ON TRACE SETS OF RESTRICTED CONTINUED FRACTION SEMIGROUPS

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Abstract. We record an argument due to Jean Bourgain which gives lower bounds on the size of the trace sets of certain semigroups related to continued fractions on finite alphabets. These bounds are motivated by the "Classical Arithmetic Chaos" Conjecture of McMullen [McM12]. Specifically, a power is gained in the asymptotic size of the trace set over a "trivial" exponent. The proof involves a new application of the Balog-Szemerédi-Gowers Lemma from additive combinatorics.

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1. Introduction

We begin with some personal remarks and reminiscences on the occasion of this Dedicated Volume; we allow ourselves to be descriptive, returning to precision and science in §1.1. My collaboration with Jean Bourgain began in the fall of 2008, when I applied for the 2009-10 IAS Special Year in Analytic Number Theory. To explain properly what we were trying to accomplish, I have to back up to my 2007 thesis. There I was interested in a kind of mixture between the theorems of Friedlander-Iwaniec [FI98] and Piatetskii-Shapiro [Pu53], the former

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being that the polynomial

$$FI(x,y) := x^2 + (y^2)^2 \tag{1.1}$$

represents infinitely many primes, and the latter that the sequence

$$PS(n) := \lfloor n^\alpha \rfloor$$

does too, for sufficiently small values of the fixed constant $\alpha > 1$. Both sequences are "thin": the number of integers up to X represented by FI is about $X^{3/4}$, whereas for PS it is about $X^{1/\alpha}$, so it is rather difficult to produce primes in such sparse sequences[1] (the latter being still much easier than the former![2]). The nice thing about PS is that there is a parameter, α, to play with, and thus a potential range of thinness where one can succeed. The main idea of my thesis (suggested to me by Peter Sarnak, motivated by his work with Jean and Alex Gamburd on the Affine Sieve [BGS10]), was to see whether an amalgam of the two was possible in the group setting; by this we mean the following.

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be some Zariski-dense subgroup of the modular group; if it is of infinite index (or even just non-congruence!), then we have no idea exactly which pairs (c,d) arise as bottom rows, say, of elements in the group $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \overset{?}{\in} \Gamma$. We would, in principle, first need to try to write any such matrices as words in the generators of Γ. Regardless, consider the sequence

$$\mathcal{S} := \{c^2 + d^2 : \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma\}. \tag{1.2}$$

The total number of such values $c^2+d^2 < X$ *can* be counted effectively, that is, with power savings, as was done in my thesis [Kon09] (under a technical assumption that was removed in [KO12]). The answer is roughly: X^δ, where δ, assumed to exceed one-half, is the critical exponent of Γ (equivalently [Pat75], the Hausdorff dimension of the limit set of Γ; the condition $\delta > 1/2$ is needed to relate δ to the base eigenvalue $\lambda_0 = \delta(1-\delta)$ of the hyperbolic Laplacian acting on square-integrable functions on the upper half plane $\mathbb{H}$ invariant under Γ). Since one can exhibit Γ with δ arbitrarily close to 1, one can play with this "thinness" parameter, similarly to Piatetskii-Shapiro, where $1/\alpha < 1$ plays the role of δ. If instead we returned to all integer pairs (c,d) but forced $d = y^2$ to be a perfect square, then we would exactly be in the situation of Friedlander-Iwaniec (1.1). So this set S has both features, studying c^2+d^2 for restricted (by the group) values of (c,d), with the flexibility of a parameter δ. Since this phenomenon of δ being thin but not "too"

[1]Heath-Brown [HB01] was later able to do the same for the even thinner polynomial $x^3 + 2y^3$, which takes about $X^{2/3}$ values up to X.

[2]See also [Kon12] for a simpler instance of this "parity breaking."

thin will appear again and again, let me refer to it as being **slightly thin**, that is, allowing $\delta < 1$ but also requiring that $\delta > 1-\varepsilon_0$ for some (usually small) $\varepsilon_0 > 0$.

The problem of producing primes in $\mathcal{S}$ for *any* value of $\delta < 1$ is still wide open. The tools in my thesis managed to produce R-almost primes (that is, numbers with at most R prime factors) for $R = 13$ in slightly thin groups,[3] and in my application to IAS, I had proposed to use not just the linear sieve, but to introduce bilinear forms techniques into the affine sieve to attack this problem, with the hope of producing actual primes. Admittedly, this is perhaps a rather niche question, but one I enjoyed thinking about for its mixture of geometric, combinatorial, spectral, dynamical, algebraic, and number theoretic techniques.

