June 2017

Letter to Kei Nakamura about
"Apollonian Equilateral Triangles"

from Alex Kontorovich

Dear Kei,

Let me make precise what I mentioned the other day about the paper "Apollonian Equilateral Triangles" by Chen and Li (see arXiv:1303.0203; this seems to have come from a PRIMES project at MIT suggested by Stanley). Here is Figure 3 from their paper:

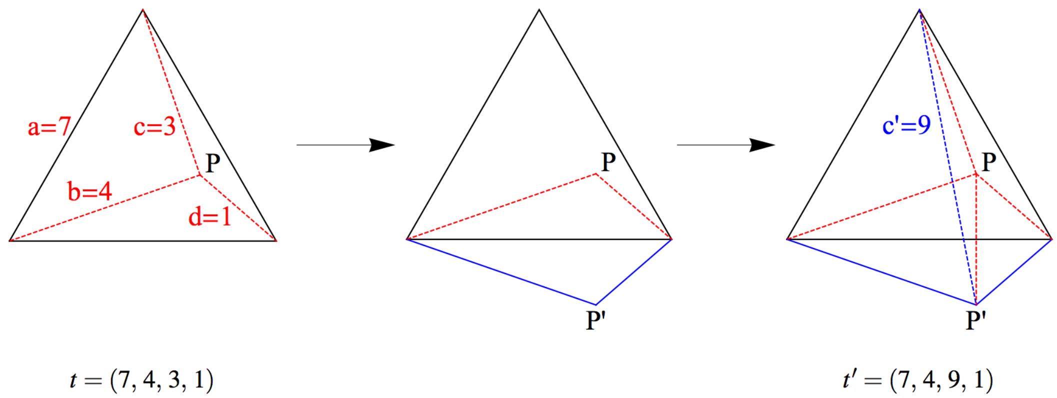


Figure 3: The new figure is generated by reflecting two of the segments across the side of the equilateral triangle. It is clear that if the original point is inside the triangle, then the new point must be outside the triangle and vice versa.

Take an equilateral triangle (here of square-side length $a = 7$, that is, the side length is $\sqrt{7}$) and some point, P, inside the triangle as shown. The square-distances from this P to the vertices of the triangle happen to be $b = 4$, $c = 3$, and $d = 1$; miraculously, they are all integers! That this is possible follows immediately from the elementary exercise in geometry that

$$Q(a, b, c, d) \ := \ 3(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 \ = \ 0. \tag{1}$$

This cone in $\mathbb{R}^4$ has lots of integral points, of which our (a, b, c, d) is but one.

You get a new point P' by reflecting P across some edge of the triangle, say the bottom edge, as shown. This doesn't change a or b or d, but $c = 3$ now becomes $c' = 9$. As is familiar from Apollonian circle packings, we can treat a, b, and d as fixed, whence (1) becomes a quadratic equation in c, the two solutions being c and c'. It is again easy to compute that the sum of the solutions is:

$$c + c' \ = \ a + b + d,$$

so we can describe this reflection $P \mapsto P'$ in terms of matrix multiplication:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & 1 & -1 & 1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \ = \ \begin{pmatrix} a \\ b \\ c' \\ d \end{pmatrix}.$$

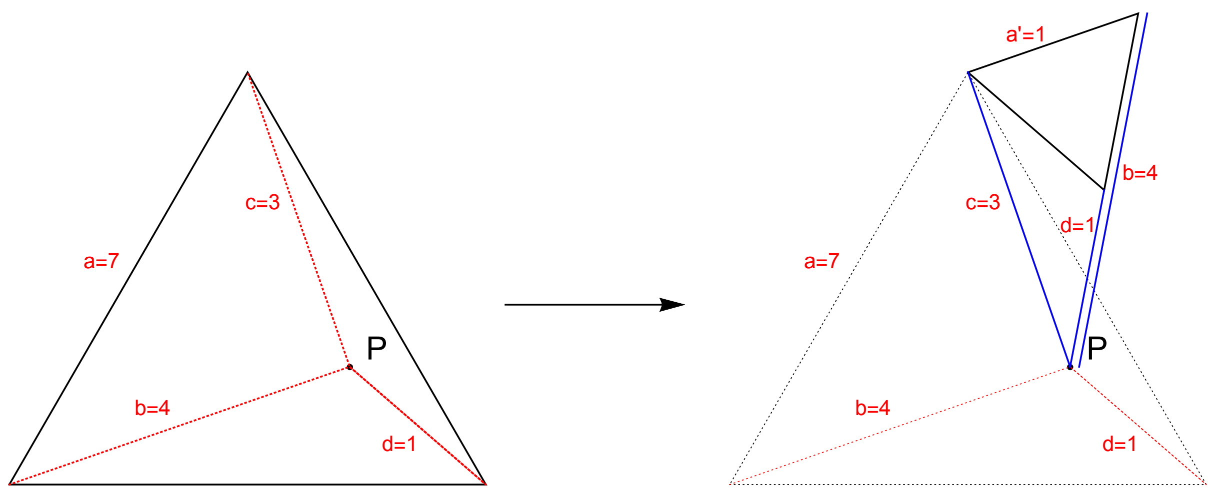
Call the above matrix M_3; there are three others like it,

$$M_1 := \begin{pmatrix} -1 & 1 & 1 & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \; M_2 := \begin{pmatrix} 1 & & & \\ 1 & -1 & 1 & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \; M_4 := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

