left(X^{-2M+1}\right)\right), \end{aligned}$$

where X and Y satisfy

$$\begin{aligned} \frac{n}{X} &= \frac{1}{b_0k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} + \zeta(0,\beta_0) - \frac{b_0^k}{2}\zeta(-k,\beta_0)\frac{1}{X} \\ &\quad - \sum_{m=1}^{M}\frac{b_0^{2mk}}{(2m-1)!}\zeta(-2m+1)\zeta(-2km,\beta_0)\frac{1}{X^{2m}}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} Y &= \frac{k+1}{2b_0k^3}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} + \frac{\zeta(0,\beta_0)}{2} \\ &\quad + \sum_{m=1}^{M}\frac{m(2m-1)b_0^{2mk}\zeta(-2m+1)\zeta(-2mk,\beta_0)}{(2m)!X^{2m}}, \end{aligned} \tag{1.5}$$

and the terms (including the error term) involving M occur only when $\beta_0 \neq 1/2, 1$.

Remarks:

- For $(a_0, b_0) = d_0 > 1$, the number $p_{A_k(a_0,b_0)}(n)$ is zero unless n is a multiple of d_0^k. In fact, $p_{A_k(a_0,b_0)}(n) = p_{A_k(a_0/d_0,b_0/d_0)}(n/d_0^k)$. Also, note that a_0/d_0 and b_0/d_0 are relatively prime. Therefore, it is sufficient to consider only those integers a_0, b_0 which are coprime to each other and satisfy $1 \leq a_0 \leq b_0$.

- Note that in Theorem 1.2, $X \sim \mathcal{S}Y^k \sim \mathcal{T}n^{k/(k+1)}$, for some constants $\mathcal{S}$ and $\mathcal{T}$. In fact, these constants can be computed explicitly from (1.4) and (1.5). Moreover, one can show that $M \leq (2(4\pi/5)^{k+1}X)^{1/k}/(4k^2)$, which is used in Sect. 4.
- In the case $\beta_0 = 1$, we recover Gafni's result [6, Theorem 1], and if we further set $k = 2$, we recover Vaughan's result [21, Theorem 1.5]. In these cases, $\beta_0 = 1$, and therefore as mentioned in Theorem 1.2, the expression for $p_{A_k(1,1)}(n)$ becomes much simpler since all the terms involving M disappear.
- Following the arguments in the proof of Theorem 1.2, one can obtain an asymptotic result for the difference $p_{A_k(a_0,b_0)}(n+1) - p_{A_k(a_0,b_0)}(n)$ as n approaches infinity.

2 Parity

In this section, we give a proof of Theorem 1.1. First, we prove two propositions which are later used in the proof, but are also interesting in their own right. For brevity, we also set $A = A_k(a_0, b_0)$, defined in (1.1).

For any positive integer l, and any set $A \subseteq \mathbb{N}$, define

$$\sigma_A(l) := \sum_{\substack{d \mid l \\ d \in A}} d.$$

Proposition 2.1 *Let c be an odd positive integer such that $c \equiv a_0 \pmod{b_0}$. Suppose that for any positive integer B, there are distinct primes $q_1, \dots, q_B$, and a positive integer l_j such that for each $j = 1, \dots, B$,*

$$q_j \geqslant B + 1, \quad q_j^{l_j} \equiv 1 \pmod{b_0}, \tag{2.1}$$

$$c^{2k} + j \equiv 0 \pmod{q_j^{k(2l_j-1)}}, \text{ and } c^{2k} + j \not\equiv 0 \pmod{q_j^{2k(2l_j-1)}}. \tag{2.2}$$

Then, $\sigma_A(c^{2k})$ is odd, and $\sigma_A(c^{2k}+j)$ is even for all $j = 1, \dots, B$.

Proof Note that

$$\begin{aligned} \sigma_A(l) &= \sum_d d : d \in \mathbb{N}, d \mid l, d = m^k, m \equiv a_0 \ (\mathrm{mod}\, b_0) \\ &\equiv \#\left\{d \in \mathbb{N} : d \mid l,\ d \text{ is odd and } d = m^k, m \equiv a_0 \ (\mathrm{mod}\, b_0)\right\} \ (\mathrm{mod}\, 2) \\ &\equiv \#\left\{m \in \mathbb{N} : m^k \mid l, m \text{ is odd and } m \equiv a_0 \ (\mathrm{mod}\, b_0)\right\} \ (\mathrm{mod}\, 2). \end{aligned} \tag{2.3}$$

Also, let l have the prime factorization

$$l = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r},$$

where $p_1, \dots, p_r$ are distinct odd primes, $\alpha_1, \dots, \alpha_r$ are positive integers, and α_0 is a non-negative integer. We consider the function $f_k : \mathbb{N} \to \mathbb{N}$ given by

$$f_k(l) := p_1^{[\alpha_1/k]} \cdots p_r^{[\alpha_r/k]}.$$

Therefore, using (2.3), we can rewrite $\sigma_A(l)$ as

$$\sigma_A(l) \equiv \#\{m \in \mathbb{N} : m \equiv a_0 \ (\mathrm{mod}\, b_0), m \mid f_k(l)\} \ (\mathrm{mod}\, 2). \tag{2.4}$$

First, we show that $\sigma_A(c^{2k}+j)$ is even for each $j=1,\ldots,B$. Note that (2.2) implies the exponent of q_j in $c^{2k}+j$ is at least $k(2l_j-1)$ but at most $2k(2l_j-1)-1$, i.e., the exponent of q_j in $f_k(c^{2k}+j)$ is exactly $2l_j-1$. In other words, for fixed $j\in\{1,\ldots,B\}$, there exists a positive integer m_j, coprime to q_j, such that

$$f_k(c^{2k}+j)=m_jq_j^{2l_j-1}. \tag{2.5}$$

Let d_j be any divisor of $f_k(c^{2k}+j)$ satisfying $d_j\equiv a_0 \pmod{b_0}$. Therefore,

$$d_j=\tilde{d}_jq_j^{\beta_j}\equiv a_0 \pmod{b_0},$$

for some $\tilde{d}_j$ coprime to q_j, and $0\le\beta_j<2l_j$. If $\beta_j<l_j$, then from (2.5), we see that $\tilde{d}_jq_j^{\beta_j+l_j}$ also divides $f_k(c^{2k}+j)$, and

$$\tilde{d}_jq_j^{\beta_j+l_j}\ne\tilde{d}_jq_j^{\beta_j}, \text{ and } \tilde{d}_jq_j^{\beta_j+l_j}=\tilde{d}_jq_j^{\beta_j}q_j^{l_j}\equiv a_0 \pmod{b_0}.$$

Similarly, if $\beta_j>l_j$, then $d_jq_j^{\beta_j-l_j}\equiv a_0 \pmod{b_0}$, and is a factor of $f_k(c^{2k}+j)$. Thus the divisors congruent to $a_0 \pmod{b_0}$ of $f_k(c^{2k}+j)$ appear in pairs, and from (2.4) we conclude that $\sigma_A(c^{2k}+j)$ is even for all $j=1,\ldots,B$.

Next, we show that $\sigma_A(c^{2k})$ is odd. Note that since c is odd, $f_k(c^{2k})=c^2$. Let u be any divisor of $f_k(c^{2k})$ so that $u\equiv a_0 \pmod{b_0}$. Then, $uv=c^2$ for some $v\in\mathbb{N}$. Also,

$$a_0^2\equiv c^2=uv\equiv a_0v \pmod{b_0}.$$

Since $(a_0,b_0)=1$, we conclude that $v_0=a_0 \pmod{b_0}$. Moreover, $u\ne v$ unless $u=c$. Therefore, once again by (2.4), we deduce that $\sigma_A(c^{2k})$ is odd. This completes the proof of the proposition. □

Proposition 2.2 *For fixed positive integers k, a_0, and b_0 such that $(a_0,b_0)=1$, let $A:=\left\{m^k: m\in\mathbb{N}, m\equiv a_0 \pmod{b_0}\right\}$. For any positive integer l, let*

$$\sigma_A(l):=\sum_{\substack{d\mid l\\ d\in A}}d.$$

Then, for any fixed positive integer B, there exists an odd positive integer l_B such that $\sigma_A(l_B)$ is odd, and $\sigma_A(l_B+j)$ is even for $j=1,\ldots,B$.

Proof Notice that once the existence of c, q_j for $j=1,\ldots,B$, as in Proposition 2.1, are established, we can simply let $l_B=c^{2k}$, and conclude the proof by invoking Proposition 2.1. Therefore, we only need to show that for each $j=1,\ldots,B$, there exist distinct primes $q_1,\ldots,q_B$, and positive integers l_j satisfying

$$q_j\ge B+1,\quad q_j^{l_j}\equiv 1 \pmod{b_0},\quad c\equiv a_0 \pmod{b_0}, \tag{2.6}$$

$$c^{2k}+j\equiv 0 \pmod{q_j^{k(2l_j-1)}}, \text{ and } c^{2k}+j\not\equiv 0 \pmod{q_j^{2k(2l_j-1)}}. \tag{2.7}$$

We construct the q_j's inductively. For a fixed $j\in\{1,\ldots,B\}$, assume $q_1,\ldots,q_{j-1}$ are already chosen, and set $q_0:=1$. Define

$$K_j:=\prod_{\substack{p_r\nmid j\\ p_r-\text{prime}\\ p_r\mid B!2kb_0q_1\ldots q_{j-1}}}p_r.$$

Fix any prime factor q_j of $K_j^{2k} + j$. Then,

- $(q_j, j) = 1$, for if $q_j | j$, then $q_j | K_j$, which further implies $q_j = p_r \nmid j$,
- $q_j \geq B + 1$, as $q_j \leqslant B$ implies q_j divides K_j, and hence j,
- $q_j \notin \{q_1, \ldots, q_{j-1}\}$,
- $q_j \nmid k$,
- $(q_j, b_0) = 1$.

Thus, for each $j = 1, \ldots, B$, the congruence $x^{2k} + j \equiv 0 \pmod{q_j}$ has a solution; for example, one can take $x = K_j$. Also, let $l_j \in \mathbb{N}$ so that $q_j^{l_j} \equiv 1 \pmod{b_0}$.

Next, for a fixed $j \in \{1, \ldots, B\}$, we define a polynomial $g_j(x) \in \mathbb{Z}[x]$ by

$$g_j(x) := x^{2k} + j + q_j^{k(2l_j-1)}. \tag{2.8}$$

Then,

$$g_j(K_j) \equiv 0 \pmod{q_j}, \text{ and } g_j'(K_j) = 2kK_j^{2k-1} \not\equiv 0 \pmod{q_j}.$$

Therefore, by Hensel's Lemma, for $m_j \in \mathbb{N}$, there exists $\beta_{j,m_j} \in \mathbb{Z}$ such that

$$\beta_{j,m_j} \equiv K_j \pmod{q_j} \quad \text{and} \quad g_j(\beta_{j,m_j}) \equiv 0 \pmod{q_j^{m_j}}.$$

In particular, set $m_j = k(2l_j - 1) + 1$. Thus using (2.8) in the last congruence above, we deduce that $\beta_{j,k(2l_j-1)+1}$ satisfies

$$\beta_{j,k(2l_j-1)+1}^{2k} + j \equiv 0 \pmod{q_j^{k(2l_j-1)}} \quad \text{and} \quad \beta_{j,k(2l_j-1)+1}^{2k} + j \not\equiv 0 \pmod{q_j^{k(2l_j-1)+1}}. \tag{2.9}$$

Using the Chinese Remainder Theorem, choose a positive integer c such that for all $j = 1, \ldots, B$,

- $c \equiv 1 \pmod 2$,
- $c \equiv a_0 \pmod{b_0}$,
- $c \equiv \beta_{j,k(2l_j-1)+1} \pmod{q_j^{k(2l_j-1)+1}}$.

