EXPOSITORY NOTE: An Arithmetic Surface
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In this note we work out a brutally explicit example of a compact (no cusps) arithmetic surface, by constructing a uniform (no unipotents) arithmetic ($\mathbb{Z}$-points) lattice, that is, a discrete $\mathbb{Q}$-subgroup $\Gamma < \text{SL}(2, \mathbb{R})$ with finite co-volume.

For $x = (x, y, z)$, let $Q(x)$ be the ternary quadratic form

$$Q(x) = x^2 + y^2 - 3z^2 = x \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} x^t.$$  

It is clearly indefinite (takes positive and negative values), and anisotropic over $\mathbb{Q}$. The latter means there are no $\mathbb{Q}$ points on the cone $Q = 0$ (it is enough to consider $\mathbb{Z}$ points (why?), and 3 is not the sum of two squares). The special orthogonal group $G = \text{SO}_Q \cong \text{SO}(2, 1)$ preserving $Q$ is the set

$$G := \{ g \in \text{SL}(3, \mathbb{R}) : Q(xg) = Q(x) \text{ for all } x \in \mathbb{R}^3 \} = \{ g \in \text{SL}(3, \mathbb{R}) : g \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} g^t = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \} .$$

In what follows, we construct the spin representation of $\text{SO}_Q$, which is double-covered by $\text{SL}(2, \mathbb{R})$. Consider symmetric matrices of the form

$$m_x := \begin{pmatrix} z\sqrt{3} - y & x \\ x & z\sqrt{3} + y \end{pmatrix} .$$

These are cooked up to have the property that $\det(m_x) = -Q(x)$. Clearly given $m_x$, we can read off $x$, e.g.:

$$\frac{1}{2\sqrt{3}} \text{tr}(m_x) = z. \quad (0.1)$$

Now, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$, we have the action on $m_x$ given by:

$$g \circ m_x := g \cdot m_x \cdot g^t,$$

which is clearly also symmetric, and satisfies

$$\det(g \circ m_x) = (ad - bc)^2 \det(m_x),$$

meaning we can write $g \circ m_x$ as $m_{x'}$ for some $x' = (x', y', z')$. It is straightforward to compute $x'$:

$$m_{x'} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z\sqrt{3} - y & x \\ x & z\sqrt{3} + y \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -ya^2 + \sqrt{3}za^2 + 2bxa + b^2y + \sqrt{3}b^2z & bcx + adx - acy + bdy + \sqrt{3}acz + \sqrt{3}bdz \\ bdx + adx - acy + bdy + \sqrt{3}acz + \sqrt{3}bdz & -yc^2 + \sqrt{3}zc^2 + 2dxc + d^2y + \sqrt{3}d^2z \end{pmatrix} ,$$

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so we can read off \( x', y', z' \) as in (0.1), and we find that

\[
\mathbf{x'} = (x, y, z) \cdot \begin{pmatrix}
bc + ad & cd - ab & \frac{ab + cd}{\sqrt{3}} \\
bd - ac & \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{-a^2 + b^2 - c^2 + d^2}{2\sqrt{3}} \\
\sqrt{3}(ac + bd) & \frac{1}{2}\sqrt{3}(-a^2 - b^2 + c^2 + d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2)
\end{pmatrix}.
\]

This of course means that the matrix above is an element of \( G \), and hence we have cooked up a map \( \iota : \text{SL}(2, \mathbb{R}) \to G = \text{SO}_Q \), sending

\[
\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad - bc} \begin{pmatrix}
bc + ad & cd - ab & \frac{ab + cd}{\sqrt{3}} \\
bd - ac & \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{-a^2 + b^2 - c^2 + d^2}{2\sqrt{3}} \\
\sqrt{3}(ac + bd) & \frac{1}{2}\sqrt{3}(-a^2 - b^2 + c^2 + d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2)
\end{pmatrix}.
\]

It’s a double-cover because \(-1\) gets mapped to the same thing as \(1\) (so we could have used \( \text{PSL}(2, \mathbb{R}) \)).

This was all over \( \mathbb{R} \). But in fact there are two \( \mathbb{Q} \) structures of \( \text{SL}_2 \), one of which is the obvious \( \text{SL}_2(\mathbb{Q}) \), and the other consists of norm 1 elements in a quaternion division algebra. It is the latter which leads to compact arithmetic surfaces, as follows.

