SPATIAL STATISTICS OF APOLLONIAN GASKETS

WEIRU CHEN, MO JIAO, CALVIN KESSLER, AMITA MALIK, AND XIN ZHANG

Abstract. Apollonian gaskets are formed by repeatedly filling the interstices between four mutually tangent circles with further tangent circles. We experimentally study the nearest neighbor spacing, pair correlation, and electrostatic energy of centers of circles from Apollonian gaskets. Even though the centers of these circles are not uniformly distributed in any 'ambient' space, after proper normalization, all these statistics seem to exhibit some interesting limiting behaviors.

1. introduction

Apollonian gaskets, named after the ancient Greek mathematician, Apollonius of Perga (200 BC), are fractal sets obtained by starting from three mutually tangent circles and iteratively inscribing new circles in the curvilinear triangular gaps. Over the last decade, there has been a resurgent interest in the study of Apollonian gaskets. Due to its rich mathematical structure, this topic has attracted attention of experts from various fields including number theory, homogeneous dynamics, group theory, and as a consequent, significant results have been obtained.

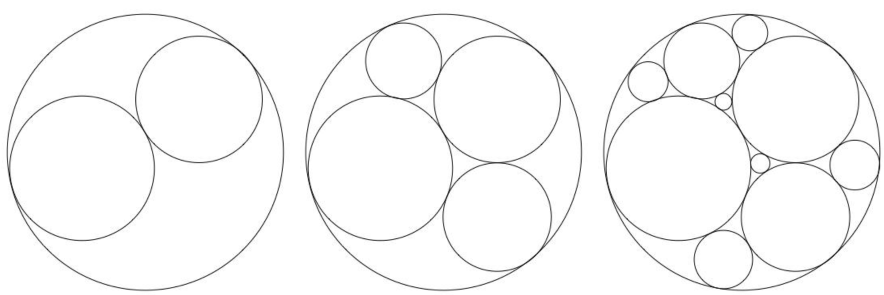

Figure 1. Construction of an Apollonian gasket

For example, it has been known since Soddy [23] that there exist Apollonian gaskets with all circles having integer curvatures (reciprocal of radii). This is due to the fact that the curvatures from any four mutually tangent circles satisfy a quadratic equation (see Figure 2). Inspired by [12], [10], and [7], Bourgain and Kontorovich used the circle method to prove a fascinating result that for any *primitive* integral (integer curvatures with *gcd 1)* Apollonian gasket, almost every integer in certain congruence classes modulo 24 is a curvature of some circle in the gasket.

In another direction, Kontorovich and Oh [16] obtained an asymptotic result for counting circles from an Apollonian gasket $\mathcal{P}$ using the spectral theory of infinite volume hyperbolic spaces, which was originally developed by Lax-Phillips [17]. Their result is stated below.

Theorem 1.1 (Kontorovich-Oh)**.** *Fix an Apollonian gasket $\mathcal{P}$, and let $\mathcal{P}_T$ be the collection of circles in this gasket with curvatures less than T. As T approaches infinity,*

$$\lim_{T\to\infty} \frac{\#\mathcal{P}_T}{T^\delta} = c_{\mathcal{P}},$$

where $c_{\mathcal{P}}$ is a positive constant depending on the gasket $\mathcal{P}$, and $\delta \approx 1.305688$ is the Hausdorff dimension of any Apollonian gasket.

The reason for all Apollonian gaskets to have the same Hausdorff dimension is that they belong to the same conformally equivalent class. In other words, for any two fixed gaskets, one can always find a Möbius transformation which maps one gasket to the other. The estimate $\delta \approx 1.305688$ was obtained in [19].

Kontorovich and Oh's result was refined by Oh and Shah [20] using homogeneous dynamics.

