GEOMETRY AND ARITHMETIC OF CRYSTALLOGRAPHIC SPHERE PACKINGS

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Abstract. We introduce the notion of a "crystallographic sphere packing," defined to be one whose limit set is that of a geometrically finite hyperbolic reflection group in one higher dimension. We exhibit for the first time an infinite family of conformally-inequivalent such with all radii being reciprocals of integers. We then prove a result in the opposite direction: the "superintegral" ones exist only in finitely many "commensurability classes," all in dimensions below 30.

The goal of this program, the details of which will appear elsewhere, is to understand the basic "nature" of the classical Apollonian gasket. Why does its integral structure exist? (Of course it follows here from Descartes' Kissing Circles Theorem, but is there a more fundamental, intrinsic explanation?) Are there more like it? (Around a half-dozen similarly integral circle and sphere packings were previously known, each given by an ad hoc description.) If so, how many more? Can they be classified? We develop a basic unified framework for addressing these questions, and find two surprising (and opposing) phenomena:

(I) there is indeed a whole infinite zoo of integral sphere packings, and
(II) up to "commensurability," there are only *finitely-many* Apollonian-like objects, over all dimensions!

Definition 1. By an S^{n-1}-*packing* (or just packing) $\mathscr{P}$ of $\widehat{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$, we mean an infinite collection of oriented $(n-1)$-spheres (or co-dim-1 planes) so that:

- the interiors of spheres are disjoint, and
- the spheres densely fill up space; that is, we require that any ball in $\widehat{\mathbb{R}^n}$ intersects the interior of some sphere in $\mathscr{P}$.

The *bend* of a sphere is the reciprocal of its (signed) radius.[1] To be dense but disjoint, the spheres in the packing $\mathscr{P}$ must have arbitrarily small radii, so arbitrarily large bends. If every sphere in $\mathscr{P}$ has integer bend, then we call the packing *integral*.

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[1]In dimensions $n = 2$, that is, for circle packings, the bend is just the curvature. But in higher dimensions $n \geq 3$, the various "curvatures" of an $(n-1)$-sphere are proportional to $1/\text{radius}^2$, not $1/\text{radius}$; so we instead use the term "bend".

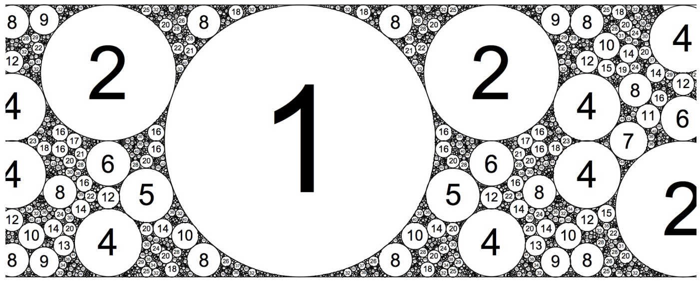


FIGURE 1. A new integral crystallographic packing. (The circles are labeled with their bend.)

Without more structure, one can make completely arbitrary constructions of integral packings. A key property enjoyed by the classical Apollonian circle packing and connecting it to the theory of "thin groups" (see [Sar14, Kon14]) is that it arises as the limit set of a geometrically finite reflection group in hyperbolic space of one higher dimension.

Definition 2. We call a packing $\mathscr{P}$ *crystallographic* if its limit set is that of some geometrically finite reflection group $\Gamma < \mathrm{Isom}(\mathbb{H}^{n+1})$.

This definition is sufficiently general to encompass all previously proposed generalizations of Apollonian gaskets found in the literature, including [Boy74, Max82, CL15, GM10, BGGM10, Sta15, Bar17]. With these two basic and general definitions in place, we may already state our first main result, confirming (I).