Note that if b_0 is even, a_0 must be odd, and therefore $c \equiv a_0 \pmod{b_0}$ implies that $c \equiv 1 \pmod 2$, and thus the Chinese Remainder Theorem does apply here. This implies that there exists an odd positive integer c such that $c \equiv a_0 \pmod{b_0}$, and for $j = 1, \ldots, B$,

$$c^{2k} + j \equiv \beta_{k(2l_j-1)+1}^{2k} + j \pmod{q_j^{k(2l_j-1)+1}}.$$

This shows the existence of c and q_j's as claimed in (2.6) and (2.7). From the discussion in the beginning of the proof, we are done. □

2.1 Proof of Theorem 1.1

Now, we give a proof of Theorem 1.1.

Proof Recall that the generating function for $p_A(n)$ is given by

$$F_A(q) := \sum_{n=0}^{\infty} p_A(n)q^n = \prod_{m \in A} \frac{1}{(1 - q^m)}.$$

Consider the formal power series $F(X)$ in the variable X defined as

$$F(X) := \prod_{m\in A} \frac{1}{1-X^m}.$$

Taking the logarithmic derivative of $F(X)$ and then multiplying both sides by X, we obtain

$$\begin{aligned} X\frac{F'(X)}{F(X)} &= \sum_{m\in A} m \sum_{n=1}^{\infty} X^{mn} \\ &= \sum_{l=1}^{\infty} X^l \sum_{\substack{m\mid l\\ m\in A}} m \\ &= \sum_{l=1}^{\infty} \sigma_A(l) X^l \\ &=: H(X), \end{aligned} \tag{2.10}$$

where for any positive integer l, $\sigma_A(l) := \sum_{d\mid l, d\in A} d$. Therefore,

$$XF'(X) = F(X)H(X). \tag{2.11}$$

□

Claim 2.3 *$p_A(n)$ is odd for infinitely many $n \in \mathbb{N}$.*

Proof Assume the contrary, and let, if possible, $p_A(n)$ be odd only for $n_i, i = 1, \ldots, r$, for some fixed positive integer r. Also, without loss of generality, we can assume that $n_1 < \cdots < n_r$. Therefore,

$$F(X) \equiv \sum_{j=1}^{r} X^{n_j} \pmod 2.$$

Using this in (2.11), and the definition of $H(X)$ in (2.10), we see that

$$\sum_{j=1}^{r} n_j X^{n_j} \equiv \sum_{j=1}^{r} X^{n_j} \sum_{l=1}^{\infty} \sigma_A(l) X^l \pmod 2. \tag{2.12}$$

For $B = n_r$ in Proposition 2.2, we obtain a positive integer l_{n_r} so that $\sigma_A(l_{n_r})$ is odd, and $\sigma_A(l_{n_r} + j)$ is even for all $j = 1, \ldots, n_r$. Therefore, comparing the coefficients of $X^{l_{n_r}+n_r}$ on both sides of (2.12) yields

$$\begin{aligned} 0 &\equiv \sum_{\substack{j=1\\ l+n_j = l_{n_r}+n_r}}^{r} \sigma_A(l) \pmod 2 \\ &\equiv \sum_{j=1}^{r} \sigma_A(l_{n_r} + n_r - n_j) \pmod 2 \\ &\equiv \sigma_A(l_{n_r}) \equiv 1 \pmod 2, \end{aligned}$$

which is a contradiction. This completes the proof of Claim 2.3. □

Claim 2.4 *$p_A(n)$ is even for infinitely many $n \in \mathbb{N}$.*

Proof Assume that $p_A(n)$ is even only for $n = m_1 < \cdots < m_v$ for some fixed positive integer v. Therefore,

$$F(X) \equiv \sum_{\substack{j=1 \\ n \neq m_j}}^{v} X^n \pmod 2.$$

In other words,

$$F(X) \equiv \sum_{n=0}^{\infty} X^n + \sum_{j=1}^{v} X^{m_j} \pmod 2.$$

This implies

$$(1 - X)F(X) \equiv 1 - (1 - X) \sum_{j=1}^{v} X^{m_j} \pmod 2. \tag{2.13}$$

Differentiating, and then multiplying both sides by $(1 - X)$, we observe that

$$(1 - X)^2 F'(X) - (1 - X)F(X) \equiv (1 - X) \sum_{j=1}^{v} X^{m_j} - (1 - X)^2 \sum_{j=1}^{v} m_j X^{m_j - 1} \pmod 2.$$

Using (2.13), we find that the above congruence becomes

$$(1 - X)^2 F'(X) \equiv 1 - (1 - X)^2 \sum_{j=1}^{v} m_j X^{m_j - 1} \pmod 2. \tag{2.14}$$

Also, recall from (2.11),

$$X(1 - X)^2 F'(X) = (1 - X)^2 F(X)H(X).$$

Employing this along with (2.10), (2.13) and (2.14), we obtain

$$\begin{aligned} X - (1 - X)^2 \sum_{j=1}^{v} m_j X^{m_j} &\equiv (1 - X) \left\{ 1 - (1 - X) \sum_{j=1}^{v} X^{m_j} \right\} \sum_{l=1}^{\infty} \sigma_A(l) X^l \pmod 2 \\ &\equiv \left\{ 1 + X + \sum_{j=1}^{v} X^{m_j} + \sum_{j=1}^{v} X^{m_j + 2} \right\} \sum_{l=1}^{\infty} \sigma_A(l) X^l \pmod 2. \end{aligned}$$

Let $B = m_v + 2$ in Proposition 2.2. So, we can find a positive integer l_{m_v+2} so that $\sigma_A(l_{m_v+2})$ is odd, while $\sigma_A(l_{m_v+2} + j)$ is even for $j = 1, \ldots, m_v + 2$. Hence, a comparison of coefficients of $X^{l_{m_v+2} + m_v + 2}$ on both sides above yields

$$\begin{aligned} 0 &\equiv \sigma_A(l_{m_v+2} + m_v + 2) + \sigma_A(l_{m_v+2} + m_v + 1) + \sum_{j=1}^{v} \sigma_A(l_{m_v+2} + m_v + 2 - m_j) \\ &\quad + \sum_{j=1}^{v} \sigma_A(l_{m_v+2} + m_v - m_j) \pmod 2 \\ &\equiv \sigma_A(l_{m_v+2}) \pmod 2, \end{aligned}$$

which is a contradiction. Thus, $p_A(n)$ is even for infinitely many positive integers n, which completes the proof of Claim 2.4. □

From Claims 2.3 and 2.4, we obtain Theorem 1.1. □

3 Asymptotics

In this section, we prove two lemmas to be used in the following section in order to compute an asymptotic formula for $p_{A_k(a_0,b_0)}(n)$, as $n \to \infty$. Recall that for a fixed integer $k \geq 2$, $p_{A_k(a_0,b_0)}(n)$ denotes the number of partitions of n with parts in $A_k(a_0, b_0)$, where for integers k, a_0, and b_0 satisfying $0 < a_0 \leq b_0$, $(a_0, b_0) = 1$, and $k \geq 2$,

$$A := A_k(a_0, b_0) = \left\{m^k \in \mathbb{N} : m \equiv a_0 \pmod{b_0}\right\}. \tag{3.1}$$

Also, recall that the generating function $\Psi(z; A)$ is given by

$$\Psi(z; A) = \sum_{n=0}^{\infty} p_A(n) z^n = \prod_{m \in A} \frac{1}{1 - z^m}, \quad |z| < 1. \tag{3.2}$$

For $|z| < 1$, define the function $\Phi(z; A)$ by

$$\Psi(z; A) = \exp(\Phi(z; A)). \tag{3.3}$$

Therefore,

$$\Phi(z; A) = \sum_{j=1}^{\infty} \sum_{m \in A} \frac{z^{jm}}{j}, \quad |z| < 1. \tag{3.4}$$

Throughout the remainder of the paper, we use the standard notation $e(x)$ for $\exp(2\pi i x)$ for any real number x.

Now, we move on to state and prove the results used in the proof of Theorem 1.2 in the next section.

Lemma 3.1 *For each sufficiently large positive number X and $\Theta \in [-3/(8\pi X), 3/(8\pi X)]$, define*

$$\Delta = (1 + 4\pi^2 \Theta^2 X^2)^{-1/2}. \tag{3.5}$$

Let $R := R(k, X, \theta)$ be defined as

$$R = \frac{(2(\pi\Delta)^{k+1} X)^{1/k}}{2k^2}. \tag{3.6}$$

Let $\rho = e^{-1/X}$, and for $a_0, b_0 \in \mathbb{N}$ with $1 \leq a_0 \leq b_0$ and $(a_0, b_0) = 1$, let $\beta_0 = a_0/b_0$. For any complex number $s = \sigma + it$, let $\zeta(s)$ and $\zeta(s, \beta_0)$ denote the Riemann zeta function and the Hurwitz zeta function, respectively. Then, for A and $\Phi(z; A)$ defined in (3.1) and (3.4), respectively, as $X \to \infty$,

$$\Phi(\rho e^{2\pi i \Theta}; A) = \frac{1}{b_0 k} \zeta(1 + 1/k) \Gamma(1/k) \left(\frac{X}{1 - 2\pi i X \Theta}\right)^{1/k} + \zeta(0, \beta_0) \log\left(\frac{b_0^{-k} X}{1 - 2\pi i X \Theta}\right)$$
$$+ k\zeta'(0, \beta_0) + \frac{b_0^k}{2} \zeta(-k, \beta_0) \left(\frac{1 - 2\pi i X \Theta}{X}\right) + \sum_{m=1}^{\lfloor R/2 \rfloor} \frac{b_0^{2mk}}{(2m)!} \zeta(-2m + 1)$$

$$\times\, \zeta(-2km, \beta_0) \left(\frac{1-2\pi i X\Theta}{X}\right)^{2m} + \mathrm{O}_{k,a_0,b_0}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\right), \quad (3.7)$$

where the expression immediately before the error term occurs only when $\beta_0 \neq 1/2, 1$.