Jean must have read my application, because the next time I visited Peter at IAS, Jean requested to speak with me. At our meeting, he outlined how to execute such bilinear forms ideas to produce primes, not in $\mathcal{S}$, but in certain algebraic traces of entries of slightly thin subgroups of the Picard group $\mathrm{SL}_2(\mathbb{Z}[i])$, the added dimension allowing for more variables.[4] Together, we whittled away at the problem until we could produce, for slightly thin subgroups of the modular group $\mathrm{SL}_2(\mathbb{Z})$, primes in the values of the *linear* map $f : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$, say (note that, for $\mathcal{S}$ in (1.2), we would instead apply a quadratic map $f : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto c^2 + d^2$). At some point during a conversation with Jean and Peter, we realized that in fact we already had almost all the tools needed to prove something much stronger: the reason we were able to produce primes values of f is because we were actually producing *almost all* numbers! (This is in contradistinction to $f = c^2 + d^2$ which is genuinely a thin subset of $\mathbb{Z}$.[5])

Thus producing an almost-all statement in the $f = d$ values of slightly thin subgroups of $\mathrm{SL}_2(\mathbb{Z})$ became our first joint paper [BK10]. It required an additional stubborn technical ingredient (of independent interest) to count effectively in bisectors in thin groups, which we proved in a companion paper jointly also with Peter [BKS10]; such ideas have since been generalized many times by many authors. We had also noticed some similarities between this problem and the local-global problem for Apollonian packings (see [BK10, Remark 1.12]), but there didn't seem to be an obvious way to transfer our technology, given that

[3]It turns out that I should have been able to product R-almost primes with $R = 7$, see [HK15].

[4]Much later, I would exploit a similar feature in [Kon19].

[5]For a formal definition of thinness in a general context, see [Kon14, p. 954].

the Apollonian group was **not** slightly thin, but had a fixed dimension, $\delta \approx 1.30$; there was no parameter to adjust!

At this point, I thought our collaboration was basically done and I could return to civilian life. But as luck would have it, Curt McMullen pointed out to us the similarity between the problem we had just attacked and Zaremba's conjecture [Zar72] on bounded continued fractions of rationals. (See, e.g., [Kon13] for a detailed discussion of this problem.) We *should* have already been aware of the connection, since years before, Jeff Lagarias had pointed out to me the similarity between Zaremba and Apollonius (see [GLM+03, p. 37]), but somehow it took Curt's urging for us to begin working on it. The Zaremba problem was nearly identical, except that we were missing a number of technical ingredients, including the main consequences of [BKS10]. The reason is that, in Zaremba, one must deal with a sub-**semi-**group of $\mathrm{SL}_2(\mathbb{Z})$, not a sub*group*, thus rendering our spectral and representation theoretic counting developments useless. Nevertheless, one could substitute the thermodynamic formalism to count [Hen89, BGS11], and we were able to show density-one for Zaremba [BK14a]. Here the analog of being "slightly thin" is having a sufficiently large allowed alphabet for the restricted partial quotients.

Again I thought that would basically be the end of things, but about a year later, we realized that using some rather different techniques (relying not on "slight thinness," but instead exploiting the existence of values of shifted binary quadratic forms inside the bend set, as observed by Sarnak [Sar07a], and taking inspiration from Jean's paper [Bou12] on prime values in Apollonian packings), we could actually extend the local-global technology to prove density-one in the Apollonian problem, see [BK14b].

It quickly became clear that the bilinear forms technology developed on our "Orbital Circle Method" phase could be applied much more widely, and we turned our attention back to the original sequence (1.2) from my thesis. There we were able to implement these ideas, along with some others (e.g., the "dispersion method" in the group context), to push past the sieve level of distribution (see [Kon14]) which follows "for free" from counting arguments and "expansion" (that is, certain families of Cayley graphs being expanders), to a level "Beyond Expansion." In the end, we could produce, for any slightly thin group, R-almost primes in $\mathcal{S}$, with $R = 4$ [BK15]. This became Part I in our Beyond Expansion program.

For Part II [BK17], we turned our attention to a problem of Einsiedler-Lindenstrauss-Michel-Venkatesh, which itself actually served as the original motivation for the Affine Sieve (see [Sar07b]). This problem, involving the same semigroup as in our Zaremba work, required "only" a square-free sieve, but as it turned out, expansion alone was just barely insufficient to solve the problem. An added difficulty was that, unlike Zaremba where the linear function on the semigroup was $f : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$, here one needed to deal with the trace, $f : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto a + d$. The nice thing about $f = d$ is that it is *already* itself bilinear, being expressible as $f(\gamma) = \langle e_2, \gamma e_2 \rangle$ (with $e_2 = (0, 1)^t$), but trace is not. Nevertheless with some new ideas, we were able to solve the problem, producing an infinitude of "low-lying" but "fundamental" closed geodesics on the modular surface. In Part III of the series [BK19], we added the adjective "reciprocal" to the closed geodesics, inching such a tad closer to the Markoff geodesics they are meant to imitate.

And the final Part IV of the series, which is related to the theorem we wish to explain in this note, was motivated by the "Classical Arithmetic Chaos Conjecture" posed by Curt McMullen, see §1.1. This problem was basically too difficult for us to say very much about at all, except that we could improve the "expansion" exponent of distribution in the trace set all the way to what it would have been, had some analog of the Ramanujan conjecture (on average) existed in this setting, see [BK18].

Thus ended my collaboration with Jean Bourgain. Echoing what others have said, with his passing, the mathematical world has lost an Archimedes, an Euler, a Gauss. It was an incredible privilege and honor to work with Jean, and I am forever grateful.

