Two examples of compact arithmetic surfaces

Brooke Ogrodnik*[1]

[1]Rutgers University

April 14, 2019

1 Introduction

The goal of this paper is to create a co-compact Dirichlet domain in Hyperbolic 3-space. This paper stems from the ideas presented in "Expository Note: An Arithmetic Surface" [1]. In Section 2, we construct the Dirichlet domain in Hyperbolic 2-space related to the quadratic form, $Q(\mathbf{x}) = x^2 + y^2 - 7z^2$, while also giving a short overview of what was discussed in [1]. Then Section 3 extends the idea to Hyperbolic 3-space and again finds a Dirchlet Domain for the new quadratic, $Q(\mathbf{x}) = w^2 + x^2 + y^2 - 7z^2$. Section 4 will go on to explain how the Dirichlet domains are actually constructed in Hyperbolic 3-space case using Mathematica and C++. Finally, in the Appendix the reader can find the 60 matrices needed to construct the domain.

2 Hyperbolic 2-Space

As mentioned earlier, we will give a quick summary of the expository note by Konorovich. To make this section a little new, we will work with a different quadratic form, $Q(\mathbf{x}) = x^2 + y^2 - 7z^2$, then what was in the note so that the reader has two different examples to reference. This quadratic form is indefinite (solutions to $Q > 0$ and $Q < 0$) and anisotropic ($Q \neq 0$) over $\mathbb{Q}$. The idea is to contract a mapping from $SL(2, \mathbb{R})$ to

$$G := \{g \in SL(3, \mathbb{R}) : Q(xg) = Q(x) \text{ for all } x \in \mathbb{R}^3\}.$$

First, we want to construct the spin representation of G. We want to create two by two matrices $m_{\mathbf{x}}$ that encode all the information of $Q(x)$. One such option is,

$$m_{\mathbf{x}} = \begin{pmatrix} z\sqrt{7} - y & x \\ x & z\sqrt{7} + y \end{pmatrix}.$$

Note that this matrix is symmetric and its determinant is $-Q(\mathbf{x})$. Next, we take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and see that

$$g \cdot m_{\mathbf{x}} \cdot g^t = \begin{pmatrix} -ya^2 + \sqrt{7}za^2 + 2bxa + b^2y + \sqrt{7}b^2z & bcx + adx - acy + bdy + \sqrt{7}acz + \sqrt{7}bdz \\ bcx + adx - acy + bdy + \sqrt{7}acz + \sqrt{7}bdz & -yc^2 + \sqrt{7}zc^2 + 2dxc + d^2y + \sqrt{7}d^2z \end{pmatrix}$$

which is again symmetric and has determinant $-Q(\mathbf{x})$. So it is easy to see that we can get new $\mathbf{x}'$ purely from $\mathbf{x}$ and our action g as follows:

$$\mathbf{x}' = (x, y, z) \cdot \begin{pmatrix} bc + ad & cd - ab & \frac{1}{\sqrt{7}}(ad + cb) \\ bd - ac & \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{1}{2\sqrt{7}}(-a^2 + b^2 - c^2 + d^2) \\ \sqrt{7}(ac + bd) & \frac{\sqrt{7}}{2}(-a^2 - b^2 + c^2 + d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) \end{pmatrix}.$$

*Email: bl481@math.rutgers.edu

So the three by three matrix above is in G and this gives a map $\iota : SL(2,\mathbb{R}) \to G$ where

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} bc+ad & cd-ab & \frac{1}{\sqrt{7}}(ad+cb) \\ bd-ac & \frac{1}{2}(a^2-b^2-c^2+d^2) & \frac{1}{2\sqrt{7}}(-a^2+b^2-c^2+d^2) \\ \sqrt{7}(ac+bd) & \frac{\sqrt{7}}{2}(-a^2-b^2+c^2+d^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}$$

and this is a double cover (γ and $-\gamma$ get mapped to the same matrix in G).

The next step is to look at elements with norm 1 in a divison algebra. We want the property that $I^2 = 7, J^2 = 7, K^2 = -1$ and $\frac{1}{7}IJ = K$. Then we get $u = a + bI + cJ + dK$ and it's norm is

$$N(u) = u\bar{u} = (a + bI + cJ + dK)(a - bI - cJ - dK) = a^2 - 7b^2 - 7c^2 + d^2.$$

