CORRIGENDUM TO "ALMOST PRIME COORDINATES FOR ANISOTROPIC AND THIN PYTHAGOREAN ORBITS"

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Let $\mathcal{Q}$ be a ternary indefinite integral quadratic form, and let

$$G := \mathrm{SO}_{\mathcal{Q}}(\mathbb{R}) \cong \mathrm{SO}_{2,1}(\mathbb{R})$$

be its group of real automorphisms. Let $\Gamma < G(\mathbb{Z})$ be a finitely-generated subgroup of the integer points of G, and assume that the critical exponent δ of Γ exceeds $1/2$. For a fixed primitive vector $\mathbf{y} \in \mathbb{Z}^3$, we consider the orbit

$$\mathcal{O} \ := \ \mathbf{y} \cdot \Gamma \ \subset \ \mathbb{Z}^3.$$

Let f be a polynomial which is integral on $\mathcal{O}$, and assume that it is "strongly primitive," that is, for all $q \geq 1$, there is an $\mathbf{x} \in \mathcal{O}$ so that $f(\mathbf{x}) \in (\mathbb{Z}/q\mathbb{Z})^{\times}$. For an integer $R \geq 1$, let $\mathcal{O}(f, R)$ denote the set of points $\mathbf{x} \in \mathcal{O}$ for which $f(\mathbf{x})$ has at most R prime divisors. The "saturation number" $R_0(\mathcal{O}, f)$ is the least R (and ∞ if none exists) for which $\mathrm{Zcl}(\mathcal{O}(f, R)) = \mathrm{Zcl}(\mathcal{O})$, where Zcl refers to Zariski closure in affine space $\mathbb{A}^3_{\mathbb{Q}}$. The main Theorems 1.10 and 1.19 of [HK15] claim to improve the values of $R_0(\mathcal{O}, f)$ for the settings of "Thin Pythagorean" and "Anisotropic" Orbits, resp; unfortunately, while Theorem 1.10 is correct, the proof of Theorem 1.19 suffers an elementary but fatal flaw and must be retracted.

The main idea, as explained in [HK15, §3.2], is that, when f is homogeneous (as is the case in most natural applications, including these), one may use a "larger" stabilizer group to count more efficiently. In particular, one defines

$$\Gamma_{\langle y \rangle}(q) \ := \ \{\gamma \in \Gamma \ : \ \mathbf{y} \cdot \gamma \in \langle y \rangle \,(\mathrm{mod}\, q)\},$$

where $\langle y \rangle$ denotes the linear span of $\mathbf{y}$; equivalently, $\Gamma_{\langle y \rangle}(q)$ contains those $\gamma \in \Gamma$ for which there exists $a \in (\mathbb{Z}/q\mathbb{Z})^{\times}$ for which

$$\mathbf{y} \cdot \gamma \equiv a\mathbf{y} (\mathrm{mod}\, q). \tag{1}$$

This group replaces the group $\Gamma_y(q) := \{\gamma \in \Gamma : \mathbf{y} \cdot \gamma \equiv y (\mathrm{mod}\, q)\}$ used previously in applications. This is all correct, up to the top of page

418 of [HK15], where it is claimed that the index

$$[\Gamma : \Gamma_{\langle y \rangle}(q)] \tag{2}$$

is of size $q^{1+o(1)}$. This index indeed has this size when $\mathbf{y}$ lies on the cone $\mathcal{Q} = 0$. But as pointed out to us by Wenjia Zhao (to whom we are grateful), when $\mathbf{y}$ lies on a quadric $\mathcal{Q} = t$ with $t \neq 0$, the condition (1) implies that $\mathcal{Q}(\mathbf{y} \cdot \gamma) \equiv a^2 t (\bmod q)$, and hence $a^2 \equiv 1 (\bmod q)$ (at least for q coprime to t). Therefore the index (2) in this case is $q^{2+o(1)}$, and there is no extra gain from using $\Gamma_{\langle y \rangle}(q)$ for the stabilizer in place of $\Gamma_{\mathbf{y}}(q)$. Hence the proof of [HK15, Theorem 1.19] is flawed; the record bound towards the saturation number in this setting remains [HK15, Theorem 1.15] (which is due to Liu-Sarnak [LS10]).

References

[HK15] Jiuzu Hong and Alex Kontorovich. Almost prime coordinates for anisotropic and thin pythagorean orbits. *Israel J. Math.*, 209(1):397–420, 2015.

[LS10] Jianya Liu and Peter Sarnak. Integral points on quadrics in three variables whose coordinates have few prime factors. *Israel J. Math*, 178:393–426, 2010.

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