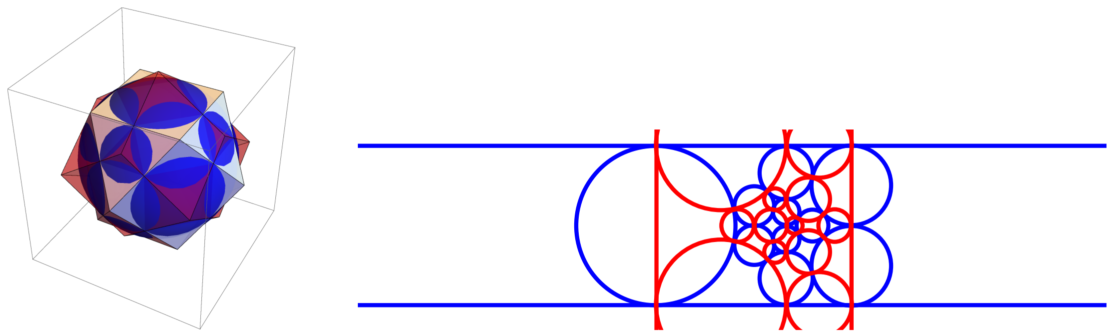
Theorem 3. *There exist infinitely many conformally-inequivalent integral crystallographic packings.*

We show in Figure 1 but one illustrative new example, whose only "obvious" symmetry is a central mirror image. It turns out (but may be hard to tell just from the picture) that this packing does indeed arise as the limit set of a Kleinian reflection group. The argument leading to Theorem 3 comes from constructing circle packings "modeled on" combinatorial types of convex polyhedra, as follows.

§(I): Polyhedral Packings

Let Π be a combinatorial type of a convex polyhedron. Equivalently, Π is a 3-connected[2] planar graph. A version of the Koebe-Andreev-Thurston Theorem says that there exists a geometrization of Π (that is, a realization of its vertices in $\mathbb{R}^3$ with straight lines as edges and faces contained in Euclidean planes) having a midsphere (meaning, a sphere tangent to all edges). This midsphere is then also simultaneously

[2]Recall that a graph is k-connected if it remains connected whenever fewer than k vertices are removed.



(a) Π, $\widehat{\Pi}$ and their midsphere

(b) An orthogonal cluster and cocluster pair with nerves Π and $\widehat{\Pi}$

FIGURE 2. Geometrization, and cluster/cocluster pair for Π = cuboctahedron with dual $\widehat{\Pi}$ = rhombic dodecahedron.

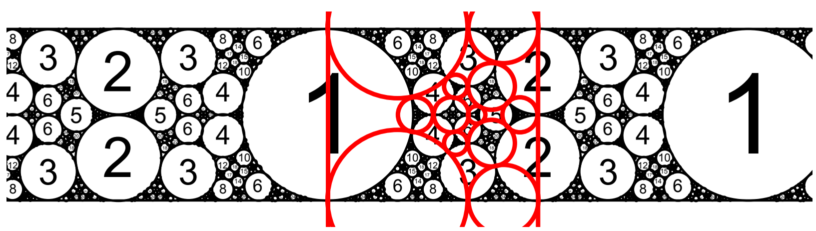


FIGURE 3. A packing modeled on the cuboctahedron, shown with cocluster

a midsphere for the dual polyhedron $\widehat{\Pi}$. Figure 2a shows the case of a cuboctahedron and its dual, the rhombic dodecahedron.

Stereographically projecting to $\widehat{\mathbb{R}^2}$, we obtain a *cluster* (just meaning, a finite collection) $\mathcal{C}$ of circles whose nerve (that is, tangency graph) is isomorphic to Π, and a *cocluster*, $\widehat{\mathcal{C}}$, with nerve $\widehat{\Pi}$ which meets $\mathcal{C}$ orthogonally. Again, the example of the cuboctahedron is shown in Figure 2b.

Definition 4. The orbit $\mathscr{P} = \mathscr{P}(\Pi) = \Gamma \cdot \mathcal{C}$ of the cluster $\mathcal{C}$ under the group $\Gamma = \left\langle \widehat{\mathcal{C}} \right\rangle$ generated by reflections through the cocluster $\widehat{\mathcal{C}}$ is said to be *modeled on* the polyhedron Π.

Lemma 5. *An orbit modeled on a polyhedron is a crystallographic packing.*

See Figure 3 for a packing modeled on the cuboctahedron. Such packings are unique up to conformal/anticonformal maps by Mostow rigidity, but Möbius transformations do not generally preserve arithmetic.