Next we realize this divison algebra as the following:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -\sqrt{7} \\ -\sqrt{7} & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$u = \begin{pmatrix} a+\sqrt{7}b & -d-\sqrt{7}c \\ d-\sqrt{7}c & a-\sqrt{7}b \end{pmatrix}$$

and here the $det(u) = N(u)$. Thus having u with norm 1 is equivalent to having elements in $SL(2,\mathbb{R})$. From here we can apply the map ι to these and get that:

$$\iota : \begin{pmatrix} a+\sqrt{7}b & -d-\sqrt{3}c \\ d-\sqrt{7}c & a-\sqrt{3}b \end{pmatrix} \mapsto \begin{pmatrix} a^2-7b^2+7c^2-d^2 & 2ad+14bc & -2(ac+bd) \\ 14bc-2ad & a^2+7b^2-7c^2-d^2 & 2cd-2ab \\ 14bd-14ac & -14(ab+cd) & a^2+7b^2+7c^2+d^2 \end{pmatrix}.$$

Letting $a, b, c, d \in \mathbb{Z}$ we get a discrete group Γ. And so we want elements in the ring of integers, $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$, of the number field, $K = \mathbb{Q}[\sqrt{3}]$. Next a brute search is done for "small" $\gamma \in \Gamma$. After this we apply the γ to some fixed point and use this to make the Dirichlet domain. Note that $tr(\begin{pmatrix} a+\sqrt{7}b & -d-\sqrt{7}c \\ d-\sqrt{7}c & a-\sqrt{7}b \end{pmatrix}) = 2a$ and so we only would have a parabolic element if $a = \pm 1$. Since we chose a Q that is anisotropic, this case never happens. GIven that it has no parabolic elements, there are no cusps and thus we can construct a co-compact Dirichlet domain.

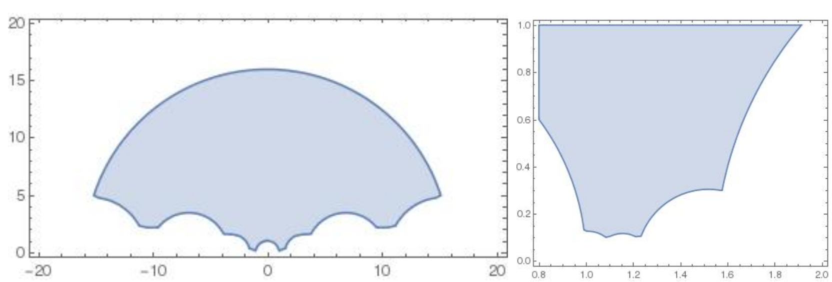

Figure 1: The Dirichlet domain associated with $Q(\mathbf{x}) = x^2 + y^2 - 7z^2$

3 Hyperbolic 3-Space

Now we will do the same thing but for a new quadratic form. Let $\mathbf{x} = (w, x, y, z)$ and

$$Q(\mathbf{x}) = w^2 + x^2 + y^2 - 7z^2 = \mathbf{x} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -7 \end{pmatrix} \mathbf{x}^t.$$

Thus $Q(\mathbf{x})$ is a quaternary quadratic form which is still anisotropic over $\mathbb{Q}$. One can see that it is anisotropic by noting that it has no solutions modulo 8.

The idea now is to construct a mapping from $SL(2,\mathbb{C})$ to

$$G := \{g \in SL(4,\mathbb{R}) : G(xg) = G(x) \text{ for all } x \in \mathbb{R}^4\}.$$

Again we wish to encode the information about $Q(\mathbf{x})$ into a two by two matrix whose determinant is equal to $-Q(\mathbf{x})$ and where we can read off (w, x, y, z). Note that the following will work:

$$m_{\mathbf{x}} = \begin{pmatrix} \sqrt{7}z - y & x + iw \\ x - iw & \sqrt{7}z + y \end{pmatrix}.$$

The determinant of $m_{\mathbf{x}}$ is $-Q(\mathbf{x})$ and the point (w, x, y, z) can be uniquely determined from this matrix. Let $g \in SL(2,\mathbb{C})$ and define $g \circ m_{\mathbf{x}} = g m_{\mathbf{x}} g^*$ where g^* represents conjugate transpose (as opposed transpose used in Section 2). Then we see that $det(g \circ m_{\mathbf{x}}) = |ad - bc|^2 det(m_{\mathbf{x}})$. As in the previous section, we can set $g \circ m_{\mathbf{x}} = m_{\mathbf{x}'}$ and read off $\mathbf{x}' = (w', x', y', z')$.

$$m_{\mathbf{x}'} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{7}z - y & x + iw \\ x - iw & \sqrt{7}z + y \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}$$