Theorem 1.2 (Oh-Shah)**.** *For any Apollonian gasket $\mathcal{P}$ placed in the complex plane $\mathbb{C}$, there exists a finite Borel measure μ, such that for any region $\mathcal{R} \subset \mathbb{C}$ with piecewise analytic boundary (see Figure 3), the cardinality of $\mathcal{P}_T(\mathcal{R})$, the set of circles from $\mathcal{P}_T$ lying in $\mathcal{R}$, satisfies*

$$\lim_{T\to\infty} \frac{\mathcal{P}_T(\mathcal{R})}{T^\delta} = \mu(\mathcal{R}).$$

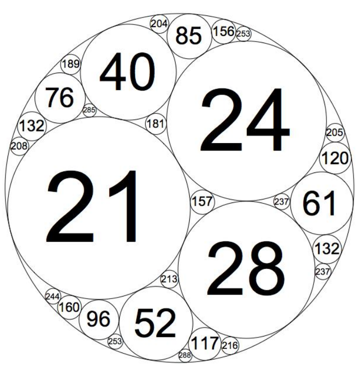

FIGURE 2. An integer Apollonian gasket

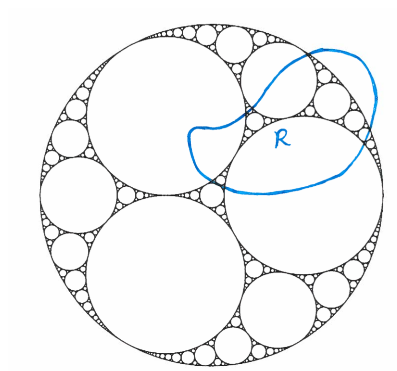

FIGURE 3. A region $\mathcal{R}$ with piecewise analytic boundary

Theorem 1.2 gives a satisfactory explanation on how circles are distributed in an Apollonian gasket on large scale. However, it yields little information on questions concerning the fine scale distribution of circles. For example, one such question is the following.

Question 1.3. *Fix $s > 0$. If one sits at the center of a random circle from $\mathcal{P}_T$, how many circles can one see within a distance of s/T?*

The fine structures of Apollonian gaskets are encoded by local spatial statistics. In this article, we report our empirical results on some of such statistics, namely, nearest neighbor spacing, pair correlation, and electrostatic energy. We find numerically that after proper normalization all these statistics exhibit some interesting limiting behavior when the growing parameter T approaches infinity. Our conjectures in this direction are based on these numerical results. In particular, a rigorous proof of Conjecture 2.2 will provide an asymptotic answer to Question 1.3 when T is large.

These spatial statistics have been widely used in disciplines such as physics and biology. For instance, in microscopic physics, the Kirkwood-Buff Solution Theory [15] links the pair correlation function of gas molecules, which encodes the microscopic details of the distribution of these molecules, to macroscopic thermodynamical properties of the gas such as pressure and potential energy. On a macroscopic level, cosmologists also use pair correlations to predict the likelihood of finding one galaxy in a neighborhood of another galaxy.

Within mathematics, there is a rich literature on the spatial statistics of point processes arising from various settings. A stunning application of pair correlation is a discovery made by Montgomery and Dyson. They found that the pair correlation function of the Riemann zeta zeros agrees with that of the eigenvalues of random Hermitian matrices. This relation (still unproven rigorously) appears to give evidences to Hilbert and Pólya's speculation that the Riemann zeta zeros correspond to the eigenvalues of a self-adjoint operator on a Hilbert space. Spatial statistics from some other point processes have otherwise been rigorously established: gap distribution of the fractional parts of $(\sqrt{n})$ by Elkies and McMullen [9], distribution of directions of lattices points [6], [5], [14], [21], [18], [8], distribution of Farey sequences [13], [4], [3], and gap distribution of saddle connection directions in translation surfaces [1], [2], [24], gap distribution of tangencies in circle packings [22]. Our list of interesting works here is far from inclusive.

There is a fundamental difference between all mentioned works above and our investigation of circles. In their work, the underlying point sequences become uniformly distributed in their 'ambient' spaces. In our case, we parametrize each circle by its center and define the distance between two circles to be the Euclidean distance between their centers. Thus, our study of circles becomes the study of their centers. However, the set of centers is clearly not even dense in any reasonable ambient space such as D, the disk bounded by the largest circle of the gasket. In fact, we notice that centers tend to cluster over some tiny regions and meanwhile we can find plenty of holes in D in which no center can be found. Consequently, we need different normalizations of parameters, as hinted in the last author's work [25] on the gap distribution of a point orbit of an infinite-covolume Schottky group.

