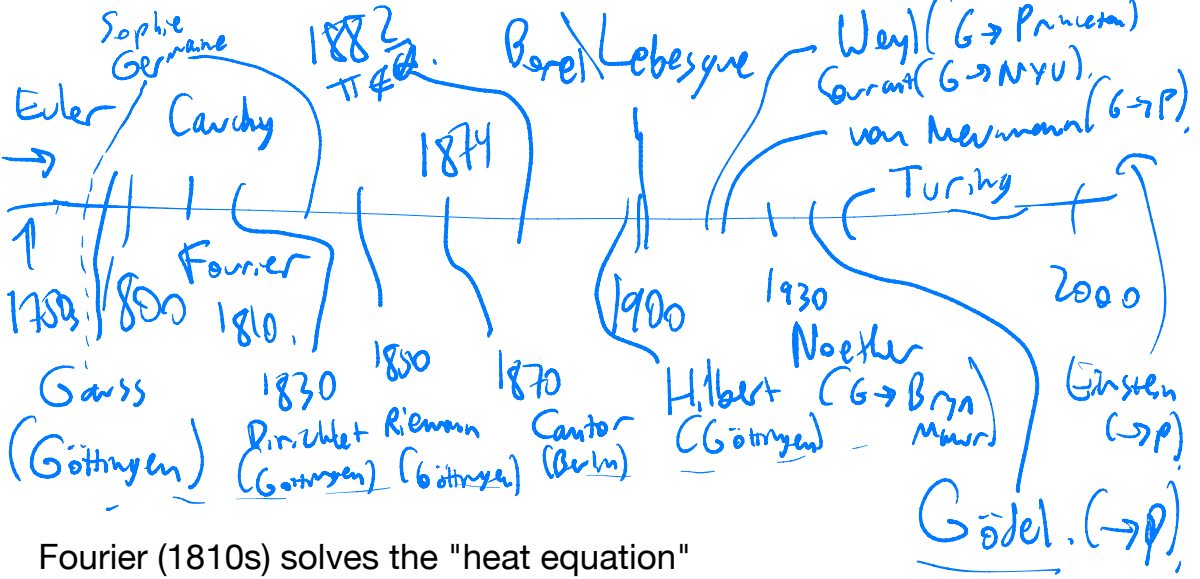
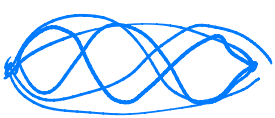


Fourier (1810s) solves the "heat equation"
Take ice pop out of freezer, heat from hand propagates through ice pop.
How? Pythagorean music theory talks about harmonics

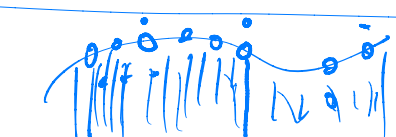


Fourier can solve the heat equation for these "waves" and solves the general equation as a superposition of harmonic waves

Needs to add infinitely many such waves up to get full solution.

(1750s d'Alembert, wave equation
D. Bernoulli, Euler only allow addition of finitely many waves)

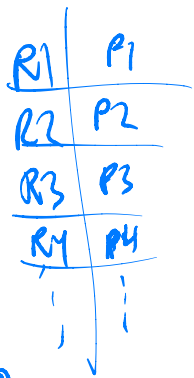
Riemann



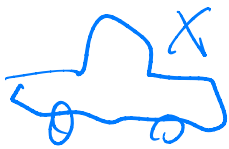
Cantor (Weierstass). can change at 2 points. can change at k points -> change at k+1 points (induction)

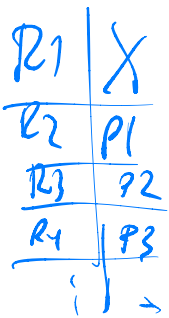
Cantor is forced to study the very nature of infinity.

Hilbert Hotel:



no vacancy





Hilbert: everybody move down a room!

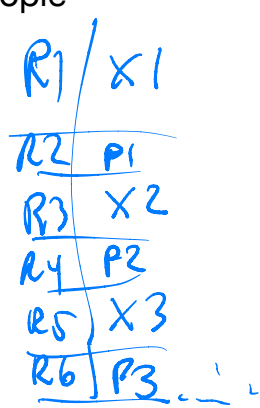
$1+\infty = \infty$. ($\neq \infty+1$).

Day 2: 10 people arrive, clerk: move down 10 rooms, no problem

$\infty + \infty = \infty$.

Day 3: bus arrives, X1, X2, X3, ... infinitely many people

Hilbert says: please double your room number!



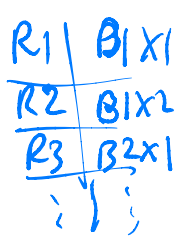
Day 4: 9 buses arrive.