Where

$$\begin{aligned}
\alpha &= -i\bar{a}bw + ia\bar{b}w + \bar{a}bx + a\bar{b}x - |a|^2 y + \sqrt{7}|a|^2 z + |b|^2 y + \sqrt{7}|b|^2 z \\
\beta &= -a\bar{c}y + \sqrt{7}a\bar{c}z + ia\bar{d}w + a\bar{d}x - ib\bar{c}w + b\bar{c}x + b\bar{d}y + \sqrt{7}b\bar{d}z \\
\gamma &= -\bar{a}cy + \sqrt{7}\bar{a}cz - i\bar{a}dw + \bar{a}dx + i\bar{b}cw + \bar{b}cx + \bar{b}dy + \sqrt{7}\bar{b}dz \\
\eta &= -i\bar{c}dw + ic\bar{d}w + \bar{c}dx + c\bar{d}x - |c|^2 y + \sqrt{7}|c|^2 z + |d|^2 y + \sqrt{7}|d|^2 z
\end{aligned}$$

Define $\iota(g) = \{m_{i,j}\}_{i,j \le 4}$ as the following:

$$\begin{aligned}
m_{1,1} &= \frac{1}{2}\left(d\bar{a} + a\bar{d} - c\bar{b} - b\bar{c}\right) & m_{1,2} &= -\frac{1}{2}i\left(d\bar{a} - a\bar{d} - c\bar{b} + b\bar{c}\right) \\
m_{1,3} &= \frac{1}{2}i\left(b\bar{a} - a\bar{b} - d\bar{c} + c\bar{d}\right) & m_{1,4} &= -\frac{i}{2\sqrt{7}}\left(b\bar{a} - a\bar{b} + d\bar{c} - c\bar{d}\right) \\
m_{2,1} &= \frac{1}{2}i\left(d\bar{a} - a\bar{d} + c\bar{b} - b\bar{c}\right) & m_{2,2} &= \frac{1}{2}\left(d\bar{a} + a\bar{d} + c\bar{b} + b\bar{c}\right) \\
m_{2,3} &= \frac{1}{2}\left(-b\bar{a} - a\bar{b} + d\bar{c} + c\bar{d}\right) & m_{2,4} &= \frac{1}{2\sqrt{7}}\left(b\bar{a} + a\bar{b} + d\bar{c} + c\bar{d}\right) \\
m_{3,1} &= -\frac{1}{2}i\left(c\bar{a} - a\bar{c} - d\bar{b} + b\bar{d}\right) & m_{3,2} &= \frac{1}{2}\left(-c\bar{a} - a\bar{c} + d\bar{b} + b\bar{d}\right) \\
m_{3,3} &= \frac{1}{2}\left(|a|^2 - |b|^2 - |c|^2 + |d|^2\right) & m_{3,4} &= \frac{1}{2\sqrt{7}}\left(-|a|^2 + |b|^2 - |c|^2 + |d|^2\right) \\
m_{4,1} &= \frac{1}{2}i\sqrt{7}\left(c\bar{a} - a\bar{c} + d\bar{b} - b\bar{d}\right) & m_{4,2} &= \frac{1}{2}\sqrt{7}\left(c\bar{a} + a\bar{c} + d\bar{b} + b\bar{d}\right) \\
m_{4,3} &= \frac{1}{2}\sqrt{7}\left(-|a|^2 - |b|^2 + |c|^2 + |d|^2\right) & m_{4,4} &= \frac{1}{2}\left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)
\end{aligned}$$

Thus we get that $\mathbf{x}' = (w', x', y', z') = (w, x, y, z) \cdot \iota(g)$. The next step is instead of looking at $SL(2, \mathbb{Q}[i])$, we are going to look at elements of norm 1 in the same divison algebra as before, but with elements in $SL(2, \mathbb{C})$. Thus we can apply the map ι to these matrices as in the previous section but this shall be omitted here. Letting $a, b, c, d \in \mathbb{Z}[i]$ we get a discrete group Γ. And so we want elements in the ring of integers, $\mathcal{O}_K = \mathbb{Z}[i][\sqrt{7}]$, of the number field, $K = \mathbb{Q}[i][\sqrt{7}]$. Next a brute search is done for small $\gamma \in \Gamma$. The same algorithm as Section 2 is done. Again we have no cusps and thus we have constructed a co-compact Dirichlet domain.