2. Experimental results and conjectures

The point process under our investigation is $\mathcal{C}_T$, the set of centers of circles from $\mathcal{P}_T$, where $\mathcal{P}_T$ is the collection of all circles in $\mathcal{P}$ with curvatures smaller than T. In this section, we provide data for the (normalized) spatial statistics such as electrostatic energy, nearest neighbor spacing, pair correlations and state our conjectures. All gaskets under consideration here come from four mutually tangent circles C_0, C_1, C_2, C_3 with C_0 bounding the other three and of radius 1. We use $\mathbb{C}$-coordinates for these circles so that $C_0 = \{z \in \mathbb{C} : |z| = 1\}$, and C_1 is tangent to C_0 at $e^{0\mathbf{i}} = 1$. Suppose that C_2 and C_3 are tangent to C_0 at $e^{\theta_1 \mathbf{i}}$ and $e^{\theta_2 \mathbf{i}}$, respectively. The pair (θ_1, θ_2) then uniquely determines an Apollonian gasket which we denote by $\mathcal{P}(\theta_1, \theta_2)$.

2.1. Nearest spacing.

For the set $\mathcal{C}_T$ and a point $x \in \mathcal{C}_T$, we let $g_T(x)$ to be the distance between x and a closest point in $\mathcal{C}_T$ to x. The nearest spacing function $H_T(s)$ for the set $\mathcal{C}_T$ is then defined as

$$H_T(s) := \frac{1}{\#\mathcal{C}_T} \sum_{x \in \mathcal{C}_T} \mathbf{1}\left\{ g_T(x) \cdot T < s \right\}, \tag{1}$$

where $\mathbf{1}\{A\}$ is the indicator function which has the value 1 if the condition A is true, and 0 otherwise.

From Theorem 1.2, we have $\#\mathcal{C}_T \sim c_{\mathcal{P}} T^{\delta}$. If these centers were randomly distributed in D, then a typical point is at the distance of $\asymp 1/T^{\delta/2}$ to its nearest neighbor, and we can normalize this distance by multiplying with $T^{\delta/2}$. Here for two quantities M_1, M_2, the relation $M_1 \asymp M_2$ means that there exists $c_1, c_2 > 0$ such that $c_1 M_1 < M_2 < c_2 M_1$.

But $\mathcal{C}_T$ is not typical of a random distribution: it tends to converge to the packing $\mathcal{P}$ as T grows. In consequence, we need to normalize g_T by multiplying with T instead of $T^{\delta/2}$. The reason is that from Theorem 1.1, most circles in the family $\mathcal{P}_T$ have radius $\asymp 1/T$, and two tangent circles in $\mathcal{C}_T$ having radius $\asymp 1/T$ also have distance $\asymp 1/T$ (recall that the distance between two circles is the Euclidean distance between their centers), so that $1/T$ is the right scale to measure the local spacing of circles. In Section 2.2 we also use the same normalization for the pair correlation functions.

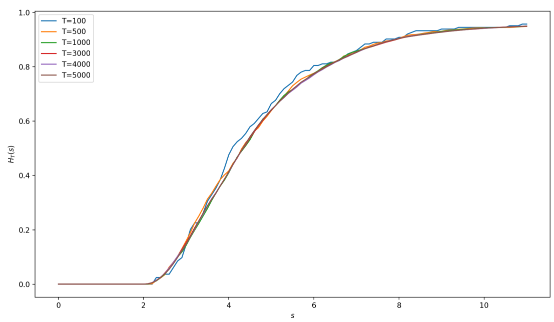

FIGURE 4. The nearest neighbor spacing function $H_T(s)$ for various T's

Figure 4 is the plot of the nearest spacing function for the gasket $\mathcal{P}(\frac{1.8}{3}\pi, \frac{3.7}{3}\pi)$ for various T's, which depicts some convergence behavior. We observe that $H_T \equiv 0$ on the interval $[0, c)$ for some $c > 0$. This can be explained by the following: any two circles $C_1, C_2 \in \mathcal{P}_T$ have radius $> 1/T$, so the distance of their centers p, q is $> 2/T$. Therefore, we always have $d(p, q)T > 2$.

Based on Figure (4), we pose the following conjecture.