The last few of our papers were being finalized as Jean was undergoing various surgeries and chemotherapies, and one only appeared posthumously (one of ours, that is; I'm sure Jean will continue coauthoring posthumously for a few more years). In our conversations over the last few years, he never once showed any signs of fear or despair at his condition, treating "mundane" things very matter-of-factly, and wanting to steer discussions back to theorems and (scientific) battles still to be waged.

Of Jean's many hand-written and scanned notes to me (as in the example below), all have been converted to publications save one, which is the one we aim to record now. To be perfectly honest, Jean thought it should be possible to do more here and wanted to return to the problem later, not publish things as they stand. But now there is no "later," so I would like to record his theorem as is. At some point, Michael Magee and I worked on this note as an appendix to our paper with Jean (which itself later became an appendix); I would like to thank Michael for his work on it, and his permission to re-use some of it here. On to the science.

ON THE SIZE OF THE TRACE SET

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(i) Returning to McMullen's question, we provide lower bounds on

$$|\operatorname{Trace}(\mathcal{S}_A \cap B_N)| \qquad (1.1)$$

that, short of establishing the 'positive density conjecture' cited earlier, at least improve on the trivial $N^{2\delta-1}$, $\delta=\delta_A$.

PROPOSITION

(1.2) For $A=2$,
$$(1.1) \gg N^{\delta-\varepsilon}$$

(1.3) For $A\geqslant 3$
$$(1.1) \gg N^{\delta+\frac{(2\delta-1)(1-\delta)}{29(5-\delta)}}$$

(1.4) For $A\geqslant 51$
$$(1.1) \gg N^{\frac{1+28\delta}{29}}$$

The proof of (1.2) is elementary, while that of (1.3), (1.4) relies on results from [B-K] and also some additive combinatorics (which we will present in a self contained way).

1.1. **McMullen's Arithmetic Chaos Conjecture.**

For $x \in \mathbb{R}$, we write its continued fraction expansion as

$$x \;=\; [a_0; a_1, \dots, a_\ell, \dots] \;= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_\ell + \ddots}}},$$

which may be either finite or infinite; here $a_0 \in \mathbb{Z}$ and for $j \geq 1$, the "partial quotients" a_j are positive integers. The bar in

$$[\overline{a_0, a_1, \dots, a_\ell}]$$

denotes periodically repeating partial quotients; it is very well-known that such numbers are quadratic surds. For a finite "alphabet" $\mathcal{A} \subset \mathbb{N}$, let $\mathfrak{C}_\mathcal{A}$ denote the Cantor-like set of numbers in the unit interval whose partial quotients lie in $\mathcal{A}$,

$$\mathfrak{C}_\mathcal{A} \;:=\; \{[0; a_1, \dots, a_\ell, \dots] \;:\; a_1, a_2, \dots \in \mathcal{A}\},$$

and let $\delta_\mathcal{A}$ be its Hausdorff dimension,

$$\delta_\mathcal{A} \;:=\; H.dim(\mathfrak{C}_\mathcal{A}) \;\in\; [0, 1).$$

Motivated by conjectures on the rigidity of higher-rank diagonal flows, McMullen [McM09, McM12] formulated the following rank-one problem.

Conjecture 1.3 (McMullen's Classical Arithmetic Chaos Conjecture). *Let $\mathcal{A}$ be any alphabet with dimension $\delta_\mathcal{A}$ exceeding $1/2$. Then for any real quadratic field K, the set*

$$\{[\overline{a_0, a_1, \dots, a_\ell}] \in K \;:\; \textit{all } a_j \in \mathcal{A}\} \tag{1.4}$$

grows exponentially as the length $\ell \to \infty$.

Exponential growth is not known for a single choice of $\mathcal{A}$ and K. Worse yet, it is unknown unconditionally whether there is an alphabet $\mathcal{A}$ such that every K has at least *one* surd with all partial quotients in $\mathcal{A}$, that is, whether the union over all ℓ of (1.4) is non-empty! On the other hand, Mercat [Mer12] has proven this last statement assuming the validity of Zaremba's Conjecture [Zar72, BK14a]. Unconditionally, Wilson [Wil80] has shown that for any K, there is some $\mathcal{A} = \mathcal{A}(K)$ so that (1.4) is non-empty infinitely often; see also [Woo78].

1.2. **Thin Semi-Groups.**

To connect this problem to "thin semigroups," let $\mathcal{G}_{\mathcal{A}} \subset \mathrm{GL}_2(\mathbb{Z})$ denote the semi-group generated by matrices of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right)$ with $a \in \mathcal{A}$,

$$\mathcal{G}_{\mathcal{A}} := \left\langle \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : a \in \mathcal{A} \right\rangle^{+},$$

where the superscript "+" indicates generation without inverses. (See [Kon16, Lecture 3] for why $\mathcal{G}_{\mathcal{A}}$ is "thin".) This matrix semi-group was introduced in [BK14a] to study Zaremba's Conjecture, but is equally germane to McMullen's problem, due to the following elementary observation: if

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_\ell \end{pmatrix} \in \mathcal{G}_{\mathcal{A}},$$

then

$$\mathbb{Q}([\overline{a_0, \dots, a_\ell}]) = K,$$

where

$$K = \mathbb{Q}(\sqrt{\mathrm{tr}^2 \gamma - 4 \det \gamma}).$$

(Recall that $\det \gamma = \pm 1$.) That is, one can read off the discriminant of the real quadratic field corresponding to adjoining $[\overline{a_0, \dots, a_\ell}]$ in terms of the trace of γ.