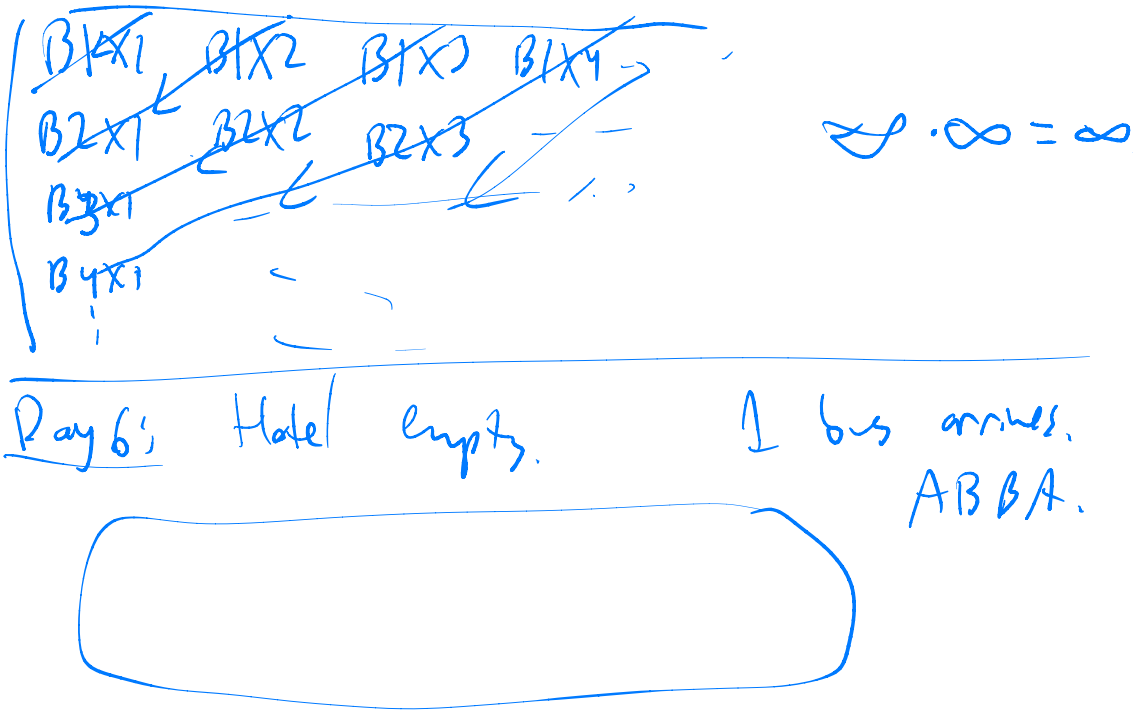
Clerk: P1 -> R10, P2 -> R20,...
B1X1 -> R1, B1X2 -> R11
B2X1 -> R2, B2X13 -> 132

$10 \cdot \infty = \infty$.

Day 5: Hotel empty.

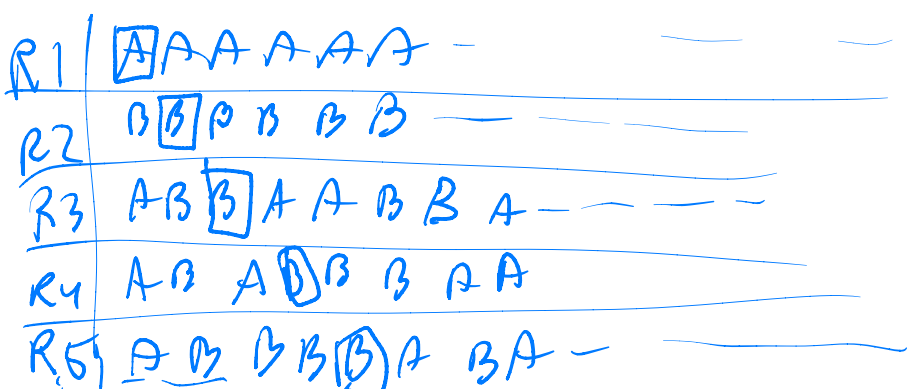
infinitely many buses, infinitely many people arrive!





Every person on this bus has an infinitely long name, using only letters A and B. One person: AAAAAAAAA... Another: BBBBBBB.... Another: ABBAABBAABBAABBA.... Another: ABBABABABABABABABBABBABBAB... Moreover, EVERY such infinite string of A's and B's is the name of some unique person on this bus.

Clerk: Hilbert, please help.
Hilbert: can't do it. Imagine that we successfully housed everyone.
Ledger:



You left off the list: BAAAA ---

Not in R1, Not R2, Not R3, ---

"Cantor Diagonalization".