Conjecture 2.1. *There exists a non-negative, monotone, continuous function H on $[0, \infty)$ which is supported away from 0 such that*

$$\lim_{T \to \infty} H_T(s) = H(s),$$

for any $s \in [0, \infty)$, where $H_T(s)$ is defined as in (1).

2.2. **Pair correlation.** The pair correlation function $F_T(s)$ for the set $\mathcal{C}_T$ is defined to be

$$F_T(s) := \frac{1}{2\#\mathcal{C}_T} \sum_{\substack{p,q \in \mathcal{C}_T \\ p \neq q}} \mathbf{1}\left\{d(p, q)T < s\right\},$$

where $d(\cdot, \cdot)$ is the Euclidean distance function. Again, a typical distance between two nearby circles in $\mathcal{P}_T$ is $\asymp 1/T$, so we need to normalize $d(\cdot, \cdot)$ by multiplying with T. We have a factor $1/2$ in $F_T(s)$ because we only want to count each pair of points once.

Let

$$N(p, T, s) = \sum_{\substack{q \in \mathcal{C}_T \\ q \neq p}} \mathbf{1}\{d(p, q)T < s\}$$

Then $N(p, T, s)$ counts the number of circles in $\mathcal{P}_T$ having distance s/T to the circle centered at p.

Writing

$$F_T(s) = \frac{1}{2\#\mathcal{C}_T} \times \sum_{p \in \mathcal{C}_T} N(p, T, s),$$

we see that $F_T(s)$ can be interpreted as one half of the expectation of the number of circles one can see within a distance s/T, when one sits at the center of a random circle in $\mathcal{P}_T$.

Figure 5 is the empirical plot for the pair correlation function F_T for the gasket $\mathcal{P}(\frac{1.8}{3}\pi, \frac{3.7}{3}\pi)$, with various values of T taken. It seems that these curves indeed stay close to each other. It suggests that the answer to Question 1.3 can be described by a continuous monotone function, when T is large.

We can also study the pair correlation function for centers restricted to some subset $\mathcal{R}$ of $\mathbb{C}$:

$$F_{T,\mathcal{R}}(s) := \frac{1}{2\#\mathcal{C}_{T,\mathcal{R}}} \sum_{\substack{p,q \in \mathcal{C}_{T,\mathcal{R}} \\ p \neq q}} \mathbf{1}\left\{d(p, q)T < s\right\}, \tag{2}$$

where $\mathcal{C}_{T,\mathcal{R}} = \mathcal{C}_T \cap \mathcal{R}$.

By convention if $\mathcal{R} = \mathbb{C}$, we can also omit $\mathcal{R}$ from this notation. Figure 6 is the plot of $F_{1000,\mathcal{R}}$ for the gasket $\mathcal{P}(\frac{1.8}{3}\pi, \frac{3.7}{3}\pi)$, with $\mathcal{R} = \mathbb{C}$, $\mathcal{R} = \{z \in \mathbb{C} | \Re z > 0\}$ and $\mathcal{R} = \{z \in \mathbb{C} | \Re z > 0, \Im z > 0\}$ respectively. These three curves indeed seem to be close to each other.

In Figure 7 we also plot "$F'_T(s)$", the empirical derivative for $F_T(s)$, defined by $F'_T(s) = \frac{F_T(s+0.1) - F_T(s)}{0.1}$, for the gasket $\mathcal{P}(\frac{1.8}{3}\pi, \frac{3.7}{3}\pi)$. Our plot suggests that the derivative of F_T exists and is continuous. The turbulent manner of the plot indicates that a rigorous proof of this claim might be difficult.

Figure 8 is the plot for the pair correlation function F_T for three different Apollonian gaskets $\mathcal{P}(\frac{1.1}{3}\pi, \frac{3.5}{3}\pi)$, $\mathcal{P}(\frac{2.5}{3}\pi, \frac{3.5}{4.2}\pi)$, and $\mathcal{P}(\frac{2.9}{3}\pi, \frac{3.2}{3}\pi)$. It appears that their limiting pair correlation should be the same.