1.3. **The Local-Global and Positive Density Conjectures.**

The above simple observation motivates one to study the set $\mathcal{T}_{\mathcal{A}}$ of traces of $\mathcal{G}_{\mathcal{A}}$,

$$\mathcal{T}_{\mathcal{A}} := \{\mathrm{tr}\, \gamma : \gamma \in \mathcal{G}_{\mathcal{A}}\}.$$

Indeed, Bourgain and the author have formulated a certain "Local-Global Conjecture" for linear forms on $\mathcal{G}_{\mathcal{A}}$ (see [Kon16, Conjecture 6.3.1]) which implies both Zaremba's Conjecture and McMullen's Conjecture 1.3, in particular predicting which traces should arise and with what multiplicity. A weaker problem, formulated already by McMullen [McM12], is the following.

Conjecture 1.5 (McMullen's Positive Density Conjecture for Traces)**.** *Let $\mathcal{A}$ be an alphabet with $\delta_{\mathcal{A}} > 1/2$. Then the trace set $\mathcal{T}_{\mathcal{A}}$ comprises a positive proportion of integers, that is,*

$$\#\mathcal{T}_{\mathcal{A}} \cap [1, N] \gg N, \tag{1.6}$$

as $N \to \infty$.

The restriction to alphabets having $\delta_{\mathcal{A}}$ exceed $1/2$ is necessary, in light of the following result of Hensley [Hen89].

Theorem 1.7 (Hensley)**.** *As $N \to \infty$,*

$$|\mathcal{G}_{\mathcal{A}} \cap B_N| \asymp N^{2\delta_{\mathcal{A}}}. \tag{1.8}$$

Indeed, if $\delta_{\mathcal{A}} < 1/2$, then $\mathcal{T}_{\mathcal{A}}$ is automatically a thin subset of the integers.

This positive density Conjecture 1.5, despite being much weaker than a full local-global statement, is also wide open, even for any choice of (finite) alphabet $\mathcal{A}$. If instead of traces, one considers the set $\mathfrak{D}_{\mathcal{A}}$ of "bottom-right" entries,

$$\mathfrak{D}_{\mathcal{A}} \ := \ \{d \in \mathbb{N} \ : \ \exists \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \mathcal{G}_{\mathcal{A}}\},$$

then one can show not just positive density but density one for "slightly thin" alphabets (ones with $\delta_{\mathcal{A}} > 1 - \varepsilon_0$), see [BK14a]. Zaremba's conjecture is equivalent to a local-global statement for $\mathfrak{D}_{\mathcal{A}}$. The proof technique there shows the following.

Theorem 1.9 ([BK14a])**.** *Let $\mathcal{A}$ be an alphabet with $\delta = \delta_{\mathcal{A}}$ sufficiently near 1, $\delta > 1 - \varepsilon_0$. Then there exist subsets $S_N \subset \mathcal{G}_{\mathcal{A}} \cap B_N$ of nearly full cardinality,*

$$\#S_N \gg N^{2\delta}$$

such that, for every $d \ll N$, the multiplicity of the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$ is bounded by:

$$\#\{\gamma \in S_N, \gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right)\} \ll N^{2\delta - 1}. \tag{1.10}$$

This estimate will be the only "black box" used; besides this, the paper is self-contained.

1.4. Statements of the Main Theorems.

Returning to the trace set $\mathcal{T}_{\mathcal{A}}$, the "trivial" bound towards (1.6) is

$$\#\mathcal{T}_{\mathcal{A}} \cap [1, N] \gg N^{2\delta_{\mathcal{A}} - 1 - o(1)}. \tag{1.11}$$

Indeed, a simple argument shows that each trace $t < N$ occurs with multiplicity $\ll N^{1+o(1)}$, whence (1.11) follows from (1.8).

Our goal here is to give Jean Bourgain's proofs of the following two results, which improve over this.

Theorem 1.12. *When $\delta_{\mathcal{A}} > 1/2$,*

$$\#\mathcal{T}_{\mathcal{A}} \cap [1, N] \gg N^{\delta - o(1)}. \tag{1.13}$$

Theorem 1.14. *Suppose $\{1, 2, 3\} \subset \mathcal{A}$ and $\delta > 1 - \varepsilon_0$ so that (1.10) holds. Then as $N \to \infty$,*

$$\#\mathcal{T}_{\mathcal{A}} \cap [1, N] \gg N^{\delta + \frac{1-\delta}{29} - o(1)}. \tag{1.15}$$