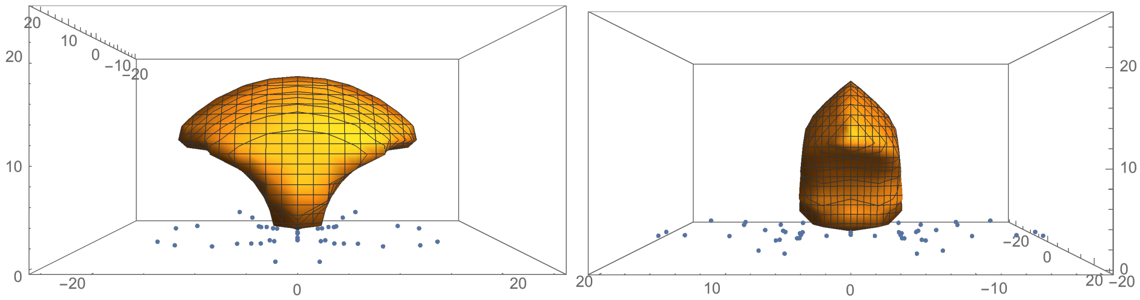

Figure 2: The Dirichlet domain associated with $Q(\mathbf{x}) = w^2 + x^2 + y^2 - 7z^2$

4 Implementation of Algorithm

The goal is to create an algorithm that finds enough matrices in our group so that we can form a Dirichlet domain. There are a few steps to do this but we start with a simple For loop. Some short hand notation will be that

$$u = \begin{pmatrix} (a+if) + \sqrt{7}(b+ih) & -(d+im) - \sqrt{7}(c+il) \\ (d+im) - \sqrt{7}(c+il) & (a+if) - \sqrt{7}(b+ih) \end{pmatrix} \mapsto [a, b, c, d, f, h, l, m].$$

In order to find elements in our group, we need to find integers $[a, b, c, d, f, h, l, m]$ with the property that

$$a^2 - 7b^2 - 7c^2 + d^2 - f^2 + 7h^2 + 7l^2 - m^2 = 1 \text{ and } af - 7bh - 7cl + dm = 0.$$

The problem is that if we want to search for integers with absolute value bounded by B (for our purposes $B = 20$), than this is $(2B+1)^8$ cases that must be checked. Using two fairly simple observations, we can speed up this search significantly.

1. If $[a, *, *, d, f, *, *, m]$ satisfies the equations then so does $[d, *, *, a, m, *, *, f]$. The same also applies to $[*, b, c, *, *, h, l, *]$ satisfying the equation and thus $[*, c, b, *, *, l, h, *]$ also will.

2. This case uses the fact that the imaginary values sum to zero. It gives us that for any solution $[a, b, c, d, f, h, l, m]$, $[-a, b, c, d, -f, h, l, m]$ would also be a solution. This is due to the fact that $af = (-a)(-f)$. The same goes with $(b, h), (c, l)$, and (d, m) and so any combination of them as well.

So now we wish to use these two facts in our code to generate a large number of small matrices in our group. The process of generating these have two steps. First, we run a search implementing the facts above. The first allows for the added condition that $a \le d \le B$ and $b \le c \le B$. Fact 2 restricts our solutions so that f, h, l, m are non-negative integers. So for every solution $[a, b, c, d, f, h, l, m]$ that we find using these new bounds, we will add four solutions to our list:

- $[a, b, c, d, f, h, l, m]$

- $[a, c, b, d, f, l, h, m]$
- $[d, b, c, a, m, h, l, f]$
- $[d, c, b, a, m, l, h, f]$

Then step two just takes these solutions and apply the 2^4 choices of negating signs. So for $x_1, x_2, x_3, x_4 \in \{0, 1\}$ we would get (possibly) new points

$$[(-1)^{x_1}a, (-1)^{x_2}b, (-1)^{x_3}c, (-1)^{x_4}d, (-1)^{x_1}f, (-1)^{x_2}, (-1)^{x_3}l, (-1)^{x_4}m].$$

Instead of having to check $(2B+1)^8$ choices we only need to solve the equations for $(2B+1)^2(B+1)^6$ (for our $B = 20$ this means we are only looking at about 2 percent of the possible combinations). Though this isn't impressive in terms of big-O, given that we want to run the search for small B, this is actually very helpful. Then to run it a little faster, this portion was implemented in C++ rather than Mathematica (where the images were made and some small checks conducted).

4.1 Constructing the Dirichlet domain

Recall that $\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in SL(2, \mathbb{C})$ maps hyperbolic 3-space to itself the following way:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} z = (\alpha z + \beta)\overline{(\gamma z + \eta)}|\gamma z + \eta|^{-2}.$$

We will take a base point, $(0, 0, 2)$, and let the matrices we generated in part 4 act on it. This will also be done in C++. Note that in the previous section we found all such matrices in SL2 when we actually want PSL2 and so we just take 1 of each to avoid the double covering. For $B = 20$ this generated points in the orbit (the entire program took 24 minutes to run). After that we will take all points in a box, say $[-30, 30] \times [-30, 30] \times [.35, 10]$, since points too far away in Euclidean distance will also be too far away in Hyperbolic distance. This will narrow the number of matrices down from 275770 to 130 which can be seen in Figure 3.

