Fun with Modular Forms:

$q$-series: \[ q + q^4 + q^9 + \cdots \]
have "mock modular" symmetries.

Prehistory: Euler solves Basel Problem: \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \pi^2 \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

Bernoulli's \[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{9} \cdot \frac{1}{3} \cdot \frac{1}{3} + \cdots \]

\[ \frac{1}{6} \pi^2 \]

Compare term \[ \frac{1}{n^2} \]

Kronecker symbol: \( \left( \frac{p}{q} \right) \) primitive poly of degree \( d \), roots \( \zeta_1, \ldots, \zeta_d \) in \( \mathbb{C} \),

\[ a \cdot \left( \frac{0}{1} \right) \Rightarrow p(x) = \left( 1 - \frac{x}{a_1} \right) \cdots \left( 1 - \frac{x}{a_d} \right) \]

Infinite product vs. infinite series vs. stuff.

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} \right) = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) \]

\[ \sum_{p} \frac{1}{p} = \infty \]

\[ \frac{1}{1 - \lambda} = 1 + \lambda + \lambda^2 + \lambda^3 + \cdots \]

Partition function: \( p(5) = 7 \), \( 5 = 4 + 1 = 3 + 2 = 2 + 2 + 1 \), etc.,
Euler: \[ 1 + p(1)q + p(2)q^2 + p(3)q^3 + \ldots = \sum p(n)q^n. \]

\[
\frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots} = \left(\frac{1}{1-q}\right) \left(\frac{1}{1-q^2}\right) \left(\frac{1}{1-q^3}\right) \left(\frac{1}{1-q^4}\right) \ldots \]

\[ \text{Coef of } q^5 = \sum p(n)q^5. \]

1. **Kepler (1611): Spherical packings, densest possible configuration?**

   Both pack 74\% of space. Can one do better?

1998 Hales proof Kepler (Fejes Tóth 40/50s)


2003 proves \( \frac{\pi}{\sqrt{18}} \)

2014 Complete found proof. Lots of collaborators.

3-D, 2-D Thue-1890s: Optimal Fejes Tóth 40.

4-D ???, 5-D ???, . . .

8-D, \( E_8 \) lattice \( (n_1, \ldots, n_8) \in \mathbb{R}^8 \) \( \sum n_i = 2 \) \( \sum n_i \text{ even} \).
\[ E: y^2 + y = x^3 - x \]

\[ y \equiv x^3 - x \pmod{p} \]

\[ y = p + a(p) \]

\[ a(3) = 1 \]

\[ f(z) = \sum_{n=0}^{\infty} a(n) q^n \] is a modular form!

SO: Taniyama-Shimura-Weil "Modularity" (Langlands).

Says: Every elliptic curve is modular.

80s Frey: If \( a^p + b^p = c^p \), \( p \geq 3 \), \((a,b,c) \neq 0\),

ie if Fermat's last theorem false, then:

\[ y^2 = x(x-a^p)(x+b^p) \] might not be modular?

Serre & Rubin

A. Wiles proved enough modularity \( \Rightarrow \) FLT. (Wiles-Taylor)
\[ X^2 + y^2 = 1 \quad \text{parametrization?} \quad x = \cos \theta, \quad y = \sin \theta, \quad \theta \in \mathbb{R}/2\mathbb{Z}. \]

\[ y^2 = x + Ax + B \quad \text{parametrization?} \]

Weierstrass \( X = P(w), \quad y = P'(w) \cdot \Lambda \quad \Lambda \text{ finite} \quad \mathbb{C} / \Delta. \)

\[ P'(w) = P(w)^2 + E_4(w) \cdot P(w) + E_6(w) \]

"\( j \)-invariant" \( j(z) = \frac{E(2)}{\Delta(2)} \)

\[ \Delta(2) = q \prod_{n=1}^{\infty} (1 - q^n)^2 \]

\[ q = e^{2\pi i z} \]

\[ j \left( \frac{1 + \sqrt{-163}}{2} \right) = -744 \]

\[ \Delta(2) = q \prod_{n=1}^{\infty} (1 - q^n)^2 \]

Aside: 70\( \frac{3}{2} \) finite simple groups \( \text{PSL}(2|17) \)

A finite list of infinite family.

Sporadic groups finite (26).
Largest: Monster $8 	imes 10^{53}$ elements.

- In tangent space to manifold (linearization)
  $\text{dim}$ (indecomposable) reps of Group $^3$.  

$\text{Ned}$ rep of M; 1, 196883, 21296876, ---

Moonshine --- vertex operator algebras...Jim Lepowsky.

Richard Borcherds proved, 1998 FM.

\[ \text{Ramanujan:} \quad j \left( \frac{1 + \sqrt{163}i}{2} \right) = -640320 \in \mathbb{Z}. \]

\[ q = \exp \left( \frac{1 + i \sqrt{163}}{e} \right) = -e^{\pi \sqrt{163}} \]

\[ -e + 744 + \ldots \]

\[ \Rightarrow e = 640320^3 + 744 + O(10^{-12}). \]