Proof The series for the Riemann zeta function $\zeta(s+1)$ and the Hurwitz zeta function $\zeta(ks, \beta_0)$ converge absolutely and uniformly for $\mathrm{Re}\, s > 1/k + \delta$ for any fixed positive number δ. Therefore, using Mellin's transform, we have, for a real number $c > 1/k$,

$$\begin{aligned}
\Phi(\rho e(\Theta); A) &= \sum_{j=1}^{\infty} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \frac{\rho^{jm^k} e(jm^k\Theta)}{j} \\
&= \sum_{j=1}^{\infty} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \frac{1}{j} \exp\left(\frac{-jm^k}{X} + 2\pi i j m^k \Theta\right) \\
&= \sum_{j=1}^{\infty} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \frac{1}{j} \exp\left(-jm^k\, \frac{1-2\pi i X\Theta}{X}\right) \\
&= \frac{1}{2\pi i} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \int_{c-i\infty}^{c+i\infty} \left(jm^k \frac{1-2\pi i X\Theta}{X}\right)^{-s} \Gamma(s)\, ds \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} b_0^{-ks} \left(\frac{1-2\pi i X\Theta}{X}\right)^{-s} \Gamma(s) \\
&\quad \times \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{m=0}^{\infty} \frac{1}{(m+a_0/b_0)^{ks}}\, ds.
\end{aligned}$$

We notice that the series above can be written in terms of the Riemann and the Hurwitz zeta functions. Thus,

$$\begin{aligned}
\Phi(\rho e(\Theta); A) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} b_0^{-ks} \zeta(s+1)\zeta(ks, \beta_0)\Gamma(s) \left(\frac{X}{1-2\pi i X\Theta}\right)^{s} ds \\
&=: \frac{1}{2\pi i} \left(\int_{c-i\infty}^{c-iR} + \int_{c-iR}^{c+iR} + \int_{c+iR}^{c+i\infty}\right) \mathcal{J}_s\, ds, \qquad (3.8)
\end{aligned}$$

where R is defined in (3.6). We compute these integrals using the residue theorem. For the middle integral on the far right side of (3.8), consider the rectangle $\mathcal{R}_m$ with vertices $-R \pm iR$ and $c \pm iR$. Therefore, by the residue theorem,

$$\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \mathcal{J}_s\, ds = \sum_{\text{poles in } \mathcal{R}_m} \mathrm{Res}\, \mathcal{J}_s - \left(\int_{c+iR}^{-R+iR} + \int_{-R+iR}^{-R-iR} + \int_{-R-iR}^{c-iR}\right) \mathcal{J}_s\, ds. \quad (3.9)$$

In order to compute the first integral on the far right side of (3.8), for any real number $L > 0$, we define the rectangle $\mathcal{R}_L$ with vertices $-R - i(R + L)$, $-R - iR$, $c - iR$, and $c - i(R + L)$. Thus, by the residue theorem,

$$\frac{1}{2\pi i} \int_{c-i(R+L)}^{c-iR} \mathcal{J}_s \, ds = \sum_{\substack{\text{poles} \\ \text{in } \mathcal{R}_L}} \operatorname{Res} \mathcal{J}_s - \left(\int_{c-iR}^{-R-iR} + \int_{-R-iR}^{-R-i(R+L)} + \int_{-R-i(R+L)}^{c-i(R+L)} \right) \mathcal{J}_s \, ds. \tag{3.10}$$

Finally, for the last integral on the far right side of (3.8), for any real number $U > 0$, we define the rectangle $\mathcal{R}_U$ given by the vertices $-R + iR$, $-R + i(R + U)$, $c + i(R + U)$, $c + iR$. Once again, by the residue theorem,

$$\frac{1}{2\pi i} \int_{c+iR}^{c+i(R+U)} \mathcal{J}_s \, ds = \sum_{\substack{\text{poles} \\ \text{in } \mathcal{R}_U}} \operatorname{Res} \mathcal{J}_s - \left(\int_{c+i(R+U)}^{-R+i(R+U)} + \int_{-R+i(R+U)}^{-R+iR} + \int_{-R+iR}^{c+iR} \right) \mathcal{J}_s \, ds. \tag{3.11}$$

Here, we can assume that the integrand $\mathcal{J}_s$ defined in (3.8) has no zeros on any of the sides of the rectangles R_m, R_L, R_U.

The only possible poles of the integrand $\mathcal{J}_s$ are at $s = 1/k, 0, -1$, and $-2j$, for each positive integer j. Thus, all the poles are real, which means $\mathcal{J}_s$ is holomorphic within the rectangles R_L and R_U. Therefore, the sum of the residues in (3.10) and (3.11) is zero. Thus, by letting L and U tend to infinity, we have

$$\int_{c-i\infty}^{c-iR} \mathcal{J}_s \, ds = -\int_{c-iR}^{-R-iR} \mathcal{J}_s \, ds - \lim_{L\to\infty} \left(\int_{-R-iR}^{-R-i(R+L)} + \int_{-R-i(R+L)}^{c-i(R+L)} \right) \mathcal{J}_s \, ds, \tag{3.12}$$

and

$$\int_{c+iR}^{c+i\infty} \mathcal{J}_s \, ds = -\int_{-R+iR}^{c+iR} \mathcal{J}_s \, ds - \lim_{U\to\infty} \left(\int_{c+i(R+U)}^{-R+i(R+U)} + \int_{-R+i(R+U)}^{-R+iR} \right) \mathcal{J}_s \, ds. \tag{3.13}$$

Using (3.9), (3.12), and (3.13) in (3.8), we deduce that

$$\begin{aligned} \Phi(\rho e(\Theta); A) = &- \lim_{L\to\infty} \left(\int_{-R-iR}^{-R-i(R+L)} + \int_{-R-i(R+L)}^{c-i(R+L)} \right) \mathcal{J}_s \, ds + \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s \\ &- \int_{-R+iR}^{-R-iR} \mathcal{J}_s \, ds - \lim_{U\to\infty} \left(\int_{c+i(R+U)}^{-R+i(R+U)} + \int_{-R+i(R+U)}^{-R+iR} \right) \mathcal{J}_s \, ds. \end{aligned} \tag{3.14}$$

Now, we show that the two integrals along the horizontal lines above approach zero as U and L approach infinity. Also,

$$\left| \left(\frac{X}{1 - 2\pi i X\Theta} \right)^s \right| = (X\Delta)^\sigma \exp(-t\phi),$$

where ϕ is the argument of $X^s/(1 - 2\pi i X\Theta)^s$. Therefore, $\tan(\pi/2 - \phi) = 1/(2\pi X\Theta)$, and $\sin(\pi/2 - \phi) = \Delta$. Using estimates for the sine function, we see that $\pi/2 - \phi > \Delta$, and therefore,

$$\left| \left(\frac{X}{1 - 2\pi i X\Theta} \right)^s \right| \leqslant (X\Delta)^\sigma \exp(|t|(\pi/2 - \Delta)). \tag{3.15}$$

Also, by Stirling's formula in a vertical strip, for $s = \sigma + it$ and $\alpha \leq \sigma \leq \beta$,

$$|\Gamma(s)| \ll |s|^{\sigma-1/2} \exp(-\pi|t|/2). \tag{3.16}$$

Combining this with (3.15) and standard bounds for $\zeta(s)$ and $\zeta(s, \beta_0)$ (for example, see [19, p. 81], [2, p. 270]), we deduce that there exist constants B and C such that

$$\int_{-R-i(R+L)}^{c-i(R+L)} \mathcal{J}_s \, ds \ll (R+L)^B e^{-(R+L)\Delta+R},$$

and

$$\int_{c+i(R+U)}^{-R+i(R+U)} \mathcal{J}_s \, ds \ll (R+U)^C e^{-(R+U)\Delta+R},$$

which both tend to zero as L and U approach infinity, since R and Δ are both fixed, positive real numbers. Therefore, from (3.14),

$$\begin{aligned}
\Phi(\rho e(\Theta); A) &= -\int_{-R-iR}^{-R-i\infty} \mathcal{J}_s \, ds + \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s - \int_{-R+iR}^{-R-iR} \mathcal{J}_s \, ds \\
&\quad - \int_{-R+i\infty}^{-R+iR} \mathcal{J}_s \, ds \\
&= \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s + \left(\int_{-R-i\infty}^{-R-iR} + \int_{-R-iR}^{-R+iR} + \int_{-R+iR}^{-R+i\infty} \right) \mathcal{J}_s \, ds. \qquad (3.17)
\end{aligned}$$

Next, we find bounds for the integrand $\mathcal{J}_s$ in order to estimate the integrals in (3.17). Using the functional equation (in its asymmetric form) for the Riemann zeta function [4, p. 73], [19, p. 16] and the functional equation for the Hurwitz zeta function [4, p. 72], [19, p. 37], we have

$$\begin{aligned}
\zeta(s+1)\zeta(ks, \beta_0) &= 4 \left\{ \sin\left(\frac{\pi ks}{2}\right) \sum_{m=1}^{\infty} \frac{\cos(2m\pi\beta_0)}{m^{1-ks}} + \cos\left(\frac{\pi ks}{2}\right) \sum_{m=1}^{\infty} \frac{\sin(2m\pi\beta_0)}{m^{1-ks}} \right\} \\
&\quad \times (2\pi)^{(k+1)s-1} \cos(\pi s/2)\zeta(-s)\Gamma(-s)\Gamma(1-ks). \qquad (3.18)
\end{aligned}$$

Using the functional equation and reflection formula for the gamma function,

$$\Gamma(1-ks) = -ks\, \Gamma(-ks), \quad -s \sin(\pi s)\Gamma(-s)\Gamma(s) = \pi,$$

we can write (3.18) in the form

$$\begin{aligned}
\zeta(s+1)\zeta(ks, \beta_0)\Gamma(s) &= 2k(2\pi)^{(k+1)s} \zeta(-s)\Gamma(-ks) \frac{\cos(\pi s/2)}{\sin(\pi s)} \\
&\quad \times \sum_{m=1}^{\infty} \left\{ \sin\left(\frac{\pi ks}{2}\right) \frac{\cos(2m\pi\beta_0)}{m^{1-ks}} + \cos\left(\frac{\pi ks}{2}\right) \frac{\sin(2m\pi\beta_0)}{m^{1-ks}} \right\}.
\end{aligned} \tag{3.19}$$

Also note that

$$\frac{\cos(\pi s/2)}{\sin(\pi s)} \sin(\pi ks/2) = \frac{\sin(\pi ks/2)}{2\sin(\pi s/2)} \ll e^{(k-1)|t|\pi/2},$$

and

$$\frac{\cos(\pi s/2)}{\sin(\pi s)}\cos(\pi ks/2) = \frac{\cos(\pi ks/2)}{2\sin(\pi s/2)} \ll e^{(k-1)|t|\pi/2}.$$

Using these bounds along with (3.15), (3.16), and (3.19), for the integrand $\mathcal{J}_s$ in (3.17), we find that

$$\begin{aligned}\mathcal{J}_s &\ll (2\pi)^{(k+1)\sigma}k^{-k\sigma}|s|^{-1/2-k\sigma}(X\Delta)^{\sigma}e^{-\Delta|t|}\\ &= \left(\frac{k^k}{(2\pi)^{k+1}X\Delta}\right)^R |R+it|^{-1/2+kR}e^{-\Delta|t|},\end{aligned} \tag{3.20}$$

since $\sigma = -R$ here. For the middle integral on the far right side of (3.17), we have $|t| \le R$. Therefore, using the foregoing estimates for the integrand $\mathcal{J}_s$, we arrive at

$$\begin{aligned}\int_{-R-iR}^{-R+iR} \mathcal{J}_s\, ds &\ll \left(\frac{k^k}{2\pi^{k+1}X\Delta}\right)^R R^{-1/2+kR}\int_0^R e^{-\Delta t}\, dt\\ &\ll \left(\frac{(2k^2)^k}{2\pi^{k+1}X\Delta}\right)^R R^{-1/2+kR}\\ &\ll \exp\left(-\frac{(2(\pi\Delta)^{k+1}X)^{1/k}}{2k}\right),\end{aligned} \tag{3.21}$$

where in the penultimate step above, we have used the definition of R. For the first and the last integrals in (3.17), we have the inequality $|t| > R$. Therefore, invoking (3.20), and making a change of variable $y = \Delta t$, we deduce that

$$\begin{aligned}\left(\int_{-R-i\infty}^{-R-iR} + \int_{-R+iR}^{-R+i\infty}\right)\mathcal{J}_s\, ds &\ll \left(\frac{k^k}{2\pi^{k+1}X\Delta}\right)^R \int_R^{\infty} t^{-1/2+kR}e^{-\Delta t}\, dt\\ &\ll \left(\frac{k^k}{2(\pi\Delta)^{k+1}X}\right)^R \int_{\Delta R}^{\infty} y^{-1/2+kR}e^{-y}\, dy\\ &\ll \left(\frac{k^k}{2(\pi\Delta)^{k+1}X}\right)^R \Gamma(kR+1/2)\\ &\ll \left(\frac{(2k^2)^k}{2(\pi\Delta)^{k+1}X}\right)^R e^{-kR}R^{kR}\\ &\ll \exp\left(-\frac{(2(\pi\Delta)^{k+1}X)^{1/k}}{2k}\right),\end{aligned} \tag{3.22}$$

where in the penultimate step above, Stirling's formula is invoked, and in the last step, we have used the definition of R. Using the fact that $\Delta \ge 4/5$ when Θ lies in the interval $[-3/(8\pi X), 3/(8\pi X)]$, and the estimates in (3.21) and (3.22) in (3.17), we obtain

$$\Phi(\rho e(\Theta); A) = \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s + O\left(\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\right). \tag{3.23}$$

Now, we compute the residues in the above sum. From [22, page 267], we know that for any non-negative integer m,

$$\zeta(-m, \beta_0) = -\frac{B_{m+1}(\beta_0)}{m+1}, \tag{3.24}$$

where $B_m(x)$ denotes the Bernoulli polynomial of degree m, and in particular, $B_m(0)$ is the mth Bernoulli number. In particular,

$$\zeta(0,\beta_0)=\frac{1}{2}-\beta_0. \tag{3.25}$$

Nörlund [14, p. 22] showed that $B_{2m+1}(x)$ has only two real zeros, 1/2 and 1, in the interval (0, 1]. Therefore, for any positive integer m, (3.24) implies that $\zeta(-2m,\beta_0)$ is zero if and only if β_0 equals 1/2 or 1.