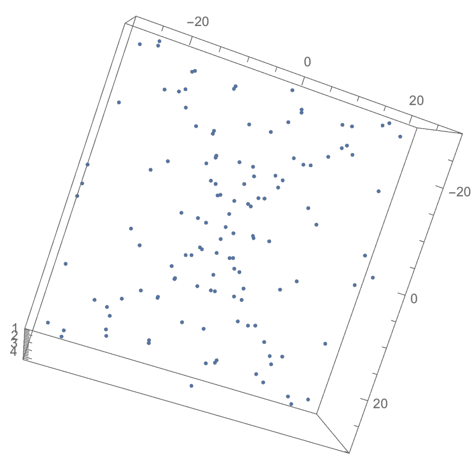

Figure 3: The smallest points in a given rectangle.

The next step is to minimize the hyperbolic distance between our base point (0,0,2) and points in its orbit. Recall that minimizing hyperbolic distance between u and v is equivalent to minimizing the function

$$\frac{|u - v|^2}{4u(3)v(3)}.$$

4.2 Introducing inversive coordinates

We want to narrow down the points used to create this fundamental domain. Thus we will use inversive coordinates to figure out if two different geodesics are nested [2]. So for all 129 points we find the geodesic that is equidistant between it and our basepoint. We will represent this geodesic by $\{a, b, r^2\}$ i.e. it is the hemisphere, $(x-a)^2 + (y-b)^2 + z^2 = r^2$. Next we change these into their inversive coordinates as follows $(\frac{a^2+b^2-r^2}{r}, \frac{1}{r}, \frac{a}{r}, \frac{b}{r})$. Finally we check if they are nested. If $\{a, b, r^2\}$ crosses the z-axis above $z = 2$, then we want to get rid of any geodesics that contains it and if it does not cross the z axis or it crosses below $z = 2$ then we want to remove any geodesics that are contained in it. We can tell if two geodesics, represented as inversive coordinates v_1, v_2, are nested if

$$v_1 \cdot \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} v_2^t < -1.$$

Doing this method gets the number of points down to 60. None of the values in these matrices exceed 16

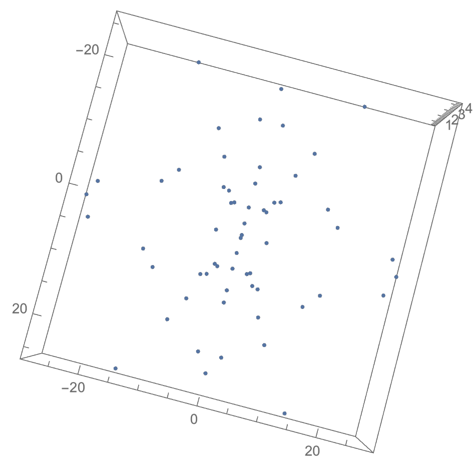

Figure 4: The remaining 60 points.

thus this could have all be done with a bound of 16 instead of 20 (the search only takes 6 minutes). The final step is to multiply these matrices together to see if we get any new points. A program in C++ was written to execute this algorithm, then imported back in to Mathematica and ran again. However, no new points remained, i.e. all of them were inside one of the previous 60 geodesics (or outside if the geodesic crossed the z-axis above 2). This entire portion of code ran in Mathematica took only 8 minutes.

5 References

1. Kontorovich, A. Expository Note: An Arithmetic Surface `sites.math.rutgers.edu/~alexk/files/UniformLattice.pdf`

2. Kontorovich, A. Letter to Bill Duke `http://sites.math.rutgers.edu/~alexk/files/LetterToDuke.pdf`

6 Appendix

7 8-tuple Form

This is a list of the 60 matrices used to construct the Dirichlet domain shown in Figure 2 of the form $[a, b, c, d, f, h, l, m]$.