Definition 6. We call a polyhedron Π *integral* if there exists *some* packing modeled on Π which is integral.

It is not hard to see that the cuboctahedron is indeed integral. So is the tetrahedron, which corresponds to the classical Apollonian gasket. It is a fundamental problem to classify all integral polyhedra.

Let us point out some basic difficulties with this problem. First of all, it is nontrivial to determine whether, given a particular polyhedron, there exists some packing modeled on it which is integral. Indeed, Koebe-Andreev-Thurston geometrization is an infinite limiting process, and how is one to know whether 3.9999 is really 4? To the rescue is Mostow rigidity again, which implies that one can always find cluster/cocluster configurations with all centers and radii *algebraic*. This means that after computing enough decimal places, one can *guess* what the nearby algebraic values might be, and then *rigorously* verify whether the guess gives the correct tangency data. This algorithm works for small examples, but once Π is sufficiently complicated, it may take a very long time for the guessing process to halt.

Despite these difficulties, we are able to show the following towards (I).

Theorem 7. *Infinitely many polyhedra are integral, and give rise to infinitely many conformally-inequivalent integral polyhedral packings.*

This of course implies Theorem 3. To explain the main ideas in the proof, we need some more notation.

Returning to the general setting of crystallographic packings, recall that $\mathscr{P}$ is assumed to arise as the limit set of a discrete group Γ; we call the latter a *symmetry group* of $\mathscr{P}$.

Definition 8. Given a packing $\mathscr{P}$ with symmetry group Γ, we define its *supergroup*, $\widetilde{\Gamma}$, to be the group generated by Γ itself, plus reflections through all spheres in $\mathscr{P}$. Abusing notation, we may write this as

$$\widetilde{\Gamma} \ := \ \langle \Gamma, \mathscr{P} \rangle \ < \ \mathrm{Isom}(\mathbb{H}^{n+1}).$$

In the case of a polyhedral packing $\mathscr{P} = \mathscr{P}(\Pi)$, the supergroup is simply the group generated by reflections in both the cluster and cocluster, $\widetilde{\Gamma} = \left\langle \widehat{\mathcal{C}}, \mathcal{C} \right\rangle$.

Definition 9. The *superpacking*, $\widetilde{\mathscr{P}}$, of $\mathscr{P}$ with symmetry group Γ is the orbit of $\mathscr{P}$ under its supergroup, that is,

$$\widetilde{\mathscr{P}} \ := \ \widetilde{\Gamma} \cdot \mathscr{P}.$$

Note that the superpacking is *not* a packing by our definition as the sphere interiors are no longer disjoint.[3]

[3] A related notion of superpacking for the classical Apollonian gasket arose already in work of Graham-Lagarias-Mallows-Wilks-Yan [GLM+06]; see also the viewpoint of "Schmidt arrangements" in work of Stange [Sta15] and Sheydvasser [She17].

Definition 10. We call a packing $\mathscr{P}$ *superintegral* if every bend in its superpacking $\widetilde{\mathscr{P}}$ is an integer.[4]

Remark 11. While different symmetry groups Γ lead to different (but commensurate) supergroups $\widetilde{\Gamma}$, the superpackings are universal, the same for all choices of Γ.

Returning to polyhedral packings, we say that a polyhedron is *superintegral* if some packing modeled on it is. To prove Theorem 7, we actually prove the following stronger statement.

Theorem 12. *Infinitely many polyhedra are superintegral, and give rise to infinitely many conformally-inequivalent superintegral crystallographic packings.*

Remark 13. We stress the conformal-inequivalence here because it turns out that infinitely many polyhedra give rise to the *same* crystallographic packing; so the first part of Theorem 12, that infinitely many polyhedra are superintegral, does not by itself imply Theorem 3.

Though every previously known integral packing was also superintegral, we discover for the first time that the latter is a strictly stronger condition.

Lemma 14. *There exist infnitely-many conformally-inequivalent crystallographic packings that are integral but not superintegral.*

Remark 15. Just one example of an integral but not superintegral polyhedron is the hexagonal pyramid. See also Remark 24.