Remark 1.16. The proof of Theorem 1.12 is elementary, and yet it already improves upon (1.11), sometimes dramatically so. Indeed, when $\mathcal{A} = \{1, 2\}$, we have $\delta_{\{1,2\}} \approx 0.531$ [Goo41]; the trivial bound (1.11) gives only

$$\#\mathcal{T}_{\{1,2\}} \cap [1, N] \gg N^{0.062},$$

while (1.13) gives

$$\#\mathcal{T}_{\{1,2\}} \cap [1, N] \gg N^{0.531}.$$

Remark 1.17. The original work [BK14a] showed (1.10) as long as $\delta > 0.984$, and this bound was relaxed in [FK14] and [Kan17] to $\delta > 0.781$; the latter holds already for $\mathcal{A} = \{1, 2, 3, 4\}$ which has dimension $\delta_{\{1,2,3,4\}} \approx 0.789$, see [Jen04]. Thus for this alphabet Theorem 1.14 improves from (1.13) that

$$\#\mathcal{T}_{\{1,2,3,4\}} \cap [1, N] \gg N^{0.789},$$

to

$$\#\mathcal{T}_{\{1,2,3,4\}} \cap [1, N] \gg N^{0.796}.$$

The proof of Theorem 1.14 applies more generally to give an improvement in the exponent whenever $\delta > 1/2$ and $\mathcal{A}$ contains a 3-term progression; but for ease of exposition, we state this simpler version.

The core of Theorem 1.14 is the following version of the Balog-Szemerédi-Gowers Lemma with polynomial dependencies of constants on one another. The original version of the Balog-Szemerédi-Gowers Lemma with polynomial dependencies of constants appeared in Gowers' work on arithmetic progressions [Gow98].

For subsets A, B of an ambient additive group and $G \subset A \times B$ an arbitrary subset, we use the notation

$$A \overset{G}{+} B := \{a + b : (a, b) \in G\}.$$

Lemma 1.18. *Let $A \subset \mathbb{Z}$ be a finite set and $G \subset A \times A$ satisfy*

$$|G| > \frac{1}{K}|A|^2$$

and

$$|A \overset{G}{+} A| \leq |A|. \tag{1.19}$$

Then there is a subset $A' \subset A$ such that

$$|(A' \times A') \cap G| \gg K^{-2}|A|^2$$

and

$$|A' - A'| \ll K^{13}|A|,$$

where the implied constants are absolute.

This version is a refinement of Bourgain's work [Bou99, Lemma 2.1] on the dimension of Kakeya sets. To make the argument almost self-contained (modulo Theorem 1.9), we give a quick proof of Lemma 1.18 in §5.

1.5. **Notation.**

Whenever we write B_N we mean the ball in the space of 2×2 matrices with respect to the ℓ^1 norm on their entries, and when we write $\|g\|$ we mean the ℓ^1 norm. We write sqf for the squarefree part of a number and ω for the number of distinct prime factors of a number. We use Vinogradov notation $\ll$,$\gg$,O,o in the standard way and indicate dependence of implied constants on other parameters by subscripts, e.g. $\ll_\epsilon$. We use $f \asymp g$ to mean $f \ll g$ and $g \ll f$. Normally we view $\mathcal{A}$ as fixed so any implied constant may depend on $\mathcal{A}$. For a subset A of an ambient additive group we write $A+A$ and $A-A$ for setwise sums and differences, e.g. $A-A=\{a_1-a_2 \,:\, a_1,a_2\in A\}$, etc.

2. Preliminary remarks

Write

$$\gamma_a := \left(\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right),$$

for a generator of the semigroup $\mathcal{G}_{\mathcal{A}}$. A ping-pong argument using the action of γ_a on $[0,1]$ by Möbius transformations shows that $\mathcal{G}_{\mathcal{A}}$ is freely generated by the γ_a, for $a\in\mathcal{A}$. Let

$$\Gamma_{\mathcal{A}} := \mathcal{G}_{\mathcal{A}} \cap \mathrm{SL}_2$$

be the sub-semigroup of orientation-preserving elements; equivalently these are even words in the generators (each of the latter has determinant $\det \gamma_a = -1$).

The key (trivial) observation used throughout is the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix} = \begin{pmatrix} b & a+\alpha b \\ d & c+\alpha d \end{pmatrix}, \tag{2.1}$$

whence $b+c+\alpha d$ is a trace in $\mathcal{G}_{\mathcal{A}}$. Taking $\alpha = 1,2,3$, (if we assume that $\{1,2,3\}\subset\mathcal{A}$), we see that all three of

$$b+c+d,\ b+c+2d,\ b+c+3d \in \mathcal{T}_{\mathcal{A}}, \tag{2.2}$$

whenever $\left(\begin{smallmatrix} * & b \\ c & d \end{smallmatrix}\right) \in \mathcal{G}_{\mathcal{A}}$.