Therefore, for $\beta_0=1/2,1$, and any positive integer m, because $\Gamma(s)$ has a simple pole, and $\zeta(s,\beta_0)$ has a simple zero at $s=-2m$, the product $\Gamma(s)\zeta(s,\beta_0)$ has a removable singularity at $s=-2m$. Moreover, from (3.25), we know that $\zeta(0,\beta_0)=0$ if and only if $\beta_0=1/2$.

Also, for any positive integer m, the product $\Gamma(s)\zeta(s+1)$ has a removable singularity at $s=-2m+1$ because of the trivial zeros of the Riemann zeta function at the negative even integers.

Thus for $\beta_0=1/2$, and the integrand $\mathcal{J}_s$, defined in (3.8), the only poles are at $s=1/k,0,-1$, and all are simple. For $\beta_0=1$, there is a double pole at $s=0$, and there are simple poles at $s=1/k$ and -1. And for $\beta_0\neq 1/2,1$, there is a double pole at $s=0$, and there are simple poles at $s=1/k,-1$, and $-2l\leq R$, where l is any positive integer.

The residue of $\mathcal{J}_s$ for the pole at $s=1/k$ is given by

$$\operatorname{Res}\mathcal{J}_s|_{s=1/k}=\frac{\zeta(1+1/k)}{b_0k}\Gamma(1/k)\left(\frac{X}{1-2\pi iX\Theta}\right)^{1/k}. \tag{3.26}$$

The function $\zeta(s+1)\Gamma(s)$ has a Laurent expansion of the form, (see [19, p. 16] and [15, p. 139]),

$$\left(\frac{1}{s}-\gamma+\sum_{j=1}^{\infty}a_js^j\right)\left(\frac{1}{s}+\gamma+\sum_{j=1}^{\infty}b_js^j\right)=\frac{1}{s^2}+\sum_{j=0}^{\infty}c_js^j,$$

where a_j, b_j and c_j are constants, and γ is Euler's constant. Thus, the residue of $\mathcal{J}_s$ for the pole at $s=0$ can be written as

$$\operatorname{Res}\mathcal{J}_s|_{s=0}=\zeta(0,\beta_0)\log\left(\frac{X}{1-2\pi iX\Theta}\right)-k\zeta(0,\beta_0)\log b_0+k\zeta'(0,\beta_0). \tag{3.27}$$

Also, the residue at $s=-1$ is given by

$$\operatorname{Res}\mathcal{J}_s|_{s=-1}=-b_0^k\zeta(0)\zeta(-k,\beta_0)\left(\frac{1-2\pi iX\Theta}{X}\right). \tag{3.28}$$

Lastly, for each positive integer $l\leq M$, where M is defined in the statement of this lemma, the residue at the pole at $s=-2l$ lying inside the rectangle R_m is given by

$$\operatorname{Res}\mathcal{J}_s|_{s=-2l}=\frac{b_0^{2lk}}{(2l)!}\zeta(-2l+1)\zeta(-2kl,\beta_0)\left(\frac{1-2\pi iX\Theta}{X}\right)^{2l}. \tag{3.29}$$

Recall from the discussion above that $\zeta(-2kl,\beta_0)$ equals zero for $\beta_0=1/2,1$. Therefore, for these values of β_0, the expression on the right side of (3.29) has the value zero. With this in mind, and using (3.26)–(3.29) in (3.23), we obtain the desired result. □

Lemma 3.2 *For any two natural numbers q and l with $(q,l)=1$, define*

$$S(k;q,l):=\sum_{m=1}^{q}e(l^km/q).$$

Suppose that $X, \Theta \in \mathbb{R}$, $X > 1$, $u \in \mathbb{Z}$, $q \in \mathbb{N}$, $(u, q) = 1$ *and* $\theta = \Theta - u/q$. *Then, for any* $\epsilon > 0$, *and* $\Phi(z; A)$ *defined in* (3.4), *as* $X \to \infty$,

$$\Phi(\rho e(\Theta); A) = \frac{1}{b_0}\,\Gamma(1+1/k)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k}\sum_{j=1}^{\infty}\frac{S(k; q_j, u_j)}{j^{1+1/k} q_j} + O_\epsilon\left(q^{1/2+\epsilon}(1+|\theta|^{1/2}X^{1/2})\log X\right),$$

where $q_j = q/(q, j)$, $u_j = uj/(q, j)$.

Proof Recall the definition of $\Phi(z; A)$ given in (3.4),

$$\Phi(\rho e(\Theta); A) = \sum_{j=1}^{\infty}\frac{1}{j}\sum_{\substack{n=1\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} e^{-jn^k/X} e(jn^k\Theta).$$

Employing

$$e^{-jn^k/X} = \int_n^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X}\,dx \tag{3.30}$$

in the above sum, we obtain

$$\Phi(\rho e(\Theta); A) = \sum_{j=1}^{\infty}\frac{1}{j}\int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X}\sum_{\substack{n\leqslant x\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\,dx. \tag{3.31}$$

Using trivial bounds, integrating by parts, and lastly, making the substitution $y = jx^k/X$, we obtain

$$\begin{aligned}&\int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X}\sum_{\substack{n\leqslant x\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\,dx\\ &\ll \int_0^{\infty} x(kx^{k-1} jX^{-1} e^{-jx^k/X})\,dx\\ &= \int_0^{\infty} e^{-jx^k/X}\,dx = \left(\frac{X}{j}\right)^{1/k}\int_0^{\infty} e^{-y^k}\,dx \ll \left(\frac{X}{j}\right)^{1/k}.\end{aligned}$$

Therefore, invoking the estimates above in (3.31), for a fixed positive integer N, we find that

$$\begin{aligned}&\sum_{j=N+1}^{\infty}\frac{1}{j}\int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X}\sum_{\substack{n\leqslant x\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\,dx\\ &\qquad\ll \sum_{j=N+1}^{\infty}\frac{1}{j}\left(\frac{X}{j}\right)^{1/k} \ll \left(\frac{X}{N}\right)^{1/k}.\end{aligned}$$

Using this in (3.31), we have

$$\Phi(\rho e(\Theta); A) = \Sigma_N + O\left(\left(\frac{X}{N}\right)^{1/k}\right), \tag{3.32}$$

where

$$\Sigma_N := \sum_{j=1}^{N} \frac{1}{j} \int_0^\infty k x^{k-1} j X^{-1} e^{-jx^k/X} \sum_{\substack{n \leqslant x \\ n \equiv a_0 \ (\mathrm{mod}\ b_0)}} e(jn^k \Theta)\, dx. \tag{3.33}$$

By a variation of Theorem 4.1 [20, p. 43], (which can be justified using the Euler-Maclaurin summation formula and standard techniques), we can write, for any real number $\epsilon > 0$,

$$\sum_{\substack{n \leqslant x \\ n \equiv a_0 \ (\mathrm{mod}\ b_0)}} e(jn^k \Theta) = \frac{S(k; q_j, u_j)}{b_0 q_j} \int_0^x e(j\gamma^k \theta)\, d\gamma + \mathrm{O}_\epsilon \left(q_j^{1/2+\epsilon} (1 + x^k j |\theta|)^{1/2} \right).$$

Employing the above estimate in (3.33), and applying (3.30) after interchanging the order of integration below, we obtain

$$\begin{aligned} \Sigma_N &= \frac{1}{b_0} \sum_{j=1}^{N} \frac{S(k; q_j, u_j)}{j q_j} \int_0^\infty k x^{k-1} j X^{-1} e^{-jx^k/X} \int_0^x e(j\gamma^k \theta)\, d\gamma\, dx + \mathrm{O}_\epsilon \left(E_N(X) \right) \\ &= \frac{1}{b_0} \sum_{j=1}^{N} \frac{S(k; q_j, u_j)}{j q_j} \int_0^\infty e^{-j\gamma^k/X} e(j\gamma^k \theta)\, d\gamma + \mathrm{O}_\epsilon \left(E_N(X) \right), \end{aligned} \tag{3.34}$$

with

$$\begin{aligned} E_N(X) :&= \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} \int_0^\infty k x^{k-1} j X^{-1} e^{-jx^k/X} (1 + x^k j |\theta|)^{1/2}\, dx \\ &\ll \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} \left(1 + \int_0^\infty \frac{j|\theta| k x^{k-1}}{2\sqrt{x^k j |\theta|}} \frac{e^{-x^k j/X}}{\sqrt{1 + 1/(x^k j |\theta|)}}\, dx \right) \\ &\ll \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} \left(1 + \frac{k}{2} \sqrt{j|\theta|} \int_0^\infty x^{k/2-1} e^{-x^k j/X}\, dx \right), \end{aligned} \tag{3.35}$$

where in the second step, we have integrated by parts. Using the substitution $y = jx^k/X$ in the integral above, we deduce that

$$\begin{aligned} \int_0^\infty x^{k/2-1} e^{-x^k j/X}\, dx &= (X/j)^{1/k} k^{-1} \int_0^\infty (yX/j)^{1/2-1/k} e^{-y} y^{1/k-1}\, dy \\ &= \frac{1}{k} \left(\frac{X}{j} \right)^{1/2} \int_0^\infty y^{-1/2} e^{-y}\, dy = \frac{1}{k} \sqrt{\frac{\pi X}{j}}. \end{aligned}$$

Using this in (3.35), we see that

$$\begin{aligned} E_N(X) &\ll \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} (1 + \sqrt{\pi X |\theta| / 4}) \\ &\ll q^{1/2+\epsilon} (1 + \sqrt{|\theta| X}) \log N. \end{aligned} \tag{3.36}$$

We now turn our attention to the main term of the expression on the far right side of (3.34). First, we rewrite the integrand there as

$$e^{-j\gamma^k/X} e(j\gamma^k \theta) = \exp(-j\gamma^k X^{-1} (1 - 2\pi i X \theta)), \tag{3.37}$$

and set

$$z = (j\gamma^k X^{-1}|1 - 2\pi i X\theta|e^{i\phi})^{1/k},$$

where ϕ is the argument of $1 - 2\pi i X\theta$, and $|\phi| \leq \pi/2$. This gives

$$\begin{aligned} \int_0^\infty e^{-j\gamma^k/X} e(j\gamma^k\theta))\, d\gamma &= \int_0^\infty \exp(-j\gamma^k X^{-1} e(j\gamma^k X\theta))\, d\gamma \\ &= \left(\frac{X}{j(1-2\pi i X\theta)}\right)^{1/k} \int_{\mathcal{L}} e^{-z^k}\, dz, \end{aligned} \tag{3.38}$$

where $\mathcal{L}$ is the ray $\{z = ue^{i\phi/k} : 0 \leq u < \infty\}$. By Cauchy's theorem, the integral along $\mathcal{L}$ is given by

$$\begin{aligned} \int_{\mathcal{L}} e^{-z^k}\, dz &= \int_0^\infty e^{-u^k}\, du \\ &= \frac{1}{k}\int_0^\infty t^{\frac{1}{k}-1} e^{-t}\, dt = \frac{1}{k}\Gamma\left(\frac{1}{k}\right). \end{aligned}$$