To prove Theorem 12, we define certain operations on "seed" polyhedra which we call "growths," including doubling the seed along a vertex or a face, and observe that, while these generally wreak havoc on the resulting packings $\mathscr{P}$, so $\mathscr{P}(growth)$ and $\mathscr{P}(seed)$ are usually conformally inequivalent, the superpackings $\widetilde{\mathscr{P}}$ are essentially preserved, in fact

$$\widetilde{\mathscr{P}}(growth) \subset \widetilde{\mathscr{P}}(seed).$$

In particular, if a polyhedron is superintegral, then all of its growths are also superintegral, and hence integral! This proves Theorem 12, and hence Theorem 3.

§(II): Classifying Superintegral Crystallographic Packings

Towards the opposite general problem of classifying integral and superintegral crystallographic packings, we make two basic observations. The first, having nothing to do with integrality, shows that the entire theory of crystallographic packings is "low"-dimensional.

Theorem 16. *Crystallographic packings can only exist in dimensions $n < 996$.*

To prove this, we need the following

[4]Note that an unrelated notion of "superintegrality" is defined in [GLM+06, §8].

Lemma 17. *The supergroup $\widetilde{\Gamma}$ of a crystallographic packing $\mathscr{P}$ with symmetry group Γ is a* lattice*, that is, it acts on $\mathbb{H}^{n+1}$ with finite covolume.*

We first sketch a proof of this lemma. Let Γ be a symmetry group for $\mathscr{P}$; then it is assumed to be geometrically finite (recall that this means some uniform thickening of the convex core of Γ has finite volume). Since Γ is a reflection group, it has an essentially unique fundamental polyhedron $\mathscr{F} := \Gamma\backslash\mathbb{H}^{n+1}$. The domain of discontinuity Ω of Γ (that is, the complement in $\partial\mathbb{H}^{n+1}$ of its limit set Λ_Γ) is the union of disjoint open geometric balls, since the limit set Λ_Γ is assumed to coincide with the set of limit points of $\mathscr{P}$. The quotient Ω/Γ is then a disjoint union of finitely many open ends. For each end, we develop the domain under the Γ-action and fill an open ball, the boundary of which is then an (un-oriented) sphere in $\mathscr{P}$. A geodesic hemisphere above such a ball is a frontier of the flare, cutting the walls it meets of $\mathscr{F}$ either tangentially or at right angles (for otherwise the spheres in $\mathscr{P}$ would overlap). Hence when we form the supergroup $\widetilde{\Gamma}$ by adjoining to Γ reflections through all the spheres in $\mathscr{P}$, we obtain a discrete action, and moreover the original domain of discontinuity Ω has been entirely cut out, rendering $\widetilde{\Gamma}$ a lattice.

Returning to Theorem 16, Vinberg [Vin81] and Prokhorov [Pro86] showed that hyperbolic reflection lattices can only exist in dimensions $n < 996$, and hence crystallographic packings are similarly bounded in dimension, proving the theorem. (The number 996 is not expected to be sharp.)

Next we show that not only is the dimension bounded, but if we assume superintegrality, then (up to commensurability) there are only *finitely many* Apollonian-like objects, period!

Definition 18. Two crystallographic packings are said to be *commensurate* if their supergroups are.

Theorem 19. *There are only* finitely-many *commensurability classes of superintegral crystallographic packings, all of dimension $n < 30$.*

To prove this theorem, we show the following

Theorem 20. *If $\mathscr{P}$ is a superintegral crystallographic packing, then its supergroup $\widetilde{\Gamma}$ is arithmetic*[5]*!*

In fact, to conclude arithmeticity, it is sufficient that the orbit under the supergroup $\widetilde{\Gamma}$ of a single sphere $S \in \mathscr{P}$ has all integer bends. Let us sketch a proof. To a

[5]Recall that a real hyperbolic reflection group is arithmetic if it is commensurate with the automorphism group of a hyperbolic quadratic form over the ring of integers of a totally real number field; see, e.g., [Bel16].