How large is the set of all names of length 3 letters in A's B's?
AAA, AAB, ABA, ABB, BAA, BAB, BBA,BBB = 8 people.
2^3.
If 10 letters: number of names: 2^10

$$\#\mathbb{R}s = 2^{\infty} > \infty$$

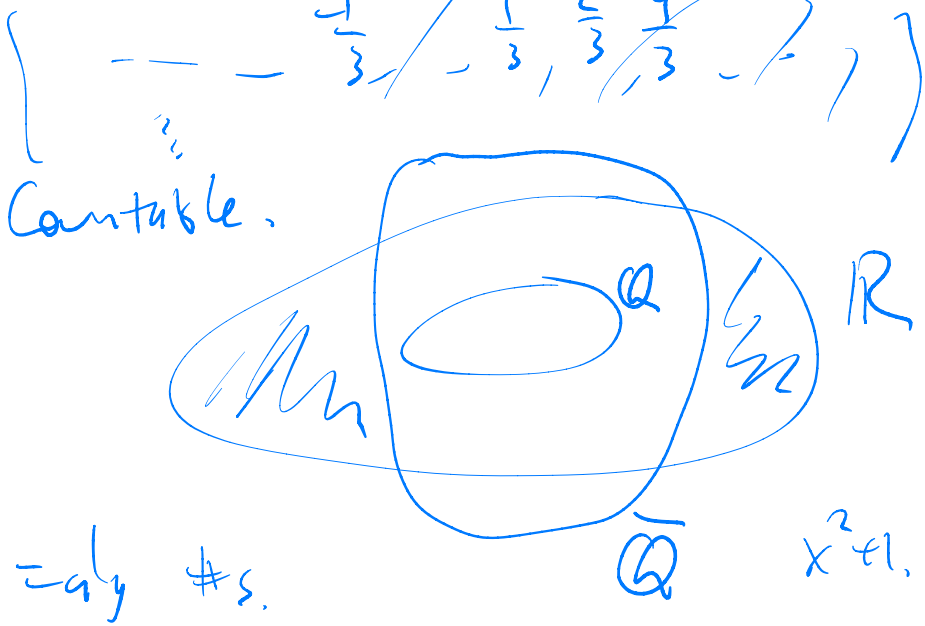
"Uncountable infinity" "Countable infinity".

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

↳ $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$

Countable.

$$\mathbb{Q} = \{0, \ldots, \tfrac{-3}{1}, \tfrac{-2}{1}, \tfrac{-1}{1}, \tfrac{1}{1}, \tfrac{2}{1}, \tfrac{3}{1}, \ldots$$
$$\ldots, \tfrac{-3}{2}, \tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \ldots\}$$



$\bar{\mathbb{Q}}$ = only #s.

= solutions to polynomials with integer coefficients

$4x^7 - 3x^5 + 12x - 3$ degree 7.

Order polynomials by degree. Then by previous methods.
Infinitude of algebraic numbers is countable!!!

What about R? $\mathbb{R} \supset [0,1] \supset$ {0.111..., 0.22222..., 0.12211221..., 0.12112...} Uncountable!!!

$\Rightarrow$

Transcendental numbers not only exist, but are the vast majority of all real numbers!!!

Cantor's work 1874 (re?)-invigorates research into transcendence, and shortly thereafter, Hermite proves the e is transcendental, and 1882 Lindemann proves that pi is transcendental, resolving the 2000 year

old problem of whether you can square the circle.

Many famous people HATED this, (Kronnecker) eventually Cantor was supported by Hilbert, and is now revered

Think back to Hippasus, Gauss withholding research on non-euclidean geometry fearing ridicule

(Aside: crucial ingredient to go from e transc $\Rightarrow$ π B.S.) $e^{i\pi}+1=0$.

Hilbert's dream: Theorem proving can become mechanical, imagine a machine crank, starts with axioms, makes all possible logical inferences, and hence proves all true theorems.
Imagines: 1) Axioms are consistent (no falsehoods like Axiom of Commensurability)
2) Axioms are complete: ANY true theorem has a proof within this axiomatic framework

Gödel. what really is the proof of a theorem?
statement : exists N so that for all ....

Every single theorem that has ever been proved is written down, using finitely many characters -> ASCII -> binary 0's and 1's -> "just" a number
Proof: Also a number!!
To say that a statement is not provable: "There does not exist a number N which proves statement X."
The thing in ""'s is itself a statement. With a variable. That statement itself has a corresponding number Y. Apply the statement to the number Y.

Totally unrelated. Town with one barber.
Some people cut their own hair. Some go to the barber.
Anyone who does not cut their own hair goes to the barber, and vice versa
Who cuts the barber's hair?

"This statement is FALSE"

Godel creates : "This statement has NO PROOF"

So: If this statement can indeed be proved from the axioms of the theory, then the statement is FALSE and the theory is inconsistent.
If this statement is TRUE, then there are true statements that have NO PROOF. And your theory is incomplete. Destroys Hilbert's dream.
We DO NOT have a RIGHT to knowledge!!!

We started out the course with a dream that through the axiomatic method as our epistemology (mechanism for finding truth), we would reach enlightenment. And though we succeeded to remove quite a bit of our ignorance, we realize that such full "enlightenment" is unattainable!!!