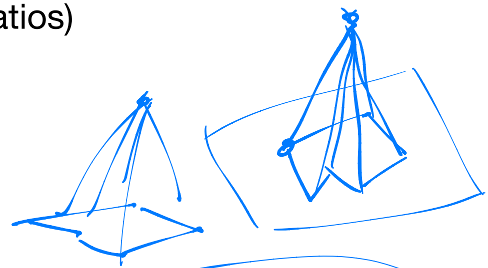
History of Math, Princeton University, Fall 2024, Prof. Kontorovich

Last time: Finished Euclid's number theory (VII-IX), Perfect numbers given "Mersenne" primes

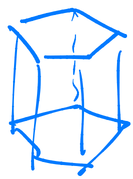
Book X: incommensurability (irraitional ratios)

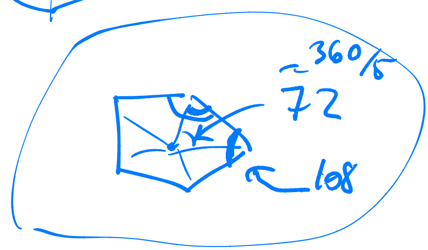
Books XI - XIII : solid geometry

Def 12: pyramid



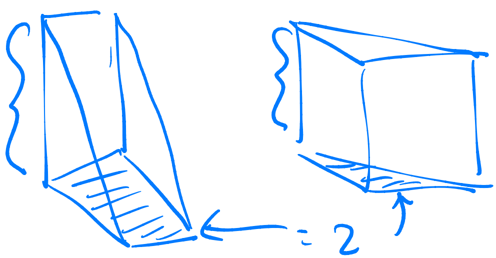
Def 13: prism





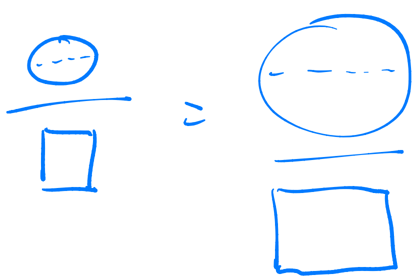
XI.21 : a solid angle (total plane angles around a corner "vertex") is less than four right angles.

By Boss XI. 39:



Prisms have same volume.

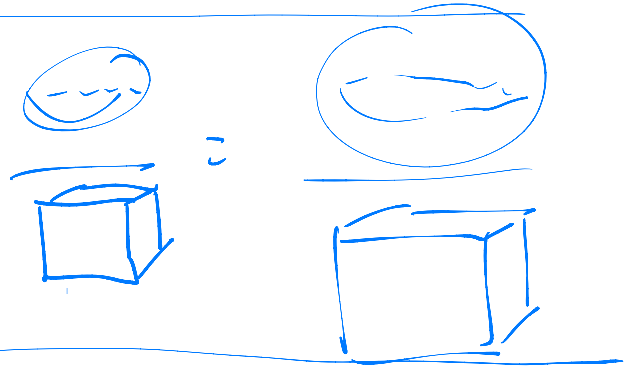
Book XII, 2:



(Hippocrates' Quadrature Lune).

$A = C \cdot d^2$.

By Boss XII, 18:



$V = C' \cdot d^3$.

Book XIII, 17 Dodecahedron.

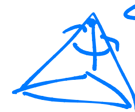
"By Boss" Remark: Has constructed ALL "platonic" solids.

Recall Def: a polygon (in 2D) is "regular" if all sides are the same, and all angles are the same. (Started Book I.1 with regular 3-gon, i.e. equilateral triangle)

Def: "regular" solid (aka Platonic solid) is a solid having all sides the same, all angles the same, all corners the same, and all faces the same.

Try to construct some of these.
Faces must be regular n-gons. Must have at least three n-gons meeting at each corner.

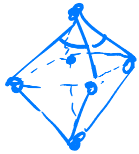
Try 3 triangles : makes a pyramid called a "Tetrahedron"
Tetra = 4



Ex's:
360
= 180
< 360

Try 4 triangles around every corner: forms an "OCTAhedron"

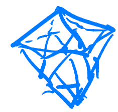
faces = 8, corners = 6



$\sum$ plane $\measuredangle$'s $= 4 \cdot 60 = 240$
< 360

Try 5 triangles around each corner

$5 \cdot 60 = 300. < 360.$



20 sides.

icosahedron.