Combining this evaluation along with (3.38), (3.34), and (3.36), we obtain

$$\begin{aligned} \Sigma_N &= \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k} \sum_{j=1}^{N} \frac{S(k; q_j, u_j)}{j^{1+1/k} q_j} \\ &\quad + \mathrm{O}_\epsilon\left(q^{1/2+\epsilon}(1+\sqrt{|\theta|X})\log N + \left(\frac{X}{N}\right)^{1/k}\right). \end{aligned} \tag{3.39}$$

Since $|S(k; q_j, u_j)| \leqslant q_j$, for each j, we have

$$\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k} \sum_{j=N}^{\infty} \frac{S(k; q_j, u_j)}{j^{1+1/k} q_j} \ll \left(\frac{X}{N}\right)^{1/k}.$$

Using this in (3.39), we conclude that

$$\begin{aligned} \Sigma_N &= \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k} \sum_{j=1}^{\infty} \frac{S(k; q_j, u_j)}{j^{1+1/k} q_j} \\ &\quad + \mathrm{O}_\epsilon\left(q^{1/2+\epsilon}(1+\sqrt{|\theta|X})\log N + \left(\frac{X}{N}\right)^{1/k}\right). \end{aligned}$$

Setting $N = \lfloor X \rfloor$ in the above expression and invoking (3.32), we obtain the desired result. □

4 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. The proof relies on the Hardy–Littlewood circle method. First, we write the function $p_A(n)$ as an integral, i.e., by (3.2), (3.3), and Cauchy's theorem,

$$\begin{aligned} p_A(n) &= \int_0^1 \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta \\ &= \int_{\mathcal{U}} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta, \end{aligned} \tag{4.1}$$

where in the last step, using the periodicity of the integrand, we have replaced the unit interval (0, 1] by the unit interval $\mathcal{U} = (-X^{-1+1/k}, 1 - X^{-1+1/k}]$, with X as in (1.4). Now, we define the major and the minor arcs. For $u, q \in \mathbb{N}$ with $(u, q) = 1$, define the major arcs by

$$\mathfrak{M}(q, u) = \{\Theta \in \mathcal{U} : |\Theta - u/q| \le q^{-1} X^{1/k-1}\},$$

and let

$$\mathfrak{M} = \cup_{1 \le u \le q \le X^{1/k}} \mathfrak{M}(q, u).$$

The minor arcs $\mathfrak{m}$ are defined to be the complement of the major arcs in the interval $\mathcal{U}$, i.e.,

$$\mathfrak{m} = \mathcal{U} \backslash \mathfrak{M}. \tag{4.2}$$

First, we compute the integral in (4.1) over the sub-interval $[-3/(8\pi X), 3/(8\pi X)]$, a portion of the major arc $\mathfrak{M}(1, 0)$, i.e., we consider

$$\int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta. \tag{4.3}$$

By Lemma 3.1,

$$\begin{aligned} &\rho^{-n} \exp(\Phi(\rho e(\Theta); A)) \\ &\quad = \rho^{-n} \exp(\widetilde{\Xi}(\rho e(\Theta); A)) \left(1 + \mathrm{O}_{k,a_0,b_0}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1} X)^{1/k}}{2k}\right)\right)\right), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \widetilde{\Xi}(\rho e(\Theta); A) &= \frac{1}{b_0 k} \zeta(1 + 1/k)\Gamma(1/k) \left(\frac{X}{1 - 2\pi i X\Theta}\right)^{1/k} + \zeta(0, \beta_0) \log\left(\frac{b_0^{-k} X}{1 - 2\pi i X\Theta}\right) \\ &\quad + k\zeta'(0, \beta_0) + \frac{b_0^k}{2} \zeta(-k, \beta_0) \left(\frac{1 - 2\pi i X\Theta}{X}\right) + \sum_{m=1}^{\lfloor R/2 \rfloor} \frac{b_0^{2mk}}{(2m)!} \zeta(-2m + 1) \\ &\quad \times \zeta(-2km, \beta_0) \left(\frac{1 - 2\pi i X\Theta}{X}\right)^{2m} \\ &= \frac{1}{b_0 k} \zeta(1 + 1/k)\Gamma(1/k) \left(\frac{X}{1 - 2\pi i X\Theta}\right)^{1/k} + \zeta(0, \beta_0) \log\left(\frac{b_0^{-k} X}{1 - 2\pi i X\Theta}\right) \\ &\quad + k\zeta'(0, \beta_0) + \frac{b_0^k}{2} \zeta(-k, \beta_0) \left(\frac{1 - 2\pi i X\Theta}{X}\right) + \sum_{m=1}^{M-1} \frac{b_0^{2mk}}{(2m)!} \zeta(-2m + 1) \\ &\quad \times \zeta(-2km, \beta_0) \left(\frac{1 - 2\pi i X\Theta}{X}\right)^{2m} + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X^{2M-1}}\right) \\ &=: \Xi(\rho e(\Theta); A) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X^{2M-1}}\right), \end{aligned} \tag{4.5}$$

with

$$R = \frac{(2(\pi\Delta)^{k+1}X)^{1/k}}{2k^2}, \quad \Delta = (1 + 4\pi^3\Theta^2X^2)^{-1/2},$$

and a fixed positive integer M satisfying $M \leq R/2$. This can be seen by combining the fact that $\Delta \geq 4/5$ for $\Theta \in [-3/(8\pi X), 3/(8\pi X)]$, and that $M \leq (2(4\pi/5)^{k+1}X)^{1/k}/(4k^2)$ as per the remark following the statement of Theorem 1.2. Also, from Lemma 3.1, note that the terms (including the error term) in (4.5) involving M disappear when β_0 equals 1/2 or 1.

Thus, using (4.5), we can rewrite $\exp(\Phi(\rho e(\Theta); A))$ in (4.4) as

$$\rho^{-n}\exp(\Phi(\rho e(\Theta); A)) = \rho^{-n}\exp(\Xi(\rho e(\Theta); A))\left(1 + \mathrm{O}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\right) + \mathrm{O}_{k,a_0,b_0}\left(1/X^{2M-1}\right)\right), \tag{4.6}$$

Also,

$$\frac{X}{1 - 2\pi i X\Theta} = X\Delta e^{i\phi},$$

where $\phi = \arg(1 + 2\pi i X\Theta)$. Note that $0 < |\phi| \leq \pi/2$, so $0 < \cos(\phi/k) < 1$. Hence,

$$\left|\left(\frac{X}{1 - 2\pi i X\Theta}\right)^{1/k}\right| = (X\Delta)^{1/k}. \tag{4.7}$$

Therefore, for the first error term in (4.6), we note that, by (4.5) and (4.7),

$$\begin{aligned}
&\rho^{-n}\exp(\Xi(\rho e(\Theta); A))\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\\
&= X^{\zeta(0,\beta_0)}\exp\left(\frac{n}{X} + \frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)(X\Delta)^{1/k} + \frac{b_0^k}{2}\zeta(-k,\beta_0)(X\Delta)^{-1}\right.\\
&\quad\left. + \sum_{m=1}^{M}\frac{b_0^{2mk}}{(2m)!}\zeta(-2m+1)\zeta(-2km,\beta_0)(X\Delta)^{-2m} - \frac{1}{2k}(2(4\pi/5)^{k+1}X)^{1/k}\right)\\
&\ll X^{\zeta(0,\beta_0)}\exp\left(\frac{n}{X} + \frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} - \delta X^{1/k}\right),
\end{aligned} \tag{4.8}$$

where $\delta := \frac{1}{2k}(2(4\pi/5)^{k+1})^{1/k} > 0$. Similarly, for the second error term in (4.6), we have

$$\begin{aligned}
&\rho^{-n}\exp(\Xi(\rho e(\Theta); A))X^{-2M+1}\\
&\quad\ll_{k,a_0,b_0} X^{-2M+1}\exp\left(\frac{n}{X} + \frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k}\right).
\end{aligned} \tag{4.9}$$

Therefore, by (4.8), (4.9), (4.3), and (4.6), we deduce that

$$\begin{aligned}
&\int_{-3/(8\pi X)}^{3/(8\pi X)}\rho^{-n}\exp(\Phi(\rho e(\Theta); A)) - 2\pi i n\Theta)\,d\Theta\\
&\qquad = \rho^{-n}\int_{-3/(8\pi X)}^{3/(8\pi X)}\exp(\Xi(\rho e(\Theta))) - 2\pi i n\Theta)\,d\Theta
\end{aligned}$$

$$+ \mathrm{O}_{k,a_0,b_0}\left(X^{\zeta(0,\beta_0)} \exp\left(\frac{n}{X} + \frac{1}{b_0 k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} - \delta X^{1/k}\right)\right)$$
$$+ \mathrm{O}_{k,a_0,b_0}\left(X^{-2M+1} \exp\left(\frac{n}{X} + \frac{1}{b_0 k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k}\right)\right). \tag{4.10}$$

We now turn our attention to the main term in (4.10). Since $|\Theta| < 1/(2\pi X)$, we can rewrite the expression

$$\Xi(\rho e(\Theta); A) - 2\pi i n\Theta$$

in the integrand as a power series in Θ by expanding the terms in (4.5) using the binomial formula and the Taylor series expansion for the logarithm. Using the definition of X in (1.4), we note that the coefficient of Θ in this power series is equal to zero. Hence, with Y defined in (1.5), the main term in (4.10) is given by

$$\begin{aligned}
&\rho^{-n}\int_{-3/(8\pi X)}^{3/(8\pi X)} \exp(\Xi(\rho e(\Theta); A)) - 2\pi i n\Theta)\, d\Theta \\
&= \rho^{-n} e^{\mathcal{C}} \int_{-3/(8\pi X)}^{3/(8\pi X)} \exp(-Y(2\pi X\Theta)^2 + G(\Theta))\, d\Theta \\
&=: \mathcal{I},
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\mathcal{C} := \frac{1}{b_0 k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} + \zeta(0,\beta_0)\log(b_0^{-k}X) + k\zeta'(0,\beta_0) + \frac{b_0^k}{2}\zeta(-k,\beta_0)X^{-1} \\
+ \sum_{m=1}^{M} \frac{b_0^{2km}}{(2m)!}\zeta(-2m+1)\zeta(-2km,\beta_0)X^{-2m},
\end{aligned} \tag{4.12}$$

and

$$G(\Theta) := \sum_{j=3}^{\infty}(a_j Y + b_j)(2\pi i X\Theta)^j,$$

with

$$\begin{aligned}
a_j &:= \binom{j-1+1/k}{j}\binom{1+1/k}{2}^{-1}, \\
b_j &:= \zeta(0,\beta_0)\left(\frac{1}{j} - \frac{a_j}{2}\right) + \sum_{m=1}^{M} \frac{b_0^{2mk}\zeta(-2m+1)\zeta(-2mk,\beta_0)}{(2m)!X^{2m}}\left(\binom{2m}{j} - a_j\binom{2m}{2}\right).
\end{aligned}$$

Note that since X is large, and M is a fixed positive integer,

$$b_j = \zeta(0,\beta_0)\left(\frac{1}{j} - \frac{a_j}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right).$$

Thus,

$$\frac{b_j}{a_j} = \zeta(0,\beta_0)\left(\frac{1}{j a_j} - \frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right).$$

Also, $ja_j \geq 1$ for any $j \geq 3$, and since $\zeta(0, \beta_0) = 1/2 - \beta_0$ [2, p. 264], we deduce that $|\zeta(0, \beta_0)| \leq 1/2$, since $0 < \beta_0 \leq 1$. Hence,

$$\left|\zeta(0, \beta_0)\left(\frac{1}{ja_j} - \frac{1}{2}\right)\right| \leq \frac{3}{4}.$$

Therefore, for X large, we conclude that $\left|b_j/a_j\right| \leq 1$, i.e., $|b_j| \leq |a_j|$ for all $j \geq 3$.