$\{3, 1, 5, 14, 7, 2, -3, -8\}$	$\{3, 1, 5, 14, -7, -2, 3, 8\}$	$\{3, 1, 2, 5, 1, 1, -6, -16\}$
$\{3, 1, 2, 5, 1, 0, 2, 5\}$	$\{3, 1, 2, 5, -1, 0, -2, -5\}$	$\{3, 1, 2, 5, -1, -1, 6, 16\}$
$\{3, 1, 1, 2, 1, 0, 1, 2\}$	$\{3, 1, 1, 2, -1, 0, -1, -2\}$	$\{3, 1, -1, -2, 1, 0, -1, -2\}$
$\{3, 1, -1, -2, -1, 0, 1, 2\}$	$\{3, 1, -2, -5, 1, 1, 6, 16\}$	$\{3, 1, -2, -5, 1, 0, -2, -5\}$
$\{3, 1, -2, -5, -1, 0, 2, 5\}$	$\{3, 1, -2, -5, -1, -1, -6, -16\}$	$\{3, 1, -5, -14, 7, 2, 3, 8\}$
$\{3, 1, -5, -14, -7, -2, -3, -8\}$	$\{2, 1, 1, 3, 2, 1, 3, 8\}$	$\{2, 1, 1, 3, 2, 1, 0, 1\}$
$\{2, 1, 1, 3, -2, -1, 0, -1\}$	$\{2, 1, 1, 3, -2, -1, -3, -8\}$	$\{2, 1, -1, -3, 2, 1, 0, -1\}$
$\{2, 1, -1, -3, 2, 1, -3, -8\}$	$\{2, 1, -1, -3, -2, -1, 3, 8\}$	$\{2, 1, -1, -3, -2, -1, 0, 1\}$
$\{2, 0, 4, 10, 5, 2, 5, 13\}$	$\{2, 0, 4, 10, -5, -2, -5, -13\}$	$\{2, 0, 2, 5, 0, 0, 0, 0\}$
$\{2, 0, 1, 2, 0, 0, 0, 0\}$	$\{2, 0, -1, -2, 0, 0, 0, 0\}$	$\{2, 0, -2, -5, 0, 0, 0, 0\}$
$\{2, 0, -4, -10, 5, 2, -5, -13\}$	$\{2, 0, -4, -10, -5, -2, 5, 13\}$	$\{1, 1, 3, 8, 2, 1, -1, -2\}$
$\{1, 1, 3, 8, -2, -1, 1, 2\}$	$\{1, 1, -3, -8, 2, 1, 1, 2\}$	$\{1, 1, -3, -8, -2, -1, -1, -2\}$
$\{1, 0, 6, 15, 12, 5, 6, 16\}$	$\{1, 0, 6, 15, -12, -5, -6, -16\}$	$\{1, 0, 3, 8, 2, 1, -6, -16\}$
$\{1, 0, 3, 8, 2, 0, 2, 5\}$	$\{1, 0, 3, 8, -2, 0, -2, -5\}$	$\{1, 0, 3, 8, -2, -1, 6, 16\}$
$\{1, 0, 0, 0, 0, 0, 0, 0\}$	$\{1, 0, -3, -8, 2, 1, 6, 16\}$	$\{1, 0, -3, -8, 2, 0, -2, -5\}$
$\{1, 0, -3, -8, -2, 0, 2, 5\}$	$\{1, 0, -3, -8, -2, -1, -6, -16\}$	$\{1, 0, -6, -15, 12, 5, -6, -16\}$
$\{1, 0, -6, -15, -12, -5, 6, 16\}$	$\{0, 0, 4, 11, 6, 2, 0, 0\}$	$\{0, 0, 4, 11, -6, -2, 0, 0\}$
$\{0, 0, 1, 3, 1, 0, 0, 0\}$	$\{0, 0, 1, 3, -1, 0, 0, 0\}$	$\{0, 0, 0, 1, 0, 0, 0, 0\}$
$\{0, 0, 0, 0, 3, 1, 2, 5\}$	$\{0, 0, 0, 0, 3, 1, 1, 2\}$	$\{0, 0, 0, 0, 3, 1, -1, -2\}$
$\{0, 0, 0, 0, 3, 1, -2, -5\}$	$\{0, 0, 0, 0, 2, 1, 1, 3\}$	$\{0, 0, 0, 0, 2, 1, -1, -3\}$

7.1 Matrix Form

This is a list of the 60 matrices used to construct the Dirichlet domain shown in Figure 2 of the form $\begin{pmatrix} (a+if)+\sqrt{7}(b+ih) & -(d+im)-\sqrt{7}(c+il) \\ (d+im)-\sqrt{7}(c+il) & (a+if)-\sqrt{7}(b+ih) \end{pmatrix}$.

