Last time: Finished Euclid's elements,

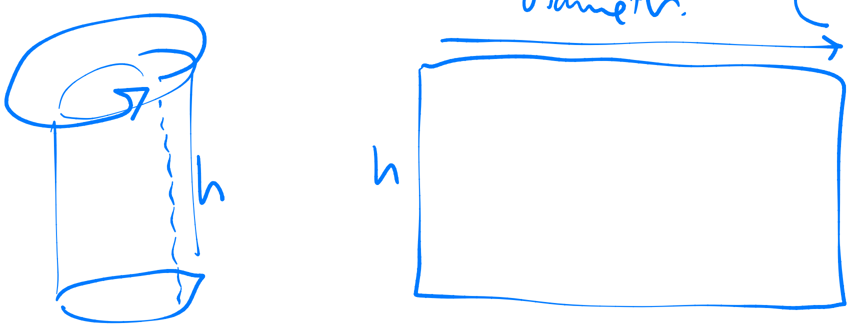
XII.2: Area (disk) = C r^2

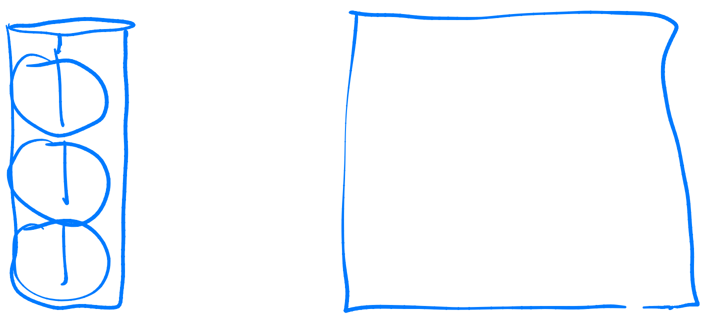
XII.18 : Volume (ball) = C' r^3

Euclid does not know these constants, if he did, he would SURELY tell us what they are.

$$\pi = 3.14\ldots = \frac{\text{circumf}}{\text{diameter}}$$

$C > h.$





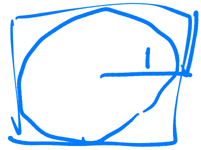
π is BIG!

Thm: $\pi > 3.$

diam = 2.

circumf > 6.

Thm: $\pi < 4$.



diam = 2

circum < 8.

Bible: Kings 7:23. Solomon's Temple.

Molten Sea

diam = 10 cubits

} 5 cubits

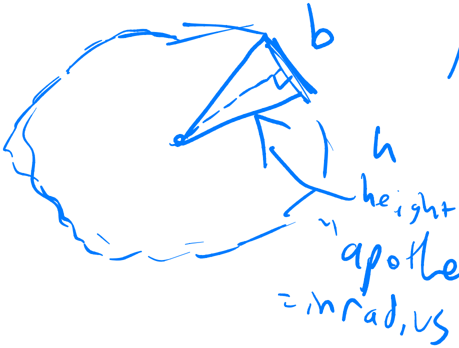
circumf = 30 cubits

2nd century commentary: this line holds a secret! Thickness of walls!

Archimedes changes the game, giving rigorous estimates for pi, by changes its meaning from 1D objects to 2D!!!

Thm: $\text{Area(disk)} = \cancel{\pi r^2} = \underbrace{\tfrac{1}{2}\,\text{circumf}}_{\pi r} \cdot \underbrace{\text{rad}}_{r}.$

Pf: △-gon: Compare circle to inscribed & circumscribed



Area of n-gon

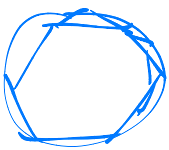
$= n \cdot (\text{Area triangles})$,

$= \underline{n} \cdot \frac{1}{2} \cdot \underline{b} \cdot h$,

$= \frac{1}{2}$ circumf $\cdot$ inradius.

$n \to \infty$.

Thm: Archimedes was able to give rigorous estimates for pi by



replacing arguments involving lengths of curves with ones involving areas where there's nothing subtle

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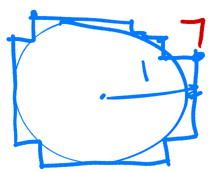
"Thm": $\underline{\pi = 4}$.