We rewrite the integral on the right side in (4.11) as

$$\begin{aligned}\rho^n e^{-\mathcal{C}}\mathcal{I} &= \int_{-3/(8\pi X)}^{3/(8\pi X)} \exp(-Y(2\pi X\Theta)^2 + G(\Theta))\, d\Theta \\ &= \int_0^{3/(8\pi X)} (\exp(G(\Theta)) + \exp(G(-\Theta)))\exp(-Y(2\pi X\Theta)^2)\, d\Theta \\ &= 2\int_0^{3/(8\pi X)} \Re \exp(G(\Theta) - Y(2\pi X\Theta)^2)\, d\Theta \\ &= \frac{1}{2\pi X\sqrt{Y}} \int_0^{9Y/16} t^{-1/2}e^{-t}\, \Re \exp(H(t))\, dt, \end{aligned} \tag{4.13}$$

where in the last step, we made the substitution $t = Y(2\pi X\Theta)^2$, and where

$$\begin{aligned} H(t) &:= \sum_{j=3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \\ &= \sum_{j=3}^{2J+2} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} + \sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \\ &=: H_J(t) + \sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2}, \end{aligned} \tag{4.14}$$

for any fixed positive integer J. Note that for $j \geq 2$,

$$|b_{2j}| \leq a_{2j} \leq a_4 = \frac{6k^2 + 5k + 1}{12k^2}.$$

Therefore, for $0 \leq t \leq 9Y/16$, where Y is sufficiently large, and $k \geq 2$,

$$\begin{aligned} \Re H_J(t) &= \Re \sum_{j=3}^{2J+2} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} = \sum_{j=2}^{J+1}(-1)^j(a_{2j}Y + b_{2j})t^jY^{-j} \\ &\leq a_4(Y+1)\sum_{j=2}^{2J+2}\left(\frac{t}{Y}\right)^j \leq a_4(Y+1)\sum_{j=2}^{\infty}\left(\frac{t}{Y}\right)^j \\ &= \frac{6k^2 + 5k + 1}{12k^2}(Y+1)\frac{(t/Y)^2}{1 - t/Y} \\ &\leq \frac{9n}{7} \times \frac{6k^2 + 5k + 1}{12k^2}(1 + Y^{-1})t = \frac{18k^2 + 15k + 3}{28k^2}(1 + Y^{-1})t \\ &= \left(\frac{18}{28} + \frac{15}{28k} + \frac{3}{28k^2}\right)(1 + Y^{-1})t < \left(\frac{18}{28} + \frac{15}{56} + \frac{3}{112}\right)(1 + Y^{-1})t \\ &= \frac{105}{112}(1 + Y^{-1})t < \frac{105}{112}\left(1 + \frac{1}{105}\right)t < \left(1 - \frac{1}{2016}\right)t, \end{aligned} \tag{4.15}$$

where now we assume, at least, that $Y > 105$. Note that by the definition of $H_J(t)$ in (4.14), as we let $J \to \infty$, $H_J(t)$ approaches $H(t)$. Thus, for a fixed positive real number $Z < 9Y/16$, by (4.15),

$$\begin{aligned}\int_Z^{9Y/16} t^{-1/2}e^{-t}\Re\exp(H(t))\,dt &\le \int_Z^{9Y/16} t^{-1/2}e^{-t}e^{\Re H(t)}\,dt\\ &\le \int_Z^{9Y/16} t^{-1/2}e^{-t}\exp\left(t-\frac{t}{2016}\right)dt\\ &\ll Z^{-1/2}\int_Z^{9Y/16} e^{-t/(2016)}\,dt \ll Z^{-1/2}e^{-Z/(2016)}.\end{aligned}$$

We let $Z = 2016J\log Y$ in the above estimates to obtain

$$\int_Z^{9Y/16} t^{-1/2}e^{-t}\,\Re\exp(H(t))\,dt \ll Y^{-J}.$$

This, combined with (4.13) and (4.14), gives

$$\begin{aligned}2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} &= \int_0^Z t^{-1/2}e^{-t}\,\Re\exp(H(t))\,dt + \mathrm{O}\left(Y^{-J}\right)\\ &= \int_0^Z t^{-1/2}e^{-t}\,\Re\exp\left(H_J(t)+\sum_{j=2J+3}^{\infty} i^j(a_j+b_jY^{-1})t^{j/2}Y^{1-j/2}\right)dt\\ &\quad + \mathrm{O}\left(Y^{-J}\right).\end{aligned} \tag{4.16}$$

For $0 \le t \le Z$,

$$\sum_{j=2J+3}^{\infty} i^j(a_j+b_jY^{-1})t^{j/2}Y^{1-j/2} \ll \frac{t^{J+3/2}Y^{-1/2-J}}{1-(t/Y)^{1/2}} \ll t^{J+3/2}Y^{-1/2-J}.$$

Therefore,

$$\exp\left(\sum_{j=2J+3}^{\infty} i^j(a_j+b_jY^{-1})t^{j/2}Y^{1-j/2}\right) = 1+\mathrm{O}\left(t^{J+3/2}Y^{-1/2-J}\right). \tag{4.17}$$

Employing this in (4.16), we deduce that

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\Re\exp(H_J(t))\big(1+\mathrm{O}\left(t^{J+3/2}Y^{-1/2-J}\right)\big)\,dt. \tag{4.18}$$

From (4.15), we see that

$$\Re H_J(t) < \left(1-\frac{1}{2016k^2}\right)t.$$

So, for the error term in (4.18), we find that

$$\int_0^Z t^{J+1}e^{-t}\,\Re\exp(H_J(t))Y^{-1/2-J}\,dt \ll Y^{-1/2-J}\int_0^\infty e^{-t/(2016k^2)}t^{J+1}\,dt \ll Y^{-J}. \tag{4.19}$$

Using this in (4.18), we find that

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\,\Re\exp(H_J(t))\,dt + \mathrm{O}\left(Y^{-J}\right)$$
$$= \int_0^Z t^{-1/2}e^{-t}\,\Re\sum_{j=0}^{\infty}\frac{H_J(t)^j}{j!}\,dt + \mathrm{O}\left(Y^{-J}\right). \qquad (4.20)$$

Next, for $0 \le t \le Z = 2016J\log Y$,

$$H_J(t) = \sum_{j=3}^{2J+2} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \ll \sum_{j=3}^{\infty} Y(t/Y)^{j/2} \ll Y^{-1/2}t^{3/2} \le Y^{-1/4}.$$

Therefore,

$$\left|\int_0^Z t^{-1/2}e^{-t}\,\Re(H_J(t)^j)\,dt\right| \le Y^{-j/4}.$$

This yields

$$\sum_{j=4J+4}^{\infty}\frac{1}{j!}\int_0^Z t^{-1/2}e^{-t}\Re(H_J(t)^j)\,dt \ll Y^{-J}.$$

Using this in (4.20), we obtain

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\,\Re\sum_{j=0}^{4J+3}\frac{1}{j!}H_J(t)^j\,dt + \mathrm{O}\left(Y^{-J}\right). \qquad (4.21)$$

Recall from (4.14) that

$$H_J(t) = \sum_{l=3}^{2J+2}(a_l + b_lY^{-1})Y^{1-l/2}(it^{1/2})^l = \sum_{l=3}^{2J+2}\left(a_l(Y^{-1/2})^{l-2} + b_l(Y^{-1/2})^l\right)(it^{1/2})^l. \qquad (4.22)$$

So, for a fixed Y, $H_J(t)$ can be viewed as a polynomial in $it^{1/2}$ of degree $2J+2$ with real coefficients. Therefore,

$$\sum_{j=0}^{4J+3}\frac{1}{j!}(H_J(t))^j = \sum_{r=0}^{(2J+2)(4J+3)} f_r(Y^{-1/2})(it^{1/2})^r, \qquad (4.23)$$

where $f_r(x)$ is a real polynomial in x of degree not larger than r. Also, from (4.22) it is not hard to verify that

$$f_0(x) = 1,\ f_1(x) = f_2(x) = 0, \quad \text{and} \quad f_r(0) = 0,$$

for $r \ge 3$, and the polynomial $f_r(x)$ is even (odd) when r is even (odd). So, $f_r(x)$ is indeed a polynomial in Y^{-1} when r is even. Using these facts and (4.23) in (4.21), and replacing r by $2r$ below, we conclude that

$$
\begin{aligned}
2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} &= \int_0^Z t^{-1/2}e^{-t}\,\Re \sum_{r=0}^{(2J+2)(4J+3)} f_r(Y^{-1/2})(it^{1/2})^r\,dt + \mathrm{O}\left(Y^{-J}\right)\\
&= \int_0^Z t^{-1/2}e^{-t} \sum_{r=0}^{(J+1)(4J+3)} (-1)^r f_{2r}(Y^{-1/2})t^r\,dt + \mathrm{O}\left(Y^{-J}\right)\\
&= \sum_{r=0}^{(J+1)(4J+3)} (-1)^r f_{2r}(Y^{-1/2}) \int_0^Z t^{r-1/2}e^{-t}\,dt + \mathrm{O}\left(Y^{-J}\right)\\
&= \sum_{r=0}^{(J+1)(4J+3)} \alpha_r Y^{-r} \int_0^Z t^{r-1/2}e^{-t}\,dt + \mathrm{O}\left(Y^{-J}\right),
\end{aligned}
\tag{4.24}
$$

for certain real numbers α_r, with $\alpha_0 = 1$. Note that, since $Z = 2016k^2 J\log Y$,

$$
\begin{aligned}
\int_0^Z t^{r-1/2}e^{-t}\,dt &= \int_0^\infty t^{r-1/2}e^{-t}\,dt - \int_Z^\infty t^{r-1/2}e^{-t}\,dt\\
&= \Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}\left(e^{-Z/2}\int_Z^\infty t^{r-1/2}e^{-t/2}\,dt\right)\\
&= \Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}\left(e^{-Z/2}\Gamma\left(r+\frac{1}{2}\right)\right)\\
&= \Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}\left(Y^{-J}\right),
\end{aligned}
$$

since J is fixed. Using this in (4.24), we conclude that

$$
\begin{aligned}
2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} &= \sum_{r=0}^{(J+1)(4J+3)} \left(\alpha_r Y^{-r}\left(\Gamma(r+1/2) + \mathrm{O}\left(Y^{-J}\right)\right) + \mathrm{O}\left(Y^{-J}\right)\right)\\
&= \sum_{r=0}^{J-1} \alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right)\\
&= \sqrt{\pi} + \sum_{r=1}^{J-1} \alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right),
\end{aligned}
\tag{4.25}
$$

as $\alpha_0 = 1$. Since $\rho = e^{-1/X}$, we deduce from (4.25) that

$$
\mathcal{I} = \frac{1}{2\pi X\sqrt{Y}}\exp\left(\frac{n}{X}+\mathcal{C}\right)\left(\sqrt{\pi} + \sum_{r=1}^{J-1}\alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right)\right).
$$