(positively-oriented) sphere S of center $\mathbf{z} = (z_1, \dots z_n)$ and radius r, we attach the "inversive coordinates"

$$\mathbf{v}_S \ := \ \left(\widehat{b}, b, b\mathbf{z}\right).$$

Here $b = 1/r$ is the bend, and $\widehat{b} = 1/\widehat{r}$ is the co-bend, that is, the reciprocal of the co-radius, the latter defined as the radius of the sphere after inversion through the unit sphere; see the discussion in, e.g., [Kon17a, LMW02]. The vector $\mathbf{v}_S$ lies on a one-sheeted hyperboloid $Q = -1$, where Q is the (universal) "discriminant" form,

$$Q = \begin{pmatrix} & \frac{1}{2} & \\ \frac{1}{2} & & \\ & & -I_{n-1} \end{pmatrix}.$$

In these coordinates,

$$\widetilde{\Gamma} < O_Q(\mathbb{R}) \tag{21}$$

is a right action by Möbius transformations on the row vector $\mathbf{v}_S$. Since $\widetilde{\Gamma}$ is a lattice, it is essentially (up to finite index components) Zariski dense in O_Q; hence the orbit $\mathcal{O} = \mathbf{v}_S \cdot \widetilde{\Gamma}$ of S is essentially Zariski dense in the quadric $Q = -1$. There is then a choice of cluster $\mathcal{C}_S \subset \mathcal{O}$ of $n+2$ spheres whose matrix $\mathcal{V}$ of inversive coordinates has (full) rank $n+2$. Make such a choice arbitrarily. This cluster $\mathcal{V}$ has a Gram matrix of inversive products,

$$\mathcal{G} \ := \ \mathcal{V} \cdot Q \cdot \mathcal{V}^{\dagger}, \tag{22}$$

which is invertible (also has rank $n+2$). Let

$$\mathcal{F} := \mathcal{G}^{-1}$$

be its inverse, which also induces a quadratic form having signature $(1, n+1)$. Then $\widetilde{\Gamma}$ is conjugate to a "bends" group,

$$\widetilde{\mathcal{A}} \ := \ \mathcal{V} \cdot \widetilde{\Gamma} \cdot \mathcal{V}^{-1} \ < \ O_{\mathcal{F}}(\mathbb{R}),$$

which now acts on the left on the (second) column vector of bends $\mathbf{b} = \mathcal{V} \cdot (0, 1, 0, \dots, 0)^{\dagger}$ in $\mathcal{V}$; this vector $\mathbf{b}$ lies on the cone $\mathcal{F} = 0$, and $\widetilde{\mathcal{A}}$ is a lattice in $O_{\mathcal{F}}(\mathbb{R})$. Though *a priori* real valued, we claim that $\mathcal{F}$ is in fact *rational*. Indeed, by assumption, the $\widetilde{\mathcal{A}}$-orbit

$$\mathcal{B} = \widetilde{\mathcal{A}} \cdot \mathbf{b}$$

lies in $\mathbb{Z}^{n+2} \cap \{\mathcal{F} = 0\}$, and is Zariski dense in the cone. But a quadratic form having a Zariski dense set of *integer* points $\mathcal{B}$ on the cone $\mathcal{F} = 0$ is easily seen to be rational, as claimed. Next we observe that, since $\widetilde{\mathcal{A}}$ is a *linear* action, it in fact preserves a full rank $\mathbb{Z}$-lattice Λ. But the group

$$O_{\mathcal{F}}^{\Lambda} = \{g \in O_{\mathcal{F}}(\mathbb{R}) : g\Lambda = \Lambda\}$$

is easily seen to be congruence, and contains $\widetilde{\mathcal{A}}$. Hence $\widetilde{\mathcal{A}}$ is arithmetic, as is its conjugate $\widetilde{\Gamma}$. This proves Theorem 20.

Returning to Theorem 19, this now follows from Theorem 20, together with the amazing fact [Vin81, LMR06, Ago06, ABSW08, Nik07], that there are only *finitely-many* commensurability classes of arithmetic reflection groups, all having dimension $n < 30$. Hence the same holds for superintegral crystallographic packings by Theorem 20.