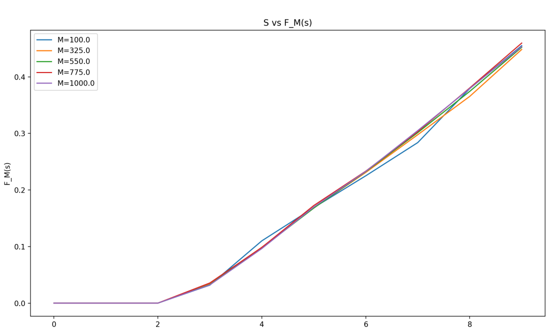

FIGURE 5. The plot for F_T with various T's

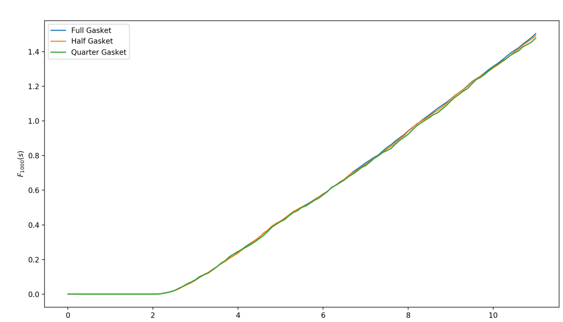

FIGURE 6. Pair correlation for the whole plane, half plane and the first quadrant

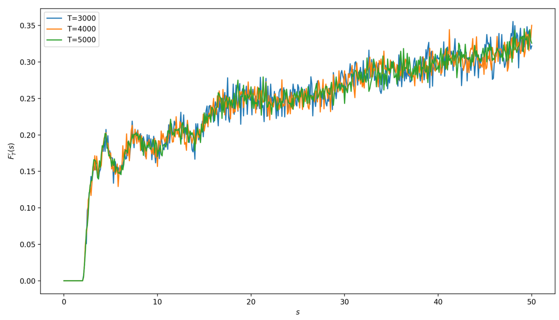

FIGURE 7. The empirical derivative $F_T'(s)$, with different T taken

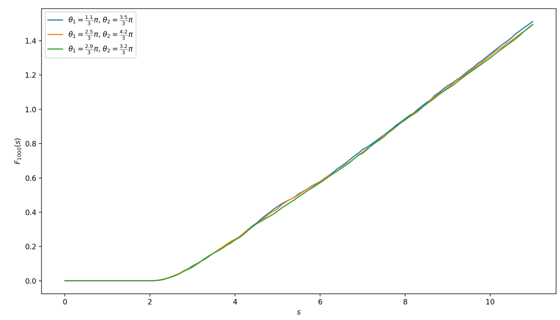

FIGURE 8. Pair correlation functions for different Apollonian gaskets

Based on these findings, we make the following conjecture.

Conjecture 2.2. *For any Apollonian gasket $\mathcal{P}$, and any $\mathcal{R} \subset \mathbb{C}$ with $\mu(\mathcal{R}) > 0$, there exists a nonnegative, monotone, continuously differentiable function F on $[0, \infty)$ which is supported away from 0 such that*

$$\lim_{T\to\infty} F_{T,\mathcal{R}}(s) = F(s)$$

for any $s \in [0, \infty)$, where $F_{T,\mathcal{R}}(s)$ is defined in (2). Moreover, the function F is independent of the chosen Apollonian gasket.

We observe that the normalized pair correlation of circles from an Apollonian gasket is close to the pair correlation of angles of hyperbolic lattice points (compare Figure 7 above and Figure 3 from [14]).

2.3. **Electrostatic Energy.** The electrostatic energy function $G(T)$ is defined by

$$G(T) := \frac{1}{T^{2\delta}} \sum_{\substack{p,q\in\mathcal{C}_T \\ p\neq q}} \frac{1}{d(p,q)}. \tag{3}$$

The definition (3) agrees with the definition of electrostatic energy for an array of electrons in physics, with an extra normalizing factor $1/T^{2\delta}$, where $\delta \approx 1.305688$ is the Hausdorff dimension of $\mathcal{P}$. This energy depends on both the global distribution of the points as well as a moderate penalty if there are points too close to each other.