Therefore, by (4.10) and (4.11),

$$
\begin{aligned}
&\int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n}\exp(\Phi(\rho e(\Theta);A) - 2\pi i n\Theta)\,d\Theta\\
&= \frac{1}{2\pi X\sqrt{Y}}\exp\left(\frac{n}{X}+\mathcal{C}\right)\left(\sqrt{\pi} + \sum_{r=1}^{J-1}\alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{Y^J}\right)\right.\\
&\quad \left. + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X^{2M-1}}\right)\right).
\end{aligned}
\tag{4.26}
$$

The remainder of the proof consists of showing that the contributions from the remaining major arcs and the minor arcs are negligible. In other words, it suffices to show that

$$\int_{\mathcal{U}\backslash[-3/(8\pi X),3/(8\pi X)]} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta \\ \ll_{k,a_0,b_0} \exp\left(\frac{n}{X} + \mathcal{C}\right) \frac{1}{XY^{J+1/2}}.$$

Suppose that $\Theta \in \mathfrak{M}(1,0)\backslash[-3/(8\pi X), 3/(8\pi X)]$, so that $\Theta > 3/(8\pi X)$. Thus,

$$\Delta = (1 + 4\pi^2\Theta^2 X^2)^{-1/2} \leq 4/5.$$

Invoking Lemma 3.2 with $q = 1$ and $u = 0$, we see that, for $\Theta \in \mathfrak{M}(1,0)$,

$$\theta = \Theta,\ |\Theta| \leq X^{1/k-1},\ q_j = 1, u_j = 0,\ S(k; q_j, u_j) = 1,$$

and

$$\begin{aligned}\Phi(\rho e(\Theta); A) &= \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\Theta}\right)^{1/k} \zeta\left(1+\frac{1}{k}\right) + \mathrm{O}_\epsilon\left((1 + X^{1/2}|\theta|^{1/2})\log X\right)\\ &= \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)(X\Delta)^{1/k} + \mathrm{O}_\epsilon\left(X^{1/(2k)+\epsilon}\right),\end{aligned}$$

where $\epsilon > 0$. Therefore, for $\rho = e^{-1/X}$,

$$\begin{aligned}\exp(\Phi(\rho e(\Theta); A)) &= \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)(X\Delta)^{1/k}\right)\left(1 + \mathrm{O}_\epsilon\left(X^{\frac{1}{2k}+\epsilon}\right)\right)\\ &\ll \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)(X\Delta)^{1/k}\right)\\ &\ll \exp\left(\left(\frac{4}{5}\right)^{1/k}\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)X^{1/k}\right). \end{aligned} \tag{4.27}$$

In other words,

$$\exp(\Phi(\rho e(\Theta); A)) \ll \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)X^{1/k} - \gamma_k X^{1/k}\right) =: \exp(\mathcal{D}),$$

where

$$\gamma_k := \left(1 - (4/5)^{1/k}\right)\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right) > 0.$$

Now we rewrite $\mathcal{D}$ in terms of $\mathcal{C}$ defined in (4.12) as follows:

$$\begin{aligned}\mathcal{D} &= \mathcal{C} - \gamma_k X^{1/k} - \zeta(0,\beta_0)\log(b_0^{-k}X) - k\zeta'(0,\beta_0) - \frac{b_0^k}{2}\zeta(-k,\beta_0)X^{-1}\\ &\quad - \sum_{m=1}^{M}\frac{b_0^{2km}}{(2m)!}\zeta(-2m+1)\zeta(-2km,\beta_0)X^{-2m}\\ &= \mathcal{C} - \gamma_k X^{1/k} - \zeta(0,\beta_0)\log(b_0^{-k}X) - k\zeta'(0,\beta_0) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right).\end{aligned}$$

Therefore,

$$\exp(\mathcal{D}) = \exp\left(\mathcal{C} - \gamma_k X^{1/k} - \zeta(0,\beta_0)\log(b_0^{-k}X) - k\zeta'(0,\beta_0)\right)\left(1 + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right)\right)$$
$$\ll X^{\zeta(0,\beta_0)}\exp\left(\mathcal{C} - \gamma_k X^{1/k}\right) \ll X^{-\mathcal{A}}\exp(\mathcal{C}),$$

for any $\mathcal{A} > 0$, since X is large. Using this in (4.27), we have

$$\exp(\Phi(\rho e(\Theta); A)) \ll X^{-\mathcal{A}}\exp(\mathcal{C}). \tag{4.28}$$

Choose $\mathcal{A}$ large enough so that $X^{-\mathcal{A}} \ll X^{-1}Y^{-J-1/2}$, which is possible since X can be written as a monomial in Y. Therefore, for $\rho = e^{-1/X}$, we can conclude that

$$\int_{\mathfrak{M}(1,0)\setminus[-3/(8\pi X),3/(8\pi X)]} \rho^{-n}\exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta \ll \frac{\exp\left(\frac{n}{X} + \mathcal{C}\right)}{XY^{J+1/2}}. \tag{4.29}$$

We now investigate the integral on the remaining major arcs. Let $\Theta \in \mathfrak{M}(q,u)$ with $q > 1$. So, $q \le X^{1/k}$, and $\theta := \Theta - u/q$ satisfies $|\theta| \le q^{-1}X^{1/k-1}$. This gives

$$q^{1/2+\epsilon}(1 + X^{1/2}|\theta|^{1/2})\log X \ll q^{1/2+\epsilon}X^{1/2+\epsilon}|\theta|^{1/2}$$
$$\ll q^{1/2+\epsilon}X^{1/2+\epsilon}q^{-1/2}X^{1/(2k)-1/2}$$
$$\ll q^{\epsilon}X^{1/(2k)+3\epsilon} \ll X^{1/(2k)+3\epsilon}.$$

Once again, by an application of Lemma 3.2, we have

$$\exp\left(\Phi(\rho e(\Theta); A)\right) = \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1 - 2\pi i X\Theta}\right)^{1/k}\sum_{j=1}^{\infty}\frac{S(k; q_j, u_j)}{j^{1+1/k}q_j}\right)$$
$$\times\left(1 + \mathrm{O}_{\epsilon}\left(X^{1/(2k)+\epsilon}\right)\right). \tag{4.30}$$

Recall, from Lemma 3.2 the notation

$$S(k; q_j, u_j) = \sum_{l=1}^{q_j} e(u_j^k l/q_j), \quad q_j = q/(q,j), \quad u_j = uj/(q,j).$$

If $q|j$, then we have $q = (q,j)$, i.e., $q_j = 1$ and $S(k; q_j, u_j) = 1$. On the other hand, if $q \nmid j$, then $q_j > 1$ and it is not difficult to see ([6, Lemma 1]) that there is a constant $\delta_k > 0$ such that $|S(k; q_j, u_j)| \le (1 - \delta_k)q_j$. Thus,

$$\sum_{j=1}^{\infty}\frac{|S(k; q_j, u_j)|}{j^{1+1/k}q_j} = \sum_{\substack{j=1\\ q|j}}^{\infty}\frac{|S(k; q_j, u_j)|}{j^{1+1/k}q_j} + \sum_{\substack{j=1\\ q\nmid j}}^{\infty}\frac{|S(k; q_j, u_j)|}{j^{1+1/k}q_j}$$
$$\le \sum_{\substack{j=1\\ q|j}}^{\infty}\frac{1-\delta_k}{j^{1+1/k}} + \sum_{\substack{j=1\\ q\nmid j}}^{\infty}\frac{1}{j^{1+1/k}}$$
$$= (1-\delta_k)(1 - q^{-(k+1)/k})\zeta(1 + 1/k) + q^{-(k+1)/k}\zeta(1 + 1/k)$$
$$= (1 - \delta_k + \delta_k q^{-(k+1)/k})\zeta(1 + 1/k)$$
$$< (1 - \delta_k/2)\zeta(1 + 1/k),$$

where in the third step, we have used the fact that $\sum_{q|j} j^{-\alpha} = q^{-\alpha}\zeta(\alpha)$, $\alpha > 1$. Employing this in (4.30), we have

$$\exp(\Phi(\rho e(\Theta); A)) \ll \exp\left(\frac{1}{b_0 k}(1-\delta_k/2)\zeta(1+1/k)\Gamma(1/k)X^{1/k}\right)$$
$$\ll \exp(\mathcal{C})\, X^{-1}Y^{-J-1/2},$$

which can be justified using the arguments as in (4.27) leading up to (4.28). Let $\tilde{\mathfrak{M}} = \mathfrak{M}\backslash\mathfrak{M}(1,0)$. Then, the bounds above imply that for $\rho = e^{-1/X}$,

$$\int_{\tilde{\mathfrak{M}}} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta) d\Theta \ll \frac{\exp\left(\frac{n}{X} + \mathcal{C}\right)}{XY^{J+1/2}}. \tag{4.31}$$

To obtain an upper bound for the contribution from the minor arcs, we first prove the following lemma.

Lemma 4.1 *For X defined in (1.4), and $\mathfrak{m}$ in (4.2), let $\rho = e^{-1/X}$, $\Theta \in \mathfrak{m}$. Then for $\Phi(z; A)$, defined in (3.4),*

$$\Phi(\rho e(\Theta); A) \ll_\epsilon X^{1/k - 2^{1-k}/k + \epsilon}.$$

Proof Let K be a positive integer. As in the proof of Lemma 3.2, we have

$$\Phi(\rho e(\Theta)) = \sum_{j=1}^{K} \frac{1}{j} \int_0^\infty kx^{k-1} jX^{-1} e^{-x^k j/X} \sum_{\substack{n\le x\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\, dx + \mathrm{O}\left((X/K)^{1/k}\right)$$
$$= \sum_{j=1}^{K} \frac{1}{j} \int_0^\infty kx^{k-1} jX^{-1} e^{-x^k j/X} \sum_{m=0}^{m=\lfloor (x-a_0)/b_0 \rfloor} e(j\Theta(a_0+b_0 m)^k)\, dx$$
$$+ \mathrm{O}\left((X/K)^{1/k}\right). \tag{4.32}$$

For each j, we use Dirichlet's approximation theorem to choose $u_j \in \mathbb{Z}_{\ge 0}$, $q_j \in \mathbb{N}$, so that

$$\left| jb_0^k m^k \Theta - \frac{u_j}{q_j} \right| \le q_j^{-1} X^{1/k-1}, \text{ and } q_j \le X^{1-1/k}.$$

By Weyl's inequality [20, Lemma 2.5],

$$\sum_{m=0}^{m=\lfloor (x-a_0)/b_0 \rfloor} e(j\Theta(a_0+b_0 m)^k) \ll_\epsilon x^{1+\epsilon-2^{-(k+1)}} + x^{1+\epsilon} q_j^{-2^{-(k-1)}} + x^{1+\epsilon}(q_j/x^k)^{2^{-(k-1)}}.$$

Note that for any $\lambda > 0$, an integration by parts gives

$$\int_0^\infty x^\lambda (jkx^{k-1}X^{-1}e^{-x^k j/X})\, dx \ll \left(\frac{X}{j}\right)^{\lambda/k}. \tag{4.33}$$

Also, since $\Theta \notin \mathfrak{M}$, we have $jb_0^k m^k q_j > X^{1/k}$. Furthermore, recall that $q_j \le X^{1-1/k}$. Invoking (4.33), and using these bounds for q_j in (4.32), we conclude that

$$\sum_{j=1}^{K}\frac{1}{j}\int_{1}^{\infty}kx^{k-1}jX^{-1}e^{-x^kj/X}\sum_{m=0}^{m=\lfloor(x-a_0)/b_0\rfloor}e(j\Theta(a_0+b_0m)^k)\,dx$$
$$\ll_e \sum_{j=1}^{K}\frac{1}{j}\left(\left(\frac{X}{j}\right)^{\frac{1+\epsilon}{k}-\frac{1}{k2^{k-1}}}+\left(\frac{X}{j}\right)^{\frac{1+\epsilon}{k}}q_j^{-2^{-(k-1)}}+\left(\frac{X}{j}\right)^{\frac{1+\epsilon}{k}-\frac{1}{2^{k-1}}}q_j^{2^{-(k-1)}}\right)$$
$$\ll_e X^{\frac{1+\epsilon}{k}-\frac{1}{k2^{k-1}}}\sum_{j=1}^{K}\left(j^{-1-\frac{1+\epsilon}{k}+\frac{1}{k2^{k-1}}}+j^{-1-\frac{1+\epsilon}{k}+\frac{1}{k2^{k-1}}}\right)+\left(\frac{X}{K}\right)^{1/k}$$
$$\ll_e X^{\frac{1}{k}+\epsilon-\frac{1}{k2^{k-1}}}+\left(\frac{X}{K}\right)^{1/k}.$$