$$\begin{pmatrix} (3+7i)+(1+2i)\sqrt{7} & (-14+8i)-(5-3i)\sqrt{7} \\ (14-8i)-(5-3i)\sqrt{7} & (3+7i)-(1+2i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3-7i)+(1-2i)\sqrt{7} & (-14-8i)-(5+3i)\sqrt{7} \\ (14+8i)-(5+3i)\sqrt{7} & (3-7i)-(1-2i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3+i)+(1+i)\sqrt{7} & (-5+16i)-(2-6i)\sqrt{7} \\ (5-16i)-(2-6i)\sqrt{7} & (3+i)-(1+i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3+i)+\sqrt{7} & (-5-5i)-(2+2i)\sqrt{7} \\ (5+5i)-(2+2i)\sqrt{7} & (3+i)-\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3-i)+\sqrt{7} & (-5+5i)-(2-2i)\sqrt{7} \\ (5-5i)-(2-2i)\sqrt{7} & (3-i)-\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3-i)+(1-i)\sqrt{7} & (-5-16i)-(2+6i)\sqrt{7} \\ (5+16i)-(2+6i)\sqrt{7} & (3-i)-(1-i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3+i)+\sqrt{7} & (-2-2i)-(1+i)\sqrt{7} \\ (2+2i)-(1+i)\sqrt{7} & (3+i)-\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3-i)+\sqrt{7} & (-2+2i)-(1-i)\sqrt{7} \\ (2-2i)-(1-i)\sqrt{7} & (3-i)-\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3+i)+\sqrt{7} & (2+2i)+(1+i)\sqrt{7} \\ (-2-2i)+(1+i)\sqrt{7} & (3+i)-\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3-i)+\sqrt{7} & (2-2i)+(1-i)\sqrt{7} \\ (-2+2i)+(1-i)\sqrt{7} & (3-i)-\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3+i)+(1+i)\sqrt{7} & (5-16i)+(2-6i)\sqrt{7} \\ (-5+16i)+(2-6i)\sqrt{7} & (3+i)-(1+i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3+i)+\sqrt{7} & (5+5i)+(2+2i)\sqrt{7} \\ (-5-5i)+(2+2i)\sqrt{7} & (3+i)-\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3-i)+\sqrt{7} & (5-5i)+(2-2i)\sqrt{7} \\ (-5+5i)+(2-2i)\sqrt{7} & (3-i)-\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3-i)+(1-i)\sqrt{7} & (5+16i)+(2+6i)\sqrt{7} \\ (-5-16i)+(2+6i)\sqrt{7} & (3-i)-(1-i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (3+7i)+(1+2i)\sqrt{7} & (14-8i)+(5-3i)\sqrt{7} \\ (-14+8i)+(5-3i)\sqrt{7} & (3+7i)-(1+2i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (3-7i)+(1-2i)\sqrt{7} & (14+8i)+(5+3i)\sqrt{7} \\ (-14-8i)+(5+3i)\sqrt{7} & (3-7i)-(1-2i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (2+2i)+(1+i)\sqrt{7} & (-3-8i)-(1+3i)\sqrt{7} \\ (3+8i)-(1+3i)\sqrt{7} & (2+2i)-(1+i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (2+2i)+(1+i)\sqrt{7} & (-3-i)-\sqrt{7} \\ (3+i)-\sqrt{7} & (2+2i)-(1+i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (2-2i)+(1-i)\sqrt{7} & (-3+i)-\sqrt{7} \\ (3-i)-\sqrt{7} & (2-2i)-(1-i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (2-2i)+(1-i)\sqrt{7} & (-3+8i)-(1-3i)\sqrt{7} \\ (3-8i)-(1-3i)\sqrt{7} & (2-2i)-(1-i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (2+2i)+(1+i)\sqrt{7} & (3+i)+\sqrt{7} \\ (-3-i)+\sqrt{7} & (2+2i)-(1+i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (2+2i)+(1+i)\sqrt{7} & (3+8i)+(1+3i)\sqrt{7} \\ (-3-8i)+(1+3i)\sqrt{7} & (2+2i)-(1+i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (2-2i)+(1-i)\sqrt{7} & (3-8i)+(1-3i)\sqrt{7} \\ (-3+8i)+(1-3i)\sqrt{7} & (2-2i)-(1-i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (2-2i)+(1-i)\sqrt{7} & (3-i)+\sqrt{7} \\ (-3+i)+\sqrt{7} & (2-2i)-(1-i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (2+5i)+2i\sqrt{7} & (-10-13i)-(4+5i)\sqrt{7} \\ (10+13i)-(4+5i)\sqrt{7} & (2+5i)-2i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (2-5i)-2i\sqrt{7} & (-10+13i)-(4-5i)\sqrt{7} \\ (10-13i)-(4-5i)\sqrt{7} & (2-5i)+2i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} 2 & -5-2\sqrt{7} \\ 5-2\sqrt{7} & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -2-\sqrt{7} \\ 2-\sqrt{7} & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2+\sqrt{7} \\ -2+\sqrt{7} & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 5+2\sqrt{7} \\ -5+2\sqrt{7} & 2 \end{pmatrix}$$