These also act on (a, b, c, d), and by construction, preserve the form Q in (1) (and do so over the integers!). That is,

$$M_j \in O_Q(\mathbb{Z}).$$

A note of clarification: it is obvious what reflecting b and d means geometrically (namely, moving P across the corresponding sides of the triangle), but what does it mean to reflect a? That is, when applying M_1, what should happen to our picture? This doesn't seem to be addressed in the paper, but my interpretation is that the equilateral triangle itself moves, keeping P fixed (the square-distances b, c, d staying the same):



There are many choices for where to put the new triangle, and actually perhaps what we should be moving around is the full symmetric configuration in Chen-Li's Figure 4, reproduced below:

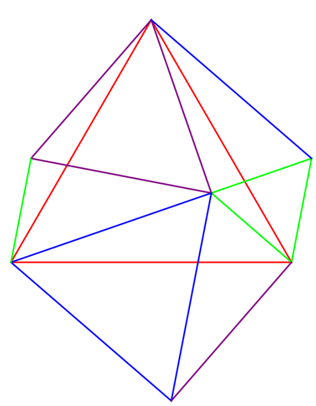


Figure 4: Segments of the same length are colored the same color.

In any event, the square-distances are what we will care about, so nevermind the pictures. They key object to study, as usual, is the symmetry group generated by the M_j,

$$\Gamma \ := \langle M_1,\ M_2,\ M_3,\ M_4\rangle \ < \ O_Q(\mathbb{Z}).$$

Chen-Li prove that the Zariski closure of Γ is all of O_Q, see their Lemma 7. In fact, much more is true. Despite the similarity in appearance between the symmetry group of Apollonian circle packings and that of Apollonian equilateral triangles, there is a very striking distinction between the two, namely, the former is *thin* while the latter is **not**!

Theorem 2. *The symmetry group Γ of Apollonian equilateral triangles is commensurate with the principal congruence group*

$$\mathrm{PSL}_2(\mathcal{O})(\varrho) = \{\gamma \in \mathrm{PSL}_2(\mathcal{O}) \ : \ \gamma \equiv I (\mathrm{mod}\, \varrho)\},$$

where $\mathcal{O} = \mathbb{Z}[e^{2\pi i/3}]$ is the ring of Eisenstein integers, and $\varrho = (3 + i\sqrt{3})/2$ is the prime above the ramified (in $\mathcal{O}$) rational prime $p = 3$.

This follows quite easily from the same (standard) techniques as in my paper "The Local-Global Principle for Integral Soddy Sphere Packings" (arXiv:1208.5441). Here is a sketch.

First consider the orientation-preserving subgroup

$$\Xi \ = \ \Gamma \cap \mathrm{SO}_Q \ = \ \langle \xi_1, \xi_2, \xi_3\rangle\,,$$

say, where

$$\xi_1 \ = \ M_1 M_2, \qquad \xi_2 \ = \ M_1 M_3, \qquad \xi_3 \ = \ M_1 M_4.$$

Changing variables, the quadratic form Q is

$$\begin{aligned} Q(a,b,c,d) \ &= \ 2(a^2+b^2+c^2+d^2-ab-ac-ad-bc-bd-cd) \\ &= \ 2\left((a-\frac{1}{2}(b+c+d))^2 - \frac{1}{4}(b+c+d)^2 - bc - bd - cd + b^2 + c^2 + d^2\right) \\ &= \ 2\left(a_1^2 + \frac{3}{4}b^2 + \frac{3}{4}c^2 + \frac{3}{4}d^2 - \frac{3}{2}bc - \frac{3}{2}bd - \frac{3}{2}cd\right) \\ &= \ 2\left(a_1^2 + 3(\frac{b-(c+d)}{2})^2 - \frac{3}{4}(c+d)^2 + \frac{3}{4}(c-d)^2\right) \\ &= \ 2\left(a_1^2 + 3b_1^2 - 3cd\right). \end{aligned}$$

Here we replaced $\mathbf{x} = (a,b,c,d)$ with

$$\mathbf{y} = (a_1, b_1, c, d) = \left(a - \frac{b+c+d}{2}, \frac{b-(c+d)}{2}, c, d\right),$$

via $\mathbf{x} = J\mathbf{y}$, where

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 2 & 1 & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Let

$$\widetilde{Q}(a,b,c,d) = a^2 + 3b^2 - 3cd$$

be the conjugate form. The corresponding matrix is

$$\widetilde{Q} = \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & & -\frac{3}{2} \\ & & -\frac{3}{2} & \end{pmatrix} = \frac{1}{2}\ {}^tJ.Q.J.$$

