Nov 2018

Letter to Stephen Wolfram on "Audioactive Decay Mod 2"

from Alex Kontorovich

Dear Stephen,

Here is the promised discussion of the fractals mentioned over dinner. As you know very well, the "Audioactive Decay" (or "Look and Say") sequence goes like this:

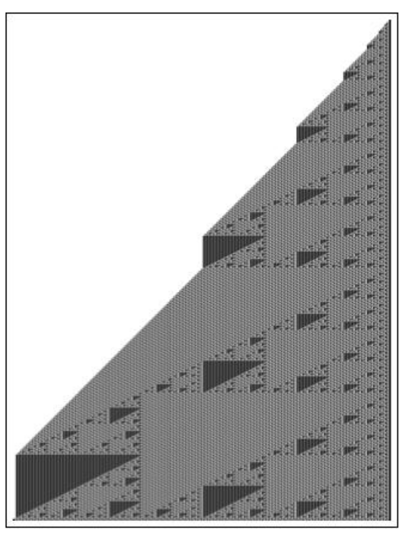
$$1, 11, 21, 1211, 111221, ...$$

The next term is obtained from the previous one by "reading it out loud," i.e., 1 is "one 1," then 11 is "two 1s", then 21 is "one 2, one 1," etc. By linear algebra, Conway shows that the sequence length grows like $c\lambda^n$, where $\lambda = 1.30\ldots$ is "Conway's constant." More surprisingly, the same exponential growth rate holds for *any* starting sequence (except "boring old" 22), and is a consequence of the "Cosmological Theorem," proved by Conway (though the proof was lost until (re?)discovered by Doron Zeilberger and his computer, Shalosh B. Ekhad). None of this is news to you.

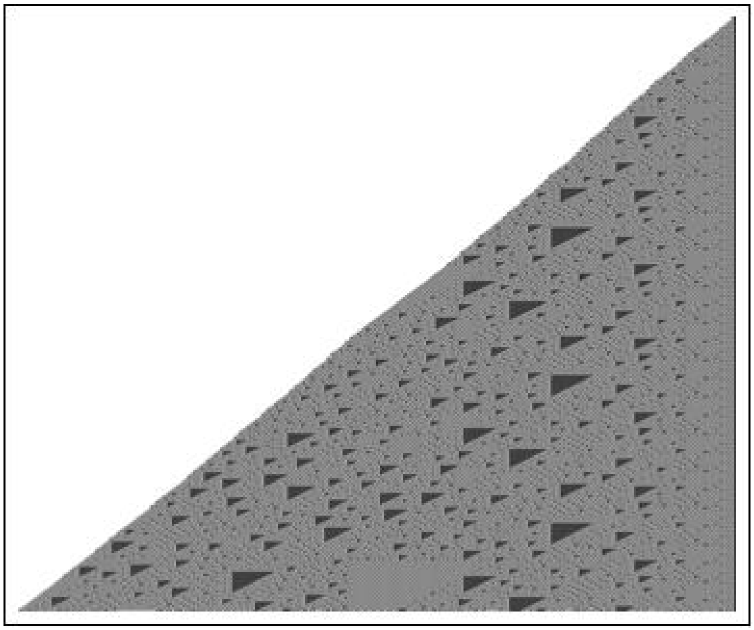
I was very fortunate to have Conway for linear algebra my freshman fall (1998), and after he showed us these theorems (as an application to the Jordan normal form!), Sam Payne and I started playing around with modular versions of this sequence. Since only the digits 1,2,3, play any substantial role, any modulus other than 2 has the same behavior as the original; hence we studied "Look and Say" mod 2. The sequence now starts like this:

$$1, 11, 01, 1011, 111001.$$

The next term is 110011, since there are three ($\equiv 1$) ones, two ($\equiv 0$) 0's and one 1. We discovered that there are now two limiting behaviors (except "boring old" 00 and "boring new" 1110) for arbitrary starting sequences. Plot 1's as black pixels and 0's as white, and right-justify the image. For the sequence starting with seed "1," we obtain the curious fractal:



The other fractal arises on starting with seed “1101”:



If I recall, we proved that these are the only 2 behaviors (length growing either linearly or piecewise-linear), though 20 years later, this proof, too, is lost (maybe Zeilberger/Ekhad can also come to our rescue). The proof should be similar: any sequence of a given length can only go backwards in time a certain distance, and then propagated forward. (A novel and serious difficulty is that, unlike the original analysis, one can have arbitrarily long concatenations of “boring new” 1110, and in fact this is responsible for the large triangles in both fractals.) I recall it being convenient to rename:

$$A = 11, \ T = 10, \ C = 01, \ G = 00,$$

(I was studying nucleotides back then...) so that A cannot be followed by C, nor T by G. It is easy to work out the decay rules:

$$\begin{aligned}
AT = (0)1110 &\rightarrow 11 = A, \\
AG = (0)1100 &\rightarrow 01 = C, \\
TA = 1011 &\rightarrow 10 = T, \\
TC = 1001 &\rightarrow 00 = G, \\
CT = 0110 &\rightarrow 01 = C, \\
CG = 0100 &\rightarrow 11 = A, \\
GA = (1)0011 &\rightarrow 00 = G, \\
GC = (1)0001 &\rightarrow 10 = T.
\end{aligned}$$

Would be fun to (re-)work out the details. The Mathematica file for constructing these and some more is available at: `http://math.rutgers.edu/~alexk/maths/Audioactive.nb`.

With best wishes,

Alex