It turns out that superintegrality is a necessary condition in Theorem 20, and mere integrality is insufficient. Indeed, we discover for the first time the following

Lemma 23. *There exist infinitely many conformally inequivalent integral (but of course not superintegral) packings whose supergroups are non-arithmetic!*

Remark 24. The supergroup of the hexagonal pyramid is *non-arithmetic*; see also Remark 15.

Remark 25. Note also that there is no contradiction with Theorem 12 (and Theorem 3), as the packings constructed there fall into finitely many commensurability classes.

Given these finiteness results, the complete classification of superintegral crystallographic packings will then rely on understanding to what extent a converse of Theorem 20 may be true.

Question 26. *Given an arithmetic reflection group, is it commensurate with the supergroup of a superintegral crystallographic packing?*

We will say that an arithmetic group "supports" a packing if the answer to the above is YES. We have investigated this question in some special cases and found the following positive results.

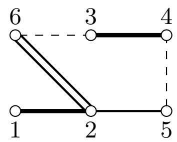
Theorem 27. *The answer to Question 26 is YES for all non-uniform lattices over $\mathbb{Q}$ in dimension $n = 2$. Namely, every reflective (that is, commensurate to a reflection group) Bianchi group supports a superintegral crystallographic packing.*

In higher dimensions, we are also able to show the following.

Theorem 28. *The answer is YES for certain lattices in all dimensions up to (at least) $n = 13$; that is, superintegral crystallographic packings exist in all these dimensions.*

Before saying more about these theorems, let us point out that we suspect that the answer may be NO in general.

Remark 29. At present, we do not know of a single superintegral (or even integral) packing whose supergroup is cocompact. In dimension $n = 2$, the integral orthogonal groups preserving the form $x_1^2 + x_2^2 + x_3^2 - dx_4^2$ are cocompact and reflective only when the coeffcient $d = 7$ or 15 [McL13]. We suspect, but do not know how to prove, that neither of these reflection groups support crystallographic packings. See Remark 33.



Vertices denote reflecting walls, and two vertices are connected by

a dotted line,	if the walls' interiors are disjoint,
a thick line,	if the walls meet at a cusp, i.e., dihedral angle 0,
$m-2$ thin lines,	if the walls intersect with dihedral angle π/m,
no line,	if the walls intersect orthogonally.

FIGURE 4. The Coxeter diagram for the reflective subgroup of the maximal discrete extension of the Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}])$.

Remark 30. Taking, e.g., $\mathfrak{o} = \mathbb{Z}[\varphi]$ the ring of the golden mean, we *can* construct $\mathfrak{o}$-superintegral packings (that is, with all bends in $\mathfrak{o}$), and having supergroup the right-angled dodecahedron (which is arithmetic and co-compact). It is an interesting problem to extend our theory to packings with bends in integer rings. (And more generally to complex hyperbolic space, $\mathrm{SU}(n,1)$, etc.)

Theorems 27 and 28 follow from our Structure Theorem:

Theorem 31 (Structure Theorem for Crystallographic Packings)**.** *Let $\widetilde{\mathcal{C}}$ be a set of walls (that is, spheres), the reflections through which generate a hyperbolic lattice, and orient these walls so that the fundamental domain is the intersection of their exteriors. Assume that $\widetilde{\mathcal{C}}$ decomposes into a cluster/cocluster pair:*

$$\widetilde{\mathcal{C}} = \mathcal{C} \bigsqcup \widehat{\mathcal{C}} \tag{32}$$

so that

- *any pair of spheres in $\mathcal{C}$ is either disjoint or tangent, and*
- *any sphere in $\mathcal{C}$ is either disjoint, tangent, or orthogonal to any in $\widehat{\mathcal{C}}$.*