3. Proof of Theorem 1.12

For simplicity, assume that $\{1,2\} \subset \mathcal{A}$; in general, we know by $\delta > 1/2$ that $|\mathcal{A}| \geq 2$, and trivial modifications are needed in what follows. In light of (2.2), we would like to know the multiplicity of the map

$$\varphi : \Gamma_{\mathcal{A}} \to \mathbb{N}^2 : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (b+c, d).$$

Lemma 3.1. *Let $(n,m) \in \mathbb{N}^2$ with $n, m \ll N$. Then the preimage of (n,m) has cardinality at most*

$$|\varphi^{-1}(n,m)| \leq \gcd(n^2+4, m)^{1/2} N^{o(1)}.$$

Proof. Suppose that $(b+c, d) = (n,m)$. Clearly $d = m$ is determined. Since $ad - bc = 1$ and $c = n - b$, we have

$$1 + b(n-b) \equiv 0 (\operatorname{mod} d).$$

The discriminant of this quadratic in b is $\Delta := n^2 + 4$, and it is elementary that the number of solutions to $b(\operatorname{mod} d)$ is $\ll_\varepsilon (\Delta, d)^{1/2} N^\varepsilon$. Since $b \leq d$, it is determined once it is known mod d; then so is $c = n - b$, and then $a = (1 + bc)/d$. □

The issue becomes to discard $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathcal{A}}$ having large $\gcd(\Delta, d)$.

Lemma 3.2. *For any $\epsilon > 0$ there is a subset $B'_N \subset \Gamma_{\mathcal{A}} \cap B_N$ satisfying*

$$|B'_N| > \frac{1}{2} |\Gamma_{\mathcal{A}} \cap B(N)| \tag{3.3}$$

and if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B'_N$ then $\gcd((b+c)^2 + 4, d) \ll_\epsilon N^\epsilon$.

Lemma 3.2 follows immediately from the following Lemma that we will also use later.

Lemma 3.4. *Suppose that $\delta > \frac{1}{2}$. For all $\epsilon > 0$, there is $\eta = \eta(\epsilon) > 0$ such that*

$$\left| \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_{\mathcal{A}} \cap B_N \ : \ \gcd((b+c)^2 + 4, d) > N^\epsilon \right\} \right| \ll_\epsilon N^{2\delta - \eta}.$$

In particular, in comparison to Theorem 1.7, these elements form a negligible subset.

Proof. Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathcal{A}} \cap B_N$, suppose that there is 'large' $q > N^\epsilon$ dividing both $(b+c)^2 + 4$ and d. Then $bc + 1 \equiv 0 \bmod d$ implies

$$(b-c)^2 \equiv b^2 + c^2 + 2 \equiv (b+c)^2 + 4 \equiv 0 \bmod q.$$

Therefore $b \equiv c \bmod q_1$ for some $q_1 | q$ with $q_1 > N^{\epsilon/2}$. Then

$$d \equiv b^2 + 1 \equiv c^2 + 1 \equiv 0 \bmod q_1. \tag{3.5}$$

We write each $g \in \Gamma_{\mathcal{A}}$ with $\|g\| \asymp N$ in the form
$$g = g_1 g_2$$
with
$$\|g_2\| \asymp N' := N^{\frac{\epsilon}{10}}.$$
(Note that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{G}_{\mathcal{A}}$, the entries a, b, c, d are all commensurate, and that wordlength in the generators is log-commensurate to the archimedean norm.) Accordingly, we write
$$g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \zeta \end{pmatrix}, \quad g_2 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
so that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha x + \beta z & \alpha y + \beta w \\ \gamma x + \zeta z & \gamma y + \zeta w \end{pmatrix}. \tag{3.6}$$
For each choice of g_1 and y we will show there are few possibilities for w if (3.5) is to hold for some $q_1 > N^{\frac{\epsilon}{2}}$. On the other hand, y and w determine g_2 and hence g_1, y, w determine g. Combining (3.5) and (3.6) we get
$$\gamma y + \zeta w \equiv 0 \bmod q_1 \tag{3.7}$$
and
$$(\gamma x + \zeta z)^2 \equiv -1 \bmod q_1.$$
Thus using $\det g_2 = 1$ gives
$$-y^2 \equiv y^2(\gamma x + \zeta z)^2 \equiv (\gamma y x + \zeta y z)^2 \equiv (\gamma y x + \zeta(xw-1))^2 \equiv \zeta^2 \bmod q_1$$
where the last equality uses (3.7). In other words
$$y^2 + \zeta^2 \equiv 0 \bmod q_1.$$
For fixed g_1, y the number of w so that (3.5) holds for some q_1 is bounded by
$$\sum_{q_1 | y^2+\zeta^2} |\{w \,:\, \gamma y + \zeta w = 0 \bmod q_1\}|. \tag{3.8}$$
For each q_1 in the sum let $q_2 = \gcd(q_1, \zeta^2)$. We have $y^2 \equiv 0 \bmod q_2$ and since $0 < y < N'$ this implies $q_2 < N'$. This means $\gcd(q_1, \zeta) < N'$ and then $\gamma y + \zeta w = 0 \bmod q_1$ specifies $w \bmod q_3 := q_1 / \gcd(q_1, \zeta)$, with $q_3 > N^{\frac{\epsilon}{2}} N^{-\frac{\epsilon}{10}}$. But since $0 < w \ll N^{\frac{\epsilon}{10}}$ this specifies w.

