Last time: $\cdot\ \vartheta_f(t) = \sum_{n\geq 1} f(nt)$, $f \in \mathbb{C}_c^\infty, n \in \mathbb{Z}$

$\cdot\ \tilde{\vartheta}_f(s) = \zeta(s) \cdot \tilde{f}(s)$.

(anal. cont., funct eq) $= -\frac{\hat{f}(0)}{2(1-s)} - \frac{f(0)}{2s} + \int_1^\infty \left(\theta_f(t) t^s + \theta_{\hat{f}}(t) t^{1-s}\right) \frac{dt}{t}$.

$\cdot\ -\frac{\zeta'}{\zeta}(s) = \sum_n \frac{\Lambda(n)}{n^s}$, $\frac{1}{2\pi i}\int_{(2)} \left(\frac{-\zeta'}{\zeta}\right)(s)\, \tilde{\varphi}(s) X^s ds$ — φ smooth.

(von Mangoldt 1895) $\sum_n \Lambda(n) \varphi\left(\frac{n}{x}\right) = \tilde{\varphi}(1) X^1 - \sum_\rho \tilde{\varphi}(\rho) X^\rho$.

Riemann 1859 observes: $\psi(x) \stackrel{\text{PNT}}{=} \sum_{n<x} \Lambda(n) \sim x$ $\left(\varphi(y) = \mathbb{1}_{\{y<1\}}\right)$.

$\Longleftrightarrow$ $\left(\tilde{\varphi}(s) = \frac{1}{s}\right)$

$\boxed{\forall \rho,\ \operatorname{Re}\rho < 1}$ $\leftarrow$ Hadamard, de la Vallée Poussin 1896.

(Conj $\operatorname{Re}\rho = \frac{1}{2}$). Uses trig identity: $3 + 4\cos\theta + \cos 2\theta \geq 0$.
$\left(\left(2(1+\cos\theta)^2\right)\right)$

More robust method 1994 Hoffstein-Lockhart, + Goldfeld-Hoffstein-Lieman.

(Classical: no zeros on $1+it$

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},\quad \log\zeta(\sigma)=\sum_p -\log\left(1-\frac{1}{p^\sigma}\right)=\sum_p\sum_{k\geq 1}\frac{1}{k\,p^{k\sigma}}$$

__Observation:__

$$0\leq\left|1+p^{it}+p^{-it}\right|^2=\underline{1}+p^{2it}+p^{-2it}+2p^{it}+2p^{-it}+\underline{2}\geq 0.$$

Let

$$1\leq F_t(\sigma):=\underbrace{\zeta(\sigma)^3}_{\ll(\sigma-1)^{-3}}\cdot\underbrace{\zeta(\sigma+it)^2\,\zeta(\sigma-it)^2}_{\ll(\sigma-1)^4}\,\underbrace{\zeta(\sigma+2it)}_{\ll 1}\,\underbrace{\zeta(\sigma-2it)}_{\ll 1}\to 0.$$

$$\log F_t(\sigma)=\sum_p\sum_{k\geq 1}\frac{1}{k\,p^{k\sigma}}\left[3+2p^{-it}+2p^{it}+p^{2it}+p^{-2it}\right]\geq 0.$$

Suppose $\exists T$: $\zeta(1+iT)=0$. Then $\zeta(\sigma+iT)=O(\sigma-1)$ as $\sigma\to 1^+$.

$\Rightarrow\zeta(1-iT)=0$. Know: $\zeta(\sigma)=O\left(\frac{1}{\sigma-1}\right)$ as $\sigma\to 1^+$.

$\Rightarrow \operatorname{Re}\rho<1$. Using __Hadamard Factorization__ $\Rightarrow$ (Exercise)

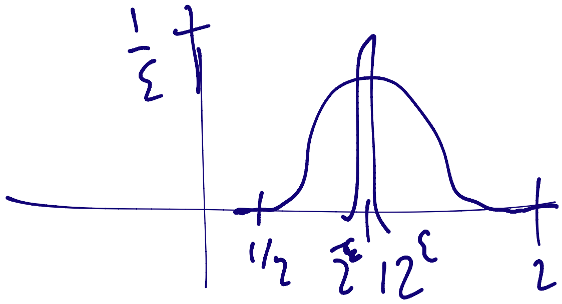
For $|t|>1$, $\sigma>1-\frac{c}{\log t}$, $\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|\ll\log t$.