Conjugate $\widetilde{\xi_j} = J^{-1}.\xi_j.J$, so that

$$\widetilde{\xi_1} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widetilde{\xi_2} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & 0 & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 1 & 3 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widetilde{\xi_3} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix},$$

and

$$\widetilde{\Xi} = \left\langle \widetilde{\xi_1}, \widetilde{\xi_2}, \widetilde{\xi_3} \right\rangle < \mathrm{SO}_{\widetilde{Q}}.$$

The spin morphism from $\mathrm{PSL}_2(\mathbb{C})$ to (the connected component of the identity of) $\mathrm{SO}_{\widetilde{Q}}(\mathbb{R})$ is given by the map $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto$

$$\frac{1}{\det^2} \begin{pmatrix} \mathfrak{Re}\left(\alpha\bar{\delta} - \beta\bar{\gamma}\right) & -\sqrt{3}\mathfrak{Im}\left(\delta\bar{\alpha} + \gamma\bar{\beta}\right) & -3\mathfrak{Im}\left(\gamma\bar{\alpha}\right) & -\mathfrak{Im}\left(\delta\bar{\beta}\right) \\ -\frac{\mathfrak{Im}\left(\gamma\bar{\beta}+\alpha\bar{\delta}\right)}{\sqrt{3}} & \mathfrak{Re}\left(\delta\bar{\alpha} + \gamma\bar{\beta}\right) & \sqrt{3}\mathfrak{Re}\left(\gamma\bar{\alpha}\right) & \frac{\mathfrak{Re}\left(\delta\bar{\beta}\right)}{\sqrt{3}} \\ -\frac{2}{3}\mathfrak{Im}\left(\alpha\bar{\beta}\right) & \frac{2\mathfrak{Re}\left(\alpha\bar{\beta}\right)}{\sqrt{3}} & |\alpha|^2 & \frac{|\beta|^2}{3} \\ -2\mathfrak{Im}\left(\gamma\bar{\delta}\right) & 2\sqrt{3}\mathfrak{Re}\left(\gamma\bar{\delta}\right) & 3|\gamma|^2 & |\delta|^2 \end{pmatrix}.$$

The preimage of $\widetilde{\xi_j}$ under this map is $\mathfrak{t}_j$, with

$$\mathfrak{t}_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \mathfrak{t}_2 = \begin{pmatrix} \omega^2 & -3i \\ 0 & \omega \end{pmatrix}, \mathfrak{t}_3 = \begin{pmatrix} \omega+1 & 0 \\ -i & \omega^2+1 \end{pmatrix},$$

where ω is a primitive cube root of unity, $\omega = e^{2\pi i/3}$. Let $\Lambda = \langle \pm\mathfrak{t}_j \rangle$. Another set of generators for Λ is: $\Lambda = \langle -I, \mathfrak{t}_1, \mathfrak{t}_4, \mathfrak{t}_5 \rangle$, where

$$\mathfrak{t}_4 := \mathfrak{t}_1.\mathfrak{t}_2^{-1}.\mathfrak{t}_1 = \begin{pmatrix} 1 & 3i \\ 0 & 1 \end{pmatrix}, \qquad \mathfrak{t}_5 := -\mathfrak{t}_1^2.\mathfrak{t}_3.\mathfrak{t}_1^2 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}.$$

Conjugate Λ by $diag(\varrho\sqrt{i}, 1/(\sqrt{i}\varrho))$ to obtain the group claimed. (See §2 in my "Soddy" paper *op. cit.* for how to finish the argument; thanks to Alan Reid for pointing out that this gives the full principal congruence group.)

In fact the original group Ξ is all of $\mathrm{SO}_Q(\mathbb{Z})$, and hence Γ is all of $O_Q(\mathbb{Z})$. By Borel/Harish-Chandra, the class number is finite, so there are only finitely many orbits of integral Apollonian triangles. In fact, the class number is one, and there is only one integral primitive Apollonian equilateral triangle! It turns out that this fact was already shown in Chen-Li's original paper, see Lemma 2, from which it should have been a red flag that Γ is a lattice (I didn't notice this lemma until after computing the spin group).

To summarize, our observation here is that, though the symmetry group of Apollonian equilateral triangles *a priori* resembles that of Apollonian circle packings, it is in fact arithmetic (and moreover, congruence), as opposed to thin, and thus all the mystery is gone; all questions one could pose reduce to classical ones.

I have not looked at what happens for the corresponding groups in higher dimensions (see Chen-Li's Theorem 8). Perhaps some of these are thin?

Best wishes,

Alex