Then each term in (3.8) is bounded by 1 and we can bound (3.8) by the number of divisors of $y^2 + \zeta^2 \leq N^2$, which is $N^{o(1)}$. It remains to sum over g_1 and y, and this gives that the number of $g \in \Gamma_{\mathcal{A}} \cap B_N$ such that (3.5) holds for some $q_1 > N^{\frac{\epsilon}{2}}$ is
$$\begin{aligned} &\leq \quad |\{g_1 : \|g_1\| \ll N^{1-\frac{\epsilon}{10}}\}| \cdot |\{y \ll N^{\frac{\epsilon}{10}}\}| \cdot N^{o(1)} \\ &\ll \quad N^{2\delta(1-\frac{\epsilon}{10})} N^{\frac{\epsilon}{10}} N^{o(1)} \ll N^{2\delta - \eta} \end{aligned}$$

for some $\eta = \eta(\epsilon) > 0$. $\square$

Proof of Theorem 1.12. For small $\epsilon > 0$, let B'_N be the family of subsets from Lemma 3.2. Combining Lemmas 3.1 and 3.2, the map

$$B'_N \to \mathcal{T}_{\mathcal{A}}(3N) \times \mathcal{T}_{\mathcal{A}}(4N) : \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (b+c+d, b+c+2d)$$

has multiplicity at most $N^{o(1)}$ and so

$$|\mathcal{T}_{\mathcal{A}}(N)|^2 > \frac{1}{2}|\Gamma_{\mathcal{A}} \cap B_N| N^{-o(1)} \gg N^{2\delta - o(1)}.$$

Taking square roots completes the proof. $\square$

4. Proof of Theorem 1.14

Assume now that $\{1,2,3\} \subset \mathcal{A}$, so we can exploit the full force of (2.2). By Theorem 1.9 there is a family of subsets

$$S(N) \subset \Gamma_{\mathcal{A}} \cap B_N$$

with $|S(N)| \gg N^{2\delta}$ and such that the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$ has multiplicity $M \ll N^{2\delta-1}$. Then using Lemma 3.4 and echoing the previous argument, we can find a subset $S'(N) \subset S(N)$ such that the map

$$\psi : S'(N) \to \mathcal{T}_{\mathcal{A}}(3N) \times \mathcal{T}_{\mathcal{A}}(5N), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (b+c+d, b+c+3d)$$

has multiplicity $< N^{o(1)}$. Let

$$T_0 = \{b+c+jd \ : \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S'(N),\ 1 \le j \le 3\} \subset \mathcal{T}_{\mathcal{A}}(5N).$$

We apply Lemma 1.18 with $A = T_0$ and

$$G = \{(b+c+d, b+c+3d) \ : \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S'(N)\} = \psi(S'(N)).$$

By our previous bound on the multiplicity of ψ,

$$|G| > N^{2\delta - o(1)}.$$

Also,

$$T_0 \overset{G}{+} T_0 = \{2(b+c+d) \ : \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S'(N)\}$$

so

$$|T_0 \overset{G}{+} T_0| \le |T_0|.$$

We can thus apply Lemma 1.18 with

$$K = |T_0|^2 N^{-2\delta + o(1)}$$

Let A' be the subset obtained from Lemma 1.18.

The key point is that for each element of $(t_1, t_2) \in (A' \times A') \cap G$ one has $t_2 - t_1 \in 2\mathcal{D}_{\mathcal{A}}$. Moreover if $(t_1, t_2) = \psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $t_2 - t_1 = 2d$.

Since the multiplicity of the denominator mapping on $S'(N)$ is at most M, the multiplicity of $(t_1, t_2) \mapsto t_2 - t_1$ is at most M on $(A' \times A') \cap G$. Therefore

$$K^{13}|A| \gg |A' - A'| \geq |(A' \times A') \cap G| M^{-1} \gg K^{-2}|A|^2 M^{-1}$$

where the outer two inequalities are the output of Lemma 1.18. This gives

$$|A| \ll K^{15} M$$

or recalling the value of K,

$$|T_0| < |T_0|^{30} N^{-30\delta + o(1)} M.$$

Substituting the value of $M = N^{2\delta - 1}$ gives the result as claimed.

5. Proof of Lemma 1.18

The following argument is a modification of [Bou10, Section 2]. By Cauchy-Schwarz and (1.19)

$$\begin{aligned} |G| &= \sum_{z \in A \overset{G}{+} A} |\{(x, y) \in G \,:\, x + y = z\}| \\ &\leq |A|^{\frac{1}{2}} |\{(x_1, y_1; x_2, y_2) \in G \times G \,:\, x_1 + y_1 = x_2 + y_2\}|^{\frac{1}{2}}, \end{aligned}$$

implying that

$$|\{(x_1, y_1; x_2, y_2) \in G \times G \,:\, x_1 + y_1 = x_2 + y_2\}| > \frac{1}{K^2}|A|^3. \tag{5.1}$$

Denote $w(x) = |\{(x_1, x_2) \in A \,:\, x_1 - x_2 = x\}|$ and set

$$D \;=\; \{x \,:\, w(x) > \frac{1}{10K^2}|A|\}, \quad R = \{(x, x') \in A^2 : x_1 - x_2 \in D\},$$

and also write

$$R_{x_1} = \{x_2 \in A \,:\, x_1 - x_2 \in D\}.$$

The set D is the 'popular differences'. Then

$$\begin{aligned} |\{(x_1, y_1; x_2, y_2) &\in A^4 \,:\, x_1 + y_1 = x_2 + y_2, \text{ either } x_1 - x_2 \notin D \text{ or } y_1 - x_2 \notin D\}| \\ &\leq 2|A|^2 \frac{1}{10K^2}|A| = \frac{1}{5K^2}|A|^3, \end{aligned}$$

since for example $x_1 - x_2 = y_1 - y_2$ so each of at most $|A|^2$ pairs (x_1, x_2) with $x_1 - x_2 \notin D$ contribute at most $\frac{1}{10K^2}|A|$ possibilities for (y_1, y_2).