Try 6 triangles at each corner

$6 \cdot 60 = 360 \not< 360$

Corner is not solid,

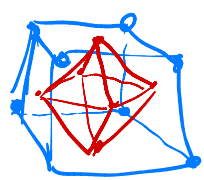
continuing to place 6 triangles at each vertex, would tile the plane by equilateral triangles = hexagons

6 triangle will make flat space (plane), has zero "curvature"
If we put 7 triangles around every corner, this will approximate negative curvature hyperbolic plane.

How about 3 squares?

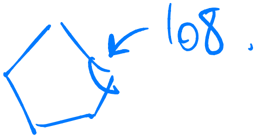
$3 \cdot 90 = 270.$

Cube (= "hexahedron").



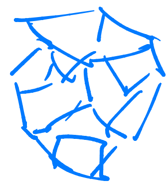
4 squares? $4 \cdot 90 = 360$.
tiles the plane.

↑ 6 faces,
8 corners



$\underline{3 \cdot 108} = \underline{324} < 360$.

Try: 3 pentagons at each corner



makes a 12 sided shape

Dodecahedron = 12 sides

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4 pentagons $4 \cdot 108 > 360$ no good

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3 hexagons.

$3 \cdot 120 \leq 360$

tiles plane.

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no options with hexagons, already 3 together are too much. So nothing else is possible. ONLY Options:

dual

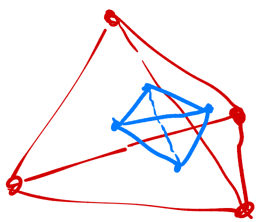
3 triangles -- tetrahedron
4 triangles -- octahedron
5 triangles -- icosahedron
3 squares -- cube
3 pentagons -- dodecahedron

5 triangles meet at corners, 20 triangles total, 12 corners.

3 pentagons / corner, 12 pentagons total 20 corners

"dual" polyhedra.

replace corners ⟷ faces.



Theorem: these 5 solids are the ONLY regular solids (called "Platonic")

500 Pythagoreans

450 Hippocrates

Socrates

400 Eudoxus

Plato

The Academy (Athens)

350 Alexander the Great

Aristotle

300 Euclid of Alexandria

250

Archimedes of Syracuse (Sicily).

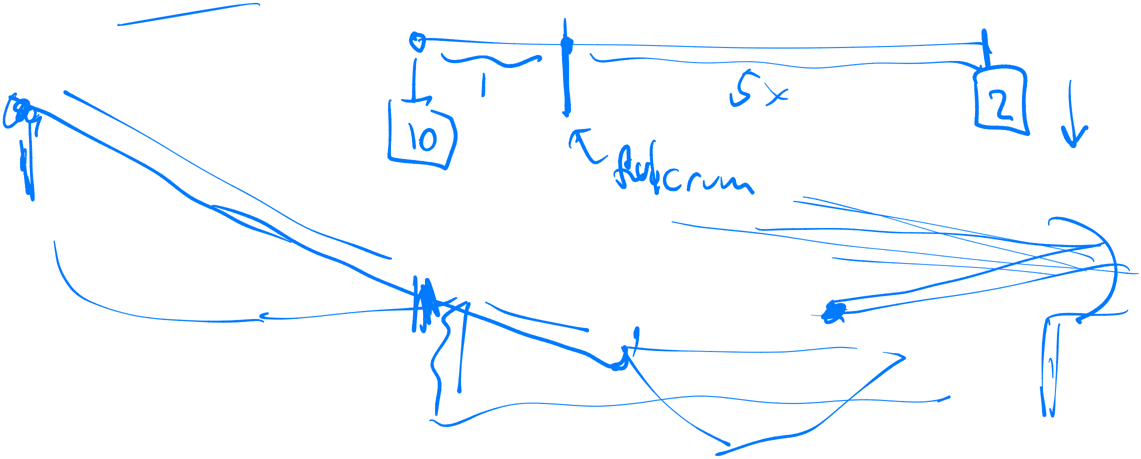
"ether"??

Earth, air, water, fire

cube, octahedron, icosahedron, tetrahedron

212 BCE Romans conquer Sicily.

Levers:



Archimedes: Give me a level long enough and I'll lift the world

Realizes to determine the density of a crown of "gold", weigh it again under water -- screams Eureka running through town naked