How to use this? Want: $\frac{1}{2\pi i}\int_{(2)} \frac{-\zeta'}{\zeta}(s)\,\tilde{\varphi}(s)\,X^s\,ds$

Where $\varphi(y) = \mathbb{1}_{y<1}$,

$\tilde{\varphi}(s) = \frac{1}{s}$. ⟵ Not enough decay, so we smooth & unsmooth

Do this by Mellin Convolution. Fix smooth χ cpt supp in $[1/2, 2]$



$\chi(y) \geq 0$.

$\tilde{\chi}(s) \ll_A \frac{1}{|s|^A}$ $(A \geq 1)$.

$\int_0^\infty \chi(y)\frac{dy}{y} = 1$.

Fix $\varepsilon > 0$, set $\chi_\varepsilon(y) = \frac{1}{\varepsilon}\chi\left(y^{1/\varepsilon}\right)$.

$\frac{1}{2} \leq y^{1/\varepsilon} \leq 2$

$1-O(\varepsilon) \leq 2^{-\varepsilon} \leq y \leq 2^{\varepsilon} \leq 1+O(\varepsilon)$

Then $\tilde{\chi}_\varepsilon(0) = \int_0^\infty \chi_\varepsilon(y)\frac{dy}{y} = \frac{1}{\varepsilon}\int_0^\infty \chi\left(y^{1/\varepsilon}\right)\frac{dy}{y}$

$u = y^{1/\varepsilon}$

$du = \frac{1}{\varepsilon}y^{\frac{1}{\varepsilon}-1}dy$

$\frac{du}{u} = \frac{1}{\varepsilon}\frac{dy}{y}$

$$= \int_0^\infty \chi(u)\frac{du}{u} = 1.$$

Mellin Convolution $f, g : \mathbb{R}_{>0} \to \mathbb{C}$, Then

$$(f * g)(y) = \int_0^\infty f(u)\cdot g\left(\frac{y}{u}\right)\frac{du}{u}$$

Exercise: $\widetilde{f*g}(s) = \tilde{f}(s)\cdot\tilde{g}(s)$.

$\boxed{\text{Exercise: } \tilde{\chi}_\varepsilon(s) = \tilde{\chi}(\varepsilon s)}$

Replace φ with $\underbrace{\varphi_\varepsilon(y)}_{} := (\varphi * \chi_\varepsilon)(y) = \int_0^1 \chi_\varepsilon\left(\frac{y}{u}\right)\frac{du}{u}$, $y>0$.

$\uparrow$ $\mathbb{1}_{y\le 1}$

$$\widetilde{\varphi}_\varepsilon(s) = \frac{1}{s}\cdot \underbrace{\widetilde{\chi}(\varepsilon s)}_{\ll \frac{1}{\varepsilon|s|}}.$$

$$= \int_0^1 \frac{1}{\varepsilon}\chi\left(\left(\frac{y}{u}\right)^{1/\varepsilon}\right)\frac{du}{u} = \int_{y^{1/\varepsilon}}^{\infty} \underbrace{\chi(v)}^{\le 1}\frac{dv}{v}$$

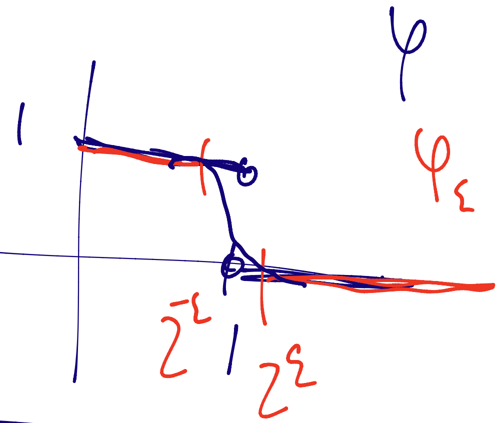
$$v = \left(\frac{y}{u}\right)^{1/\varepsilon}$$

$$\frac{dv}{v} = \cancel{y^{1/\varepsilon}}\,\frac{-1}{\varepsilon}u^{-\frac{1}{\varepsilon}-1}du$$