The other contributions are estimated similarly. This estimate together with (5.1) gives

$$\begin{aligned}\frac{1}{2K^2}|A|^3 &< |\{(x_1, y_1; x_2, y_2) \in G \times G \,:\, x_1 + y_1 = x_2 + y_2, x_1 - x_2 \in D,\, y_1 - x_2 \in D\}| \\ &\leq \sum_{(x_1,y_1)\in G} |R_{x_1} \cap R_{y_1}| \\ &= \sum_y |(R_y \times R_y) \cap G|. \end{aligned} \tag{5.2}$$

Let

$$Y = \{(x, x') \in A^2 \,:\, |R_x \cap R_{x'}| < \theta|A|\}$$

where θ is a parameter to be specified. Obviously from the definition of Y, we have

$$\sum_y |(R_y \times R_y) \cap Y| = \sum_{(x,x')\in Y} |R_x \cap R_{x'}| < \theta|A|^3.$$

Therefore from (5.2), we see that

$$\sum_y |(R_y \times R_y) \cap G| > \frac{1}{4K^2}|A|^3 + \frac{1}{4K^2\theta} \sum_y |(R_y \times R_y) \cap Y|.$$

Thus there is $y_0 \in A$ such that

$$|(R_{y_0} \times R_{y_0}) \cap G| > \frac{1}{4K^2}|A|^2 + \frac{1}{4K^2\theta}|(R_{y_0} \times R_{y_0}) \cap Y|. \tag{5.3}$$

In particular

$$|R_{y_0}| > \frac{1}{2K}|A|. \tag{5.4}$$

Let

$$A' = \{x \in R_{y_0} \,:\, |(\{x\} \times R_{y_0}) \cap Y| < \frac{1}{3}|R_{y_0}|\}. \tag{5.5}$$

Then clearly

$$\frac{1}{3}|R_{y_0}||R_{y_0} \backslash A'| < |(R_{y_0} \times R_{y_0}) \cap Y|$$

and by (5.3), (5), we have

$$\begin{aligned}|(A' \times A') \cap G| &\geq |(R_{y_0} \times R_{y_0}) \cap G| - 2|R_{y_0} \backslash A'|.|R_{y_0}| \\ &\geq \frac{1}{4K^2}|A|^2 + \left(\frac{1}{4K^2\theta} - 6\right) |(R_{y_0} \times R_{y_0}) \cap Y| \\ &\geq \frac{1}{4K^2}|A|^2 \end{aligned} \tag{5.6}$$

if we take

$$\theta = \frac{1}{24K^2}.$$

Take now $(x_1, x_2) \in (A' \times A') \cap G$. By (5.5) there are at least $\frac{1}{3}|R_{y_0}|$ values of $x \in R_{y_0}$ such that $(x_1, x) \notin Y$ and $(x_2, x) \notin Y$. For each of these x we have by the definition of Y,

$$|R_{x_1} \cap R_x| \geq \frac{1}{24K^2}|A|, \quad |R_{x_2} \cap R_x| \geq \frac{1}{24K^2}|A|$$

and then

$$\begin{aligned} x_1 - x_2 &= (x_1 - x) - (x_2 - x) \\ &= (x_1 - y_1) - (x - y_1) - (x_2 - y_2) + (x - y_2) \end{aligned} \tag{5.7}$$

for at least $\frac{|A|^2}{576K^4}$ pairs (y_1, y_2) (depending on x, x_1, x_2) with

$$(x_1, y_1), (x, y_1), (x_2, y_2), (x, y_2) \in R.$$

By definition of R and D each of the parenthetical terms in (5.7) admits a representation in at least $\frac{|A|}{10K^2}$ ways as a difference of elements of A and therefore the number of representations

$$x_1 - x_2 = (\tau_1 - \tau_2) - (\tau_3 - \tau_4) - (\tau_5 - \tau_6) + (\tau_7 - \tau_8), \quad \tau_i \in A \tag{5.8}$$

is at least

$$\frac{1}{3}|R_{y_0}|.\frac{|A|^2}{576K^4}.\left(\frac{|A|}{10K^2}\right)^4 \overset{(5.4)}{\gg} \frac{1}{K^{13}}|A|^7.$$

Considering the map on $(\tau_i)_{i=1}^8$ given by (5.8), we get

$$|A' - A'| \ll \frac{|A|^8}{K^{-13}|A|^7} = K^{13}|A|.$$

This together with the previously established (5.6) proves Lemma 1.18.

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