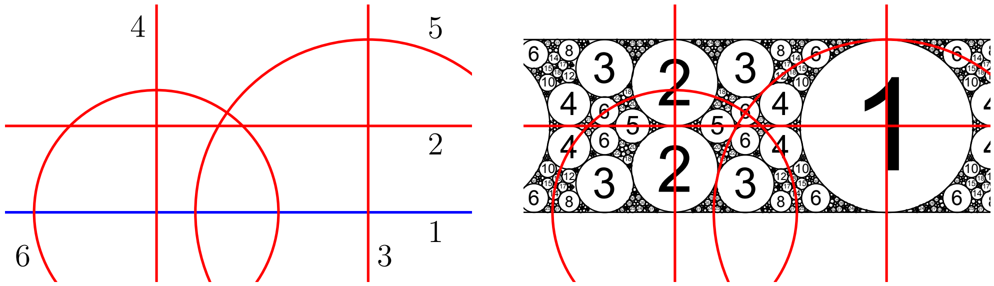
Let $\Gamma := \left\langle \widehat{\mathcal{C}} \right\rangle$ be the (thin) group generated by reflections through the cocluster. Then the cluster orbit under this group, $\mathscr{P} := \Gamma \cdot \mathcal{C}$, is a crystallographic packing.

Conversely, every crystallographic packing arises in this way.

The converse direction follows from our proof of Lemma 17, and the forward direction uses similar ideas. Hence answering Question 26 for a given reflection lattice is equivalent to finding a decomposition as in (32), or proving that one cannot exist.

Remark 33. In the case of the cocompact forms in Remark 29, we are not yet able after some effort to find a reflective subgroup (or prove it does not exist) with a suitable decomposition of the form (32).

Returning to Theorem 27, our proof of this result relies on the complete classification by Belolipetsky-Mcleod [BM13] of reflective Bianchi groups. For example, the Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}])$ is commensurate to a maximal reflection group having the Coxeter diagram illustrated in Figure 4 (we follow Vinberg's convention for the labelling, as indicated there).



(a) The reflecting walls in Figure 4.

(b) The packing resulting from the orbit generated by $\widehat{\mathcal{C}}$ on the cluster $\mathcal{C}$ in the decomposition (34).

FIGURE 5

One realization of the Coxeter diagram in Figure 4 is given by reflecting walls (circles) illustrated in Figure 5a, with the same labeling. (The reader may check that the angles of intersection are as claimed in the Coxeter diagram.) The reader may now also verify that the decomposition of labeled walls as:

$$\mathcal{C} = \{1\}, \qquad \widehat{\mathcal{C}} = \{2, 3, \dots, 6\}, \tag{34}$$

satisfies the assumptions of Theorem 31, and hence gives rise to a crystallographic packing by taking the orbit of $\mathcal{C}$ under the group of reflections through $\widehat{\mathcal{C}}$. The resulting packing is shown in Figure 5b, which is the familiar cuboctahedral packing in disguise! (Compare with Figure 3.)

All but one of the other reflective Bianchi groups can be directly verified (by the Structure Theorem) to support superintegral packings; see Figures 1 and 2 in [BM13]. The only case in which the decomposition (32) is not straightforward from studying this Coxeter data is the Bianchi group on the Eisenstein integers (that is, adjoining the cube root of unity). It turns out in this case that the Coxeter diagram in the literature has a minor mistake which can be traced to an early paper of Shaiheev [Sha90]; it has propagated in the literature ever since. The issue comes from the execution of Vinberg's algorithm for reflection subgroups which, for the Eisenstein integers, has extra stabilizers due to the larger group of units. The true diagram is

and the Eisenstein Bianchi group has a subgroup with Coxeter diagram

This last diagram supports a decomposition as in (32) by taking either $\mathcal{C} = \{1\}$ or $\mathcal{C} = \{3\}$. We are thus finished sketching the only non-immediate case of Theorem 27.

Remark 35. In fact it turns out that *all* previously known integral circle packings (and many new ones!) arise in this way as limit sets of thin subgroups of reflective Bianchi groups.

To prove Theorem 28, we simply inspect Vinberg's Coxeter diagrams [Vin72] for $n \leq 13$ for the reflective subgroup of the integer orthogonal group preserving the form $-2x_0^2 + x_1^2 + \cdots + x_{n+1}^2$ and apply the Structure Theorem.