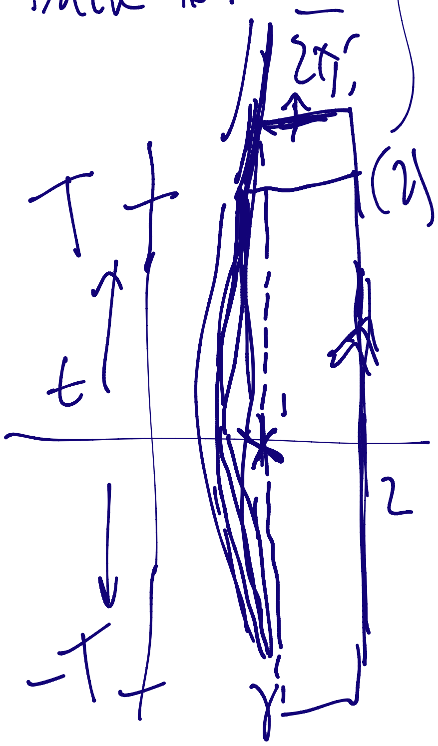
If $\underbrace{y^{1/\varepsilon} \ge 2}_{}$, $\varphi_\varepsilon(y)=0$.

$y \ge 2^\varepsilon = 1+10\varepsilon$.

If $y^{1/\varepsilon} \le \frac{1}{2}$, $\varphi_\varepsilon(y) \equiv 1$.



Back to: $\frac{1}{2\pi i}\int_{(2)} -\frac{\zeta'}{\zeta}(s)\,\underbrace{\widetilde{\varphi}_\varepsilon(s)}_{}\,x^s\,ds = \underbrace{\sum_{n\ge 1}\Lambda(n)\,\varphi_\varepsilon\left(\frac{n}{x}\right)}_{}$



$$= \underbrace{\sum_{n\ge 1}\Lambda(n)\,\varphi\left(\frac{n}{x}\right)}_{} + \underbrace{O(\varepsilon x \log x)}_{}$$

Known: $\underbrace{\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \ll \log t}_{}$ on $\underbrace{\sigma > 1-\frac{c}{\log t}}_{}$

pull contours $= \widetilde{\varphi}_\varepsilon(1)\,x + \frac{1}{2\pi i}\int_{(\gamma)}$

$$\int_{(\delta)} = \int_T^\infty + \int_{-T}^T \qquad \overset{\shortparallel}{X + O(\varepsilon X)}. \qquad \underbrace{\tilde{\varphi}(1)}_{1} \cdot \underbrace{\tilde{\chi}(\varepsilon \cdot 1)}_{\left(\underbrace{\tilde{\chi}(0)}_{1} + o(\varepsilon)\right)}.$$

$$\left|\int_T^\infty\right| \ll \int_T^\infty \log t \cdot \frac{1}{\varepsilon t^2} X^1 \, dt \ll \frac{X}{\varepsilon T^{1/2}} \quad \text{very crude.}$$

$$\left|\int_{-T}^{T}\right| \ll \int_{-T}^{T} \log(3+|t|) \frac{1}{\varepsilon(2+t^2)} X^{1-\frac{c}{\log T}} \, dt$$

$$\ll \frac{X^{1-\frac{c}{\log T}}}{\varepsilon}.$$

Put all together:

$$\sum_{n<X} \Lambda(n) + O(\varepsilon X \log X) = X + O(\varepsilon X) + O\left(\frac{X}{\varepsilon T^{1/2}} + \frac{X^{1-\frac{c}{\log T}}}{\varepsilon}\right)$$

First optimize T: $\frac{1}{T^{1/2}} = X^{-\frac{c}{\log T}}$, $\quad T^{1/2} = X^{\frac{c}{\log T}}$

$$\log T = c\sqrt{\log X}, \quad T = e^{c\sqrt{\log X}}. \qquad \cancel{\tfrac{1}{2}} \log T = \frac{c}{\log T} \log X$$

$$X^\delta = e^{\delta \log X}$$

$$= X + O\left(\varepsilon X \log X + \frac{1}{\varepsilon} X e^{-c\sqrt{\log X}}\right).$$

$$\xi X \log X = \frac{1}{\xi} X e^{\xi - c\sqrt{\log X}} \qquad \underbrace{X + O\left(X e^{-c\sqrt{\log X}}\right)}_{\Downarrow \; O\left(\frac{X}{\log^A X}\right).}$$

$$\xi^{\cancel{X}} = e^{-c\sqrt{\log X}}$$

$$(\log X)^A = e^{A \underline{\log\log X}} \qquad e^{c \underline{\sqrt{\log X}}}$$

Today: $O\left(X e^{-c(\log X)^{3/5}}\right)$ Exercise: $\frac{3}{5} \to 1$.