Letting K approach infinity, we obtain the desired bounds. □

For $\Theta\in\mathfrak{m}$, by Lemma 4.1,

$$\Phi(\rho e(\Theta);A)\ll X^{\frac{1}{k}-\frac{1}{2^{2015k}k}}.$$

Therefore, for some positive constant $\nu<1$,

$$\Phi(\rho e(\Theta);A)\le\nu\frac{1}{b_0k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)X^{1/k}.$$

Using the argument in (4.27) leading to (4.28), we conclude that

$$\int_{\mathfrak{m}}\rho^{-n}\exp(\Phi(\rho e(\Theta);A)-2\pi in\Theta)\,d\Theta\ll\exp\left(\frac{n}{X}+\mathcal{C}\right)X^{-1}Y^{-J-1/2}. \tag{4.34}$$

Combining (4.1), (4.26), (4.29), (4.31), and (4.34), we deduce that

$$p_A(n)=\frac{1}{2\pi X\sqrt{Y}}\exp\left(\frac{n}{X}+\mathcal{C}\right)$$
$$\times\left(\sqrt{\pi}+\sum_{r=1}^{J-1}\Gamma(r+1/2)\frac{\alpha_r}{Y^r}+O\left(\frac{1}{Y^J}\right)+O\left(\frac{1}{X^{2M-1}}\right)\right).$$

Substituting the values of $\mathcal{C}$ and n/X from (4.12) and (1.4), respectively, in the foregoing expression, we obtain the desired bounds for $p_A(n)$. The remark about the disappearance of the terms involving M follows from our earlier discussion after (4.5). This completes the proof of Theorem 1.2.

5 Future directions

In this paper, we are concerned with partitions into parts of the form $(a_0+mb_0)^k$, for some fixed positive integers k, a_0, and b_0 with $(a_0,b_0)=1$. It would be interesting to know whether there are versions of Theorems 1.1 and 1.2 for a more general partition function, say, where parts are of the form of a general polynomial, $\sum_{j=1}^{k}a_jm^j$ for some fixed positive integers a_j. Since the focus of this paper is about parity and asymptotics of partitions into powers of a fixed residue, we do not pursue this here, but it would be interesting to find analogues of Theorems 1.1 and 1.2.

Table 1 Counting the number of even and odd values of selected partition functions

k	a_0	b_0	Even	Odd	k	a_0	b_0	Even	Odd	k	a_0	b_0	Even	Odd
1	1	1	49800	50200	1	5	6	49850	50150	1	1	9	50133	49867
1	1	2	99484	**516**	1	1	7	50103	49897	1	2	9	50040	49960
1	1	3	49991	50009	1	2	7	49845	50155	1	4	9	50356	49644
1	2	3	50082	49918	1	3	7	49861	50139	1	5	9	50306	49694
1	1	4	49815	50185	1	4	7	50048	49952	1	7	9	49899	50101
1	3	4	49945	50055	1	5	7	50050	49950	1	8	9	50129	49871
1	1	5	49715	50285	1	6	7	50009	49991	1	1	10	49801	50199
1	2	5	50044	49956	1	1	8	49867	50133	1	3	10	50231	49769
1	3	5	50066	49934	1	3	8	50007	49993	1	7	10	50246	49754
1	4	5	49668	50332	1	5	8	50130	49870	1	9	10	49852	50148
1	1	6	50021	49980	1	7	8	50104	49896	1	1	11	49929	50071

Yang [24] considered the partition function $p_\Lambda(n)$ given by

$$\sum_{n=1}^{\infty} p_\Lambda(n)x^n = \prod_{m=1}^{\infty}(1-x^m)^{-\Lambda(m)},$$

where $\Lambda(m)$ denotes the von Mangoldt function. Improving an asymptotic formula of Richmond [18] for $p_\Lambda(n)$, Yang proved that the Riemann Hypothesis holds if and only if the error term in Richmond's theorem can be improved to a certain order. One may ask if Theorem 1.2 can be used to provide further insight into representations of integers as sums of kth powers, in analogy with Yang's theorem.

Several lower bounds have been obtained for the number of times the ordinary partition function $p(n)$ is even (odd) for $n \leq N$, as N approaches infinity (for example, see Ono [17], and Nicolas [10]). With regard to Theorem 1.1, it would be nice to obtain similar results for the function $p_{A_k(a,b_0)}(n)$ studied in this paper. In fact, numerical experiments suggest that like $p(n)$, this function also assumes even values about half the time in almost all the cases, as explained below.

For positive integers n up to 100000, and for certain values of a_0, b_0, and k, we provide two tables, Table 1 and Table 2 with the number of times $p_{A_k(a,b_0)}(n)$ is even, and odd, respectively.

We pose the following two conjectures.

Conjecture 5.1 *For positive integers $a_0 \leq b_0$ with $(a_0, b_0) = 1$, let $p_{A_1(a_0,b_0)}(n)$ be as in (1.2) with $A_1(a_0, b_0)$ defined in (1.1). Then, for $b_0 \neq 2$, $p_{A_1(a_0,b_0)}$ is even (odd) approximately half the time, i.e. for $N \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N : p_{A_1(a_0,b_0)}(n) \text{ is even}\} = \frac{1}{2}. \tag{5.1}$$

It is clear from Table 1 that for $b_0 = 2$ (hence $a_0 = 1$) and $k = 1$, (5.1) is nowhere close to being true. In fact, in this case by first applying Euler's theorem (number of partitions into distinct parts equals number of partitions into odd parts), and then Euler's pentagonal number theorem (modulo 2), we obtain that for some positive constant ν,

$$\#\{1 \leq n \leq N : p_{A_1(1,2)} \text{ is odd}\} \sim \nu\sqrt{N},$$

as N tends to infinity.

Table 2 Counting the numbers of even and odd values of selected partition functions

k	a_0	b_0	Even	Odd	k	a_0	b_0	Even	Odd	k	a_0	b_0	Even	Odd
2	1	1	50299	49701	2	1	7	50362	49638	3	1	5	49606	50394
2	1	2	49696	50304	2	2	7	49971	50029	3	2	5	50475	49525
2	1	3	49581	50419	2	3	7	50110	49890	3	3	5	51020	48980
2	2	3	50013	49987	2	4	7	50333	49667	3	4	5	54063	45937
2	1	4	50059	49941	2	5	7	50201	49879	4	1	1	50084	49916
2	3	4	50001	49999	2	6	7	50695	49305	4	1	2	50235	49765
2	1	5	50333	49667	3	1	1	50286	49714	4	1	3	49385	50614
2	2	5	49809	50191	3	1	2	50066	49934	4	2	3	54628	45372
2	3	5	50043	49957	3	1	3	49931	50069	5	1	1	50202	49798
2	4	5	50540	49460	3	2	3	50459	49541	5	1	2	48596	51404
2	1	6	50134	49866	3	1	4	50283	49717	6	1	1	49869	50131
2	5	6	50174	49826	3	3	4	52350	47650	7	1	1	50456	49544

Conjecture 5.2 *For positive integers a_0, b_0 and k with $(a_0, b_0) = 1, a_0 \leq b_0$, and $k \geq 2$, let $p_{A_k(a_0,b_0)}(n)$ be as in (1.2) with $A_k(a_0, b_0)$ defined in (1.1). Then, $p_{A_k(a_0,b_0)}$ is even approximately half the time.*

Notice that $p_{A_k(a_0,b_0)}(n)$ equals zero for all n with $1 < n < a_0^k$. Thus for "large" a_0k, one needs to compute this function for n up to a "large" number N before one can start to witness this phenomena, as is clear from the two tables above.

Note that after Conjecture 5.1 we discussed a case for which (5.1) is invalid. However, Theorem 1.1 has no such exceptions, and our proof is uniform for all k and for all arithmetic progressions.

References

1. Ahlgren, S.: Distribution of parity of the partition function in arithmetic progressions. Indag. Math. (N.S.) **10**, 173–181 (1999)
2. Apostol, T.M.: Introduction to Analytic Number Theory, vol. 1. Springer, Berlin (1976)
3. Berndt, B.C., Yee, A.J., Zaharescu, A.: New theorems on the parity of partition functions. J. Reine Angew. Math. **566**, 91–109 (2004)
4. Davenport, H.: Multiplicative Number Theory, 3rd edn. Springer, New York (2000)
5. Fabrykowski, J., Subbarao, M.V.: Some new identities involving the partition function $p(n)$. In: Mollin, R.A. (ed.) Number Theory, pp. 125–138. Walter de Gruyter, New York (1990)
6. Gafni, A.: Power partitions. J. Number Theory **163**, 19–42 (2016)
7. Hardy, G.H., Ramanujan, S.: Asymptotic formulae in combinatory analysis. Proc. Lond. Math. Soc. **17**, 75–115 (1918)
8. Kolberg, O.: Note on the parity of the partition function. Math. Scand. **7**, 377–378 (1959)
9. Newman, M.: Periodicity modulo m and divisibility properties of the partition function. Trans. Am. Math. Soc. **97**, 225–236 (1960)
10. Nicolas, J.-L.: Odd values of the partition function $p(n)$. Int. J. Number Theory **2**(4), 469–487 (2006)
11. Nicolas, J.-L., Ruzsa, I .Z., Sárközy, A.: On the parity of additive representation functions. With an appendix by J. P. Serre. J. Number Theory. **73**, 292–317 (1998)
12. Nicolas, J.-L., Sárközy, A.: On the asymptotic behaviour of general partitions functions. Ramanujan J. **4**(1), 29–39 (2000)
13. Nicolas, J.-L., Sárközy, A.: On the asymptotic behaviour of general partitions functions II. Ramanujan J. **7**(1–3), 279–298 (2003)

14. Nörlund, N.E.: Vorlesungen über Differenzenrechnung. Chelsea, New York (1954)
15. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Special Functions. Cambridge University Press, Cambridge (2010)
16. Ono, K.: Parity of the partition function in arithmetic progressions. J. Reine Angew. Math. **472**, 1–15 (1996)
17. Ono, K.: The partition function in arithmetic progressions. Math. Ann. **312**, 251–260 (1998)
18. Richmond, L.B.: A general asymptotic result for partitions. Can. J. Math. **27**(5), 1083–1091 (1975)
19. Titchmarsh, E.C.: The Theory of the Riemann Zeta-function, 2nd edn. The Clarendon Press, Oxford University Press, New York (1986)
20. Vaughan, R.C.: The Hardy–Littlewood Method, 2nd edn. Cambridge University Press, Cambridge (1997)
21. Vaughan, R.C.: Squares: additive questions and partitions. Int. J. Number Theory **11**(5), 1367–1409 (2015)
22. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis, reprint of the fourth (1927) edition. Cambridge University Press, Cambridge (1996)
23. Wright, E.M.: Asymptotic partition formulae, III, partitions into kth powers. Acta Math. **63**, 143–191 (1934)
24. Yang, Y.: Partitions into primes. Trans. Am. Math. Soc. **352**(6), 2581–2600 (2000)