§ Integral but Non-Superintegral Packings

Let us say more about what happens in Remarks 15 and 24. When Π is, e.g., the hexagonal pyramid, its supergroup $\widetilde{\Gamma} = \left\langle \mathcal{C}, \widehat{\mathcal{C}} \right\rangle$ can be computed to have Gram matrix (see (22))

$$\mathcal{G} = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \\ 1 & -1 & 1 & 5 & 7 & 5 & 1 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 \\ 1 & 1 & -1 & 1 & 5 & 7 & 5 & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 \\ 1 & 5 & 1 & -1 & 1 & 5 & 7 & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 0 \\ 1 & 7 & 5 & 1 & -1 & 1 & 5 & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 0 \\ 1 & 5 & 7 & 5 & 1 & -1 & 1 & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 0 \\ 1 & 1 & 5 & 7 & 5 & 1 & -1 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & -1 & 1 & 5 & 7 & 5 & 1 & 1 \\ 0 & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 1 & -1 & 1 & 5 & 7 & 5 & 1 \\ 0 & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 5 & 1 & -1 & 1 & 5 & 7 & 1 \\ 0 & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 2\sqrt{3} & 7 & 5 & 1 & -1 & 1 & 5 & 1 \\ 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 0 & 5 & 7 & 5 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} & 0 & 1 & 5 & 7 & 5 & 1 & -1 & 1 \\ \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}. \tag{36}$$

Vinberg's Arithmeticity Criterion [Vin67] (see also [VS93, Theorem 3.1]) says in this context that $\widetilde{\Gamma}$ is arithmetic if and only if cyclic products of $2\mathcal{G}$ are always integers. This is almost the case for (36), except for the entries $\frac{2}{\sqrt{3}}$ in the top right (and by symmetry, bottom left); hence $\widetilde{\Gamma}$ is non-arithmetic (see Lemma 23). But it is nearly so; indeed, $\widetilde{\Gamma}$, viewed as a subgroup of $O_Q(\mathbb{R})$ (see (21)), can be conjugated to lie in $O_Q(\mathbb{Z}[\frac{1}{3}])$ with unbounded denominators in its entries. The latter group is a perfectly nice S-arithmetic lattice in the product $O_Q(\mathbb{R}) \times O_Q(\mathbb{Q}_3)$, but $\widetilde{\Gamma}$ is already a lattice on projection the first factor, $O_Q(\mathbb{R})$. This too implies that $\widetilde{\Gamma}$ is non-arithmetic, and in this sense is reminiscent of constructions of non-arithmetic groups by Deligne-Mostow [DM86]. It is interesting to understand if all integral but non-superintegral packings arise this way.

§ Local-Global Principles

We conclude with a discussion of whether Local-Global Principles hold for bends of crystallographic circle ($n = 2$) packings. (For higher dimensional sphere packings, this problem becomes easier; see, e.g., [Kon17b].) As explained in [Kon13] in the case of the classical Apollonian packing, the "asymptotic" local-global principle is proved in [BK14]. This method was extended in the thesis of Zhang [Zha15] to show the same statement for packings modeled on the octahedron. Most recently, Fuchs-Stage-Zhang showed that the method extends to the following context:

Theorem 37 ([FSZ17])**.** *Let $\mathscr{P}$ be a packing with symmetry group Γ and let $C \in \mathscr{P}$. Assume that there is a circle $C' \in \mathscr{P}$ tangent to C so that the stabilizer of C' in Γ is a* congruence *(Fuchsian) group. Then the orbit $\Gamma \cdot C$ satisfies an asymptotic local-global principle.*

The assumption of the existence of such a companion circle C' is a generalization of Sarnak's observation [Sar07] in the classical Apollonian case that such leads to certain shifted binary quadratic forms representing bends in the orbit. We show that this condition is both satisfied and not satisfied infinitely often!

Theorem 38. *The assumptions (and hence conclusions) of Theorem 37 are satisfied for infinitely many conformally-inequivalent superintegral crystallographic packings. The same statement holds with "are satisfied" replaced by "are* not *satisfied."*

Thus even the asymptotic local-global problem remains open in this generality.

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