$$\begin{pmatrix} (2+5i)+2i\sqrt{7} & (10+13i)+(4+5i)\sqrt{7} \\ (-10-13i)+(4+5i)\sqrt{7} & (2+5i)-2i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (2-5i)-2i\sqrt{7} & (10-13i)+(4-5i)\sqrt{7} \\ (-10+13i)+(4-5i)\sqrt{7} & (2-5i)+2i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (1+2i)+(1+i)\sqrt{7} & (-8+2i)-(3-i)\sqrt{7} \\ (8-2i)-(3-i)\sqrt{7} & (1+2i)-(1+i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (1-2i)+(1-i)\sqrt{7} & (-8-2i)-(3+i)\sqrt{7} \\ (8+2i)-(3+i)\sqrt{7} & (1-2i)-(1-i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (1+2i)+(1+i)\sqrt{7} & (8-2i)+(3-i)\sqrt{7} \\ (-8+2i)+(3-i)\sqrt{7} & (1+2i)-(1+i)\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (1-2i)+(1-i)\sqrt{7} & (8+2i)+(3+i)\sqrt{7} \\ (-8-2i)+(3+i)\sqrt{7} & (1-2i)-(1-i)\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (1+12i)+5i\sqrt{7} & (-15-16i)-(6+6i)\sqrt{7} \\ (15+16i)-(6+6i)\sqrt{7} & (1+12i)-5i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (1-12i)-5i\sqrt{7} & (-15+16i)-(6-6i)\sqrt{7} \\ (15-16i)-(6-6i)\sqrt{7} & (1-12i)+5i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (1+2i)+i\sqrt{7} & (-8+16i)-(3-6i)\sqrt{7} \\ (8-16i)-(3-6i)\sqrt{7} & (1+2i)-i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} 1+2i & (-8-5i)-(3+2i)\sqrt{7} \\ (8+5i)-(3+2i)\sqrt{7} & 1+2i \end{pmatrix}$$

$$\begin{pmatrix} 1-2i & (-8+5i)-(3-2i)\sqrt{7} \\ (8-5i)-(3-2i)\sqrt{7} & 1-2i \end{pmatrix} \quad \begin{pmatrix} (1-2i)-i\sqrt{7} & (-8-16i)-(3+6i)\sqrt{7} \\ (8+16i)-(3+6i)\sqrt{7} & (1-2i)+i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} (1+2i)+i\sqrt{7} & (8-16i)+(3-6i)\sqrt{7} \\ (-8+16i)+(3-6i)\sqrt{7} & (1+2i)-i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} 1+2i & (8+5i)+(3+2i)\sqrt{7} \\ (-8-5i)+(3+2i)\sqrt{7} & 1+2i \end{pmatrix} \quad \begin{pmatrix} 1-2i & (8-5i)+(3-2i)\sqrt{7} \\ (-8+5i)+(3-2i)\sqrt{7} & 1-2i \end{pmatrix}$$

$$\begin{pmatrix} (1-2i)-i\sqrt{7} & (8+16i)+(3+6i)\sqrt{7} \\ (-8-16i)+(3+6i)\sqrt{7} & (1-2i)+i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} (1+12i)+5i\sqrt{7} & (15+16i)+(6+6i)\sqrt{7} \\ (-15-16i)+(6+6i)\sqrt{7} & (1+12i)-5i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} (1-12i)-5i\sqrt{7} & (15-16i)+(6-6i)\sqrt{7} \\ (-15+16i)+(6-6i)\sqrt{7} & (1-12i)+5i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} 6i+2i\sqrt{7} & -11-4\sqrt{7} \\ 11-4\sqrt{7} & 6i-2i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} -6i-2i\sqrt{7} & -11-4\sqrt{7} \\ 11-4\sqrt{7} & -6i+2i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} i & -3-\sqrt{7} \\ 3-\sqrt{7} & i \end{pmatrix}$$

$$\begin{pmatrix} -i & -3-\sqrt{7} \\ 3-\sqrt{7} & -i \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3i+i\sqrt{7} & -5i-2i\sqrt{7} \\ 5i-2i\sqrt{7} & 3i-i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} 3i+i\sqrt{7} & -2i-i\sqrt{7} \\ 2i-i\sqrt{7} & 3i-i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} 3i+i\sqrt{7} & 2i+i\sqrt{7} \\ -2i+i\sqrt{7} & 3i-i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} 3i+i\sqrt{7} & 5i+2i\sqrt{7} \\ -5i+2i\sqrt{7} & 3i-i\sqrt{7} \end{pmatrix}$$

$$\begin{pmatrix} 2i+i\sqrt{7} & -3i-i\sqrt{7} \\ 3i-i\sqrt{7} & 2i-i\sqrt{7} \end{pmatrix} \quad \begin{pmatrix} 2i+i\sqrt{7} & 3i+i\sqrt{7} \\ -3i+i\sqrt{7} & 2i-i\sqrt{7} \end{pmatrix}$$