The heuristic for the normalizing factor $1/T^{2\delta}$ can be explained as follows: For a randomly chosen $p \in \mathcal{C}_T$, from Theorem 1.2, we know $N(p, T, T) \asymp T^\delta$. Assume $N(p, T, s) \asymp s^\delta$. Then

$$\sum_{\substack{q\in\mathcal{C}_T \\ q\neq p}} \frac{1}{d(p,q)} = \int_1^{2T} \frac{1}{s/T} dN(p, T, s) \tag{4}$$

$$= \frac{N(p, T, 2T)}{2} + T \int_1^{2T} \frac{N(p, T, s)}{s^2} ds \tag{5}$$

$$\asymp T^\delta + T \int_1^{2T} s^{\delta-2} ds$$

$$\asymp T^\delta,$$

where we interpret the integral in (4) as a Stieltjes integral, and (5) follows from integration by parts.

Since $\#\mathcal{C}_T \asymp T^{\delta}$, heuristically we should have

$$\sum_{\substack{p,q\in\mathcal{C}_T \\ p\neq q}} \frac{1}{d(p,q)} = \frac{1}{2}\sum_{p\in\mathcal{C}_T}\sum_{\substack{q\in\mathcal{C}_T \\ q\neq p}} \frac{1}{d(p,q)} \asymp T^{2\delta}.$$

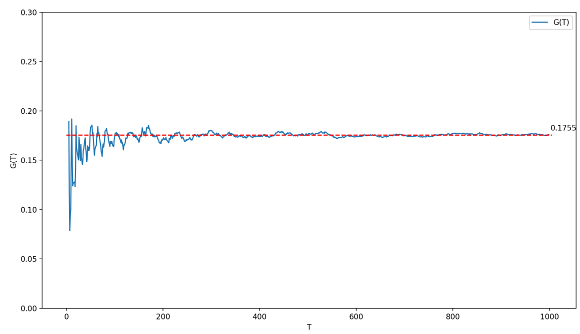

Figure 9. The electrostatic energy function $G(T)$ for $\mathcal{P}(\frac{1.8}{3}\pi, \frac{3.7}{3}\pi)$

Our experiment suggests that $G(T)$ converges to some positive constant when T gets large (see Figure 9). We formulate this as a conjecture below.

Conjecture 2.3. *There exists a constant $b > 0$, such that*

$$\lim_{T\to\infty} G(T) = b,$$

where $G(T)$ is defined in (3).

3. Conclusion

Our investigation shows that the spatial statistics of Apollonian gaskets exhibit quite regular behavior, and this is probably due to the fact that these gaskets are highly self-symmetric. A possible approach to the proposed conjectures might be via homogeneous dynamics on infinite volume hyperbolic spaces.

There are other natural problems on the fine structures of fractal sets. For instance, Figure 10 is the famous Grand Spiral Galaxy (NGC 1232), which can be simulated by a Mandelbrot set constructed from complex dynamics (see Figure 11). Both pictures are from [11].

Figure 10. A spiral galaxy

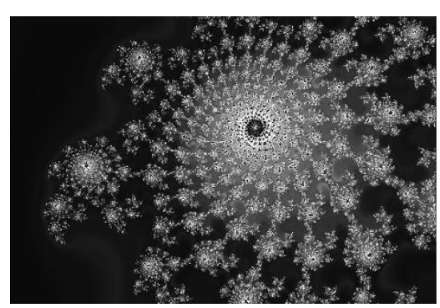

Figure 11. A Mandelbrot set

We pose the following question:

Question 3.1. *What can one say about the fine structures of star distribution in a spiral galaxy?*

Acknowledgement : Our work is the product of a research project of the Illinois Geometry Lab (IGL) for the Spring, 2017 semester supported by the National Science Foundation Grant DMS-1449269. The first three authors are the undergraduate members of our IGL team, the fourth one is the graduate mentor, and the last author is the faculty mentor for the project. We are thankful to Professor Alex Kontorovich for Figures 1 and 2, and Professor Lori Gardi for Figures 10 and 11. We are also thankful to Professors Florin Boca, Alex Kontorovich, and Zeev Rudnick for detailed comments on a previous draft.

References

[1] J. S. Athreya and J. Chaika. The distribution of gaps for saddle connection directions. *Geom. Funct. Anal.*, 22(6):1491–1516, 2012.