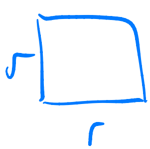


circumf = 8

diam = 2.

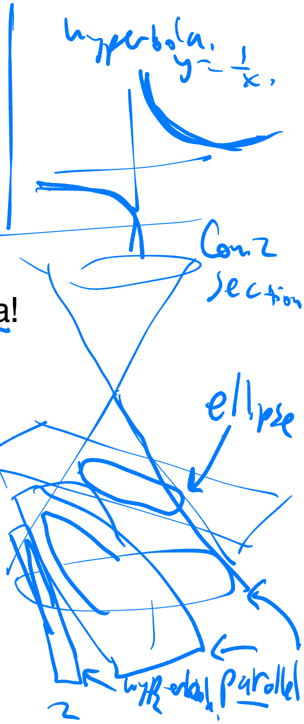
By combining this method of exhaustion (Eudoxus) and leverage, Archimedes is able to explode our understanding of curvilinear shapes

Thm: SA(sphere) $= \underline{4\pi r^2}$



= 4(Area(great disk)).

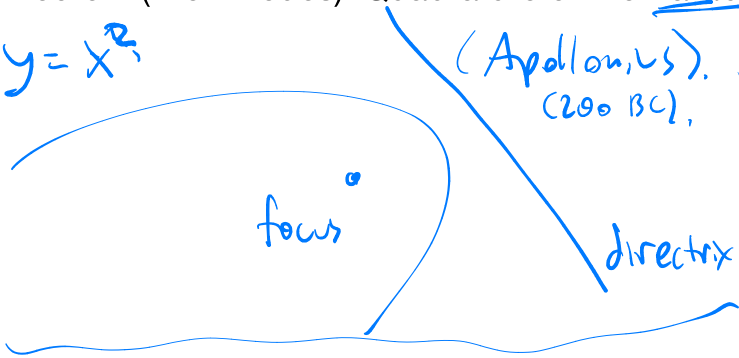
Thm: $\text{Vol}(\text{ball "sphere"}) = \frac{4}{3}\pi r^3$



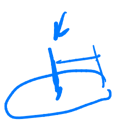
Theorem (Archimedes): Quadrature of the Parabola!

$y = x^2$,

(Apollonius). (200 BC).

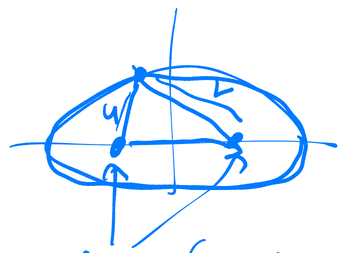


Circle: Center (stake) $x^2 + y^2 = r$.



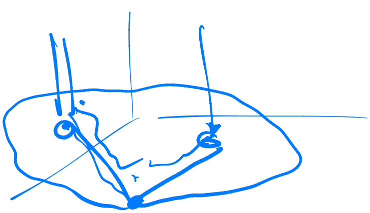
radius (rope tied to center, kept taut),

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$u + v = \text{length(rope)} = \text{fixed}$

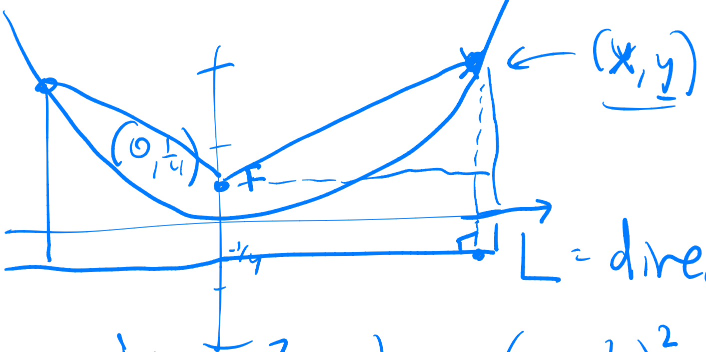
focus (foci).



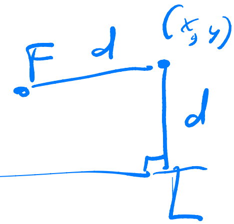
(as foci → center, ellipse → circle).

What is a parabola? It is the locus of all points equidistant to the focus and directrix

Eg: F: $(0, \frac{1}{4})$, L: $y = -\frac{1}{4}$.