Let \( I, J, K \) formally satisfy \( I^2 = 3, J^2 = 3 \), and \( K = \frac{1}{3}IJ \), so \( K^2 = -1 \). Then form the quaternion

\[
u = a + bI + cJ + dK.
\]

The norm is

\[
N(u) = uu = a^2 - 3b^2 - 3c^2 + d^2.
\]

Let \( D^1_Q \) be the elements \( u \in D_Q \) with \( N(u) = 1 \). A morphism \( \rho : D^1_Q \to G \) maps

\[
\rho : u \mapsto \begin{pmatrix}
a^2 - 3b^2 + 3c^2 - d^2 \\
6bc - 2ad & 2ad + 6bc & -2(ac + bd) \\
6bd - 6ac & a^2 + 3b^2 - 3c^2 - d^2 & 2cd - 2ab \\
6bc - 2ad & -6(ab + cd) & a^2 + 3b^2 + 3c^2 + d^2
\end{pmatrix}.
\]

One can check directly that

\[
\rho(u) \cdot \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \cdot \rho(u)^t = N(u)^2 \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix},
\]

so \( \rho(u) \in G \) if \( N(u) = 1 \). Then our co-compact lattice will come from the \( \mathbb{Z} \)-elements of \( D^1_Q \).
What’s the connection between the two morphisms \( \iota \) and \( \rho \)? Quaternion division algebras can be realized as \( 2 \times 2 \) matrices. Write

\[
\begin{align*}
1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
I &= \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}, \\
J &= \begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}, \\
K &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

and check that \( 1, I, J, K \) satisfy the above formal conditions. Then we can write \( u \) as

\[
u = a1 + bI + cJ + dK = \begin{pmatrix} a + \sqrt{3}b & -d - \sqrt{3}c \\ d - \sqrt{3}c & a - \sqrt{3}b \end{pmatrix}.
\]

Obviously with the above representation we have \( \det(u) = a^2 - \sqrt{3}b^2 - 3c^2 + d = N(u) \), so if \( u \in D_1^1 \) has norm one, then it also lives in \( \text{SL}_2(\mathbb{R}) \). That means we can apply \( \iota \), and in fact we have

\[
\iota: \begin{pmatrix} a + \sqrt{3}b & -d - \sqrt{3}c \\ d - \sqrt{3}c & a - \sqrt{3}b \end{pmatrix} \mapsto \begin{pmatrix} a^2 - 3b^2 + 3c^2 - d^2 & 2ad + 6bc & -2(ac + bd) \\ 6bc - 2ad & a^2 + 3b^2 - 3c^2 - d^2 & 2cd - 2ab \\ 6bd - 6ac & -6(ab + cd) & a^2 + 3b^2 + 3c^2 + d^2 \end{pmatrix},
\]

which is our old friend \( \rho(u) \). (When we first introduced it, we pulled it out of thin air, but now its role is clear.) To get our discrete group \( \Gamma \), we simply insist that \( a, b, c, d \in \mathbb{Z} \).

Summarizing, we see that what we really want is elements \( \alpha = a + \sqrt{3}b \), and \( \beta = d + \sqrt{3}c \) in the ring of integers \( O_K = \mathbb{Z}[\sqrt{3}] \) of the number field \( K = \mathbb{Q}([\sqrt{3}] \). We put these in a matrix of the form:

\[
M_{\alpha,\beta} := \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},
\]

and ask that \( \det M_{\alpha,\beta} = N(\alpha) + N(\beta) = a^2 - 3b^2 + d^2 - 3c^2 = 1 \).

It is easy enough to do a brute search for small elements \( \gamma \in \Gamma \), take the orbit under these elements of some fixed base point, say \( z_0 = 2i \), and construct the corresponding Dirichlet domain. The result in this example, with the orbit shown on top of the Dirichlet domain, is:
Note that for any point \( w \) in the orbit of \( z_0 = 2i \) under \( \Gamma \), one can draw the set of all points equidistant to \( w \) and \( z_0 \). This will be a geodesic, and those corresponding to the closest points will determine the bounding geodesics for the Dirichlet domain. Here they are in this case:

And the two pictures overlapped:

The group elements used in the calculation above were:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 2(1+\sqrt{3}) \\ 2(-1+\sqrt{3}) & -3 \end{pmatrix}, \begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}, \begin{pmatrix} -2 & 3 + 2\sqrt{3} \\ -3 + 2\sqrt{3} & -2 \end{pmatrix},
\]

\[
\begin{pmatrix} 2(-1+\sqrt{3}) & 2(1+\sqrt{3}) \\ -3 & -3 \end{pmatrix}, \begin{pmatrix} -3 - 2\sqrt{3} & -2 \\ 2 & -3 + 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2 - \sqrt{3} & 0 \\ 0 & 2 + \sqrt{3} \end{pmatrix},
\]

\[
\begin{pmatrix} 0 & -2 - \sqrt{3} \\ 2 - \sqrt{3} & 0 \end{pmatrix}, \begin{pmatrix} -3 - 2\sqrt{3} & 2 \\ -2 & -3 + 2\sqrt{3} \end{pmatrix}, \begin{pmatrix} -3 & -2(1+\sqrt{3}) \\ 2 - 2\sqrt{3} & -3 \end{pmatrix},
\]

\[
\begin{pmatrix} -2 & -3 - 2\sqrt{3} \\ 3 - 2\sqrt{3} & -2 \end{pmatrix}, \begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}
\]

Of course here we see the great advantage of listing these elements in their more natural structure, that of a quaternion division algebra: \( u = a\mathbf{1} + bI + cJ + dK \), where
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