[2] Jayadev S. Athreya, Jon Chaika, and Samuel Lelièvre. The gap distribution of slopes on the golden L. In *Recent trends in ergodic theory and dynamical systems*, volume 631 of *Contemp. Math.*, pages 47–62. Amer. Math. Soc., Providence, RI, 2015.

[3] Volker Augustin, Florin P. Boca, Cristian Cobeli, and Alexandru Zaharescu. The h-spacing distribution between Farey points. *Math. Proc. Cambridge Philos. Soc.*, 131(1):23–38, 2001.

[4] Florin P. Boca, Cristian Cobeli, and Alexandru Zaharescu. A conjecture of R. R. Hall on Farey points. *J. Reine Angew. Math.*, 535:207–236, 2001.

[5] Florin P. Boca, Alexandru A. Popa, and Alexandru Zaharescu. Pair correlation of hyperbolic lattice angles. *Int. J. Number Theory*, 10(8):1955–1989, 2014.

[6] Florin P. Boca and Alexandru Zaharescu. On the correlations of directions in the Euclidean plane. *Trans. Amer. Math. Soc.*, 358(4):1797–1825, 2006.

[7] Jean Bourgain and Elena Fuchs. A proof of the positive density conjecture for integer Apollonian circle packings. *J. Amer. Math. Soc.*, 24(4):945–967, 2011.

[8] Daniel El-Baz, Jens Marklof, and Ilya Vinogradov. The distribution of directions in an affine lattice: two-point correlations and mixed moments. *Int. Math. Res. Not. IMRN*, (5):1371–1400, 2015.

[9] Noam D. Elkies and Curtis T. McMullen. Gaps in $\sqrt{n} \bmod 1$ and ergodic theory. *Duke Math. J.*, 123(1):95–139, 2004.

[10] Elena Fuchs. *Arithmetic properties of Apollonian circle packings*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Princeton University.

[11] L Gardi. The mandelbrot set and the fractal nature of light, the universe, and everything. In *SPIE Optical Engineering+ Applications*, pages 883210–883210. International Society for Optics and Photonics, 2013.

[12] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: number theory. *J. Number Theory*, 100(1):1–45, 2003.

[13] R. R. Hall. A note on Farey series. *J. London Math. Soc. (2)*, 2:139–148, 1970.

[14] Dubi Kelmer and Alex Kontorovich. On the pair correlation density for hyperbolic angles. *Duke Math. J.*, 164(3):473–509, 2015.

[15] Frank P. Kirkwood, John G.; Buff. The statistical mechanical theory of solutions. I. *The Journal of Chemical Physics*, 19(6):774–777, 6 1951.

[16] Alex Kontorovich and Hee Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 24(3):603–648, 2011. With an appendix by Oh and Nimish Shah.

[17] Peter D. Lax and Ralph S. Phillips. The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Funct. Anal.*, 46(3):280–350, 1982.

[18] Jens Marklof and Ilya Vinogradov. Directions in hyperbolic lattices. *arXiv*, 09 2014.

[19] Curtis T. McMullen. Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.*, 120(4):691–721, 1998.

[20] Hee Oh and Nimish Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. *Invent. Math.*, 187(1):1–35, 2012.
[21] Morten S. Risager and Anders Södergren. Angles in hyperbolic lattices : The pair correlation density. *arXiv*, 09 2014.
[22] Zeev Rudnick and Xin Zhang. Gap distributions in circle packings. *Submitted. http://arxiv.org/abs/1509.02989*, 09 2015.
[23] Frederick Soddy. The kiss precise. *Nature*, 137:1021, 1936.
[24] Caglar Uyanik and Grace Work. The distribution of gaps for saddle connections on the octagon. *Int. Math. Res. Not. IMRN*, (18):5569–5602, 2016.
[25] Xin Zhang. The gap distribution of directions in some schottky groups. *arXiv*, 11 2015.

University of Illinois at Urbana-Champaign, Department of Mathematics, 1409 W Green Street, Urbana, IL 61801

E-mail address: wchen108@illinois.edu,
qramir2@illinois.edu,
mojiao2@illinois.edu,
amalik10@illinois.edu,
xz87@illinois.edu