L = directrix.



distance^2 to L: $(y+\frac{1}{4})^2$.

distance^2 to F: $(x-0)^2+(y-\frac{1}{4})^2$

$$(x-0)^2+(y-\frac{1}{4})^2 = (y+\frac{1}{4})^2.$$

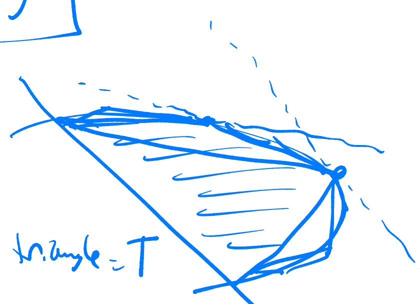
$$x^2+\cancel{y^2}-2\cdot y\cdot\frac{1}{4}+\cancel{\frac{1}{4^2}} = \cancel{y^2}+2\cdot y\cdot\frac{1}{4}+\cancel{\frac{1}{4^2}}.$$

$+\frac{1}{2}y$ $\qquad$ $+\frac{1}{2}y$.

$$\boxed{x^2 = y}$$

Thm (Archimedes):

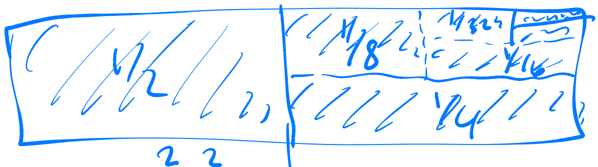
Area (parabolic sector)



$$= \frac{4}{3}\,\text{Area}(T).$$

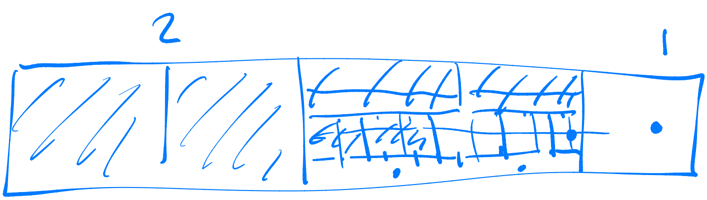
Zeno's Paradox:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$



$$\overset{\frac{2}{5}\cdot\frac{2}{5}}{}$$

$$\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \cdots + \left(\frac{2}{5}\right)^n + \cdots = \frac{2}{3}.$$



Skip to mid-1600s: Newton (1665?).
(Madhava 1400s...).

quarantines at home. "discovers" infinite series, infinite polynomials, discovers a way of computing pi that's infinitely more efficient than computing areas of inscribed and circumscribed n-gons for HUGE n.

Starts with Binomial Theorem.

Pascal's Triangle.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

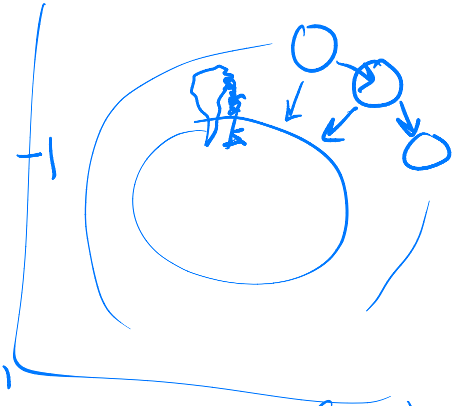
$1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1$

$11^1 = 11$

$11^2 = 121$

$11^3 = 1331$

$11^4 = 14641.$



$11 = (1+10)$

$$(1+x)^5 = (1+x)(1+x)(1+x)(1+x)(1+x).$$

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + 1 \cdot x^5,$$

$$(1+x)^n = 1 + n \cdot x + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

What about $n = -1$?

$$(1+x)^{-1} \overset{?}{=} 1 + (-1)x + \frac{(-1)(-2)}{1 \cdot 2} x^2 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

$$\frac{1}{1+x} \overset{?}{=} 1 - x + x^2 - x^3 + x^4 - \dots$$

Does multiplying RHS by $1+x$ give 1?

$$(1 - x + x^2 - x^3 + \cdots)(1 + x) = 1 + \cancel{x} - \cancel{x} + (\cancel{-x^2}) + \cancel{x^2}$$

$$- \cancel{x^3} + \cancel{x^3} + \cdots$$

$$= 1$$