Eratosthenes:

$ab = n < x$

$a < \sqrt{x}$ or $b < \sqrt{x}$.

$$\begin{array}{l} \cancel{1}\ \ 2\ \ 3\ \ \cancel{4}\ \ 5\ \ \cancel{6}\ \ 7\ \ 8\ \ 9\ \ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 100 \end{array}$$

Gauss 1790s: computes tables of

$\pi(x) = \#\{p < x\}$ $\quad Li(x) = \int_2^x \frac{dt}{\log t}$

Conjectures: $\pi(x) \overset{x/\log x}{\sim} Li(x)$ $\quad$ (RH: $\pi(x) = Li(x) + O(x^{1/2+\varepsilon})$) $\quad$ 1/2 digits accurate

(Legendre $\pi(x) \approx \frac{x}{\log x + A}$

$= \frac{x}{\log x} + \Omega\left(\frac{x}{\log^2 x}\right)$.

Last time: • Mellin transform/inversion:

$f: \mathbb{R}_{>0} \to \mathbb{C}$, $\tilde{f}(s) = \int_0^\infty f(y) y^s \frac{dy}{y}$,

$$\frac{1}{2\pi i} \int_{(2)} \tilde{f}(s) y^{-s} \, ds = f(y).$$

• Poisson summation (dilated) $\sum_{n \in \mathbb{Z}} f(nt) = \frac{1}{t} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{t}\right)$

Riemann memoir goes Tate: Say f even. $f(-x) = f(x)$

[Exercise $\Rightarrow$ $\hat{f}$ even]

$$= f(0) + 2 \sum_{n \geq 1} f(nt).$$

$$\Theta_f(t) := \sum_{n \geq 1} f(nt).$$

Big idea: Compute $\tilde{\Theta}_f(s) = \int_0^\infty \Theta_f(t) t^s \frac{dt}{t}.$

Can this converge? E.g.: $f(t) = e^{-\pi t^2}$ Gaussian.

Then $\sum_{n \geq 1} e^{-\pi n^2 t^2} \ll e^{-\pi t}$. At ∞, no problem with convergence.

What about as $t \to 0^+$?

$$\sum_n f(nt) = \frac{1}{t}\sum \hat{f}(n/t).$$
$$f(0)+2\theta_f(t) = \frac{1}{t}\hat{f}(0) + \frac{2}{t}\theta_{\hat{f}}\left(\frac{1}{t}\right).$$

$$\theta_f(t) = \sum_{n\geq 1} f(nt) = \underbrace{\frac{1}{t}\theta_{\hat{f}}\left(\frac{1}{t}\right)}_{\text{exp decay.}} + \underbrace{\frac{1}{2t}\hat{f}(0) - \frac{1}{2}f(0)}_{\text{blows up like } \frac{1}{t}.} \quad \text{as } t \to 0^+.$$

So $\displaystyle\int_0^{\varepsilon} \theta(t)\, t^s \frac{dt}{t} \simeq \int_0 \frac{1}{t}\, t^s \frac{dt}{t}$ Converges if $\mathrm{Re}\, s > 1$.

So $\displaystyle\tilde{\theta}_f(s) \overset{\text{abs convergent integral}}{=} \int_0^\infty \sum_{n\geq 1} f(nt)\, t^s \frac{dt}{t}$.

for $\underline{\mathrm{Re}\, s > 1}$

$$= \sum_{n\geq 1} \frac{1}{n^s} \cdot \underbrace{\int_0^\infty f(nt) \left(\frac{t}{n}\right)^s \frac{dt}{t}}_{t \mapsto \frac{t}{n}} = \underbrace{\zeta(s)}_{\prod_p \zeta_p(s),\ \zeta_p(s) = \frac{1}{1-\frac{1}{p^s}}} \tilde{f}(s).$$

In eg. $f(t) = e^{-\pi t^2}$, $\displaystyle\tilde{f}(s) = \int_0^\infty \underbrace{e^{-\pi t^2}}\, t^s \frac{dt}{t} = \frac{1}{2}\int_0^\infty e^{-u}\left(\frac{u}{\pi}\right)^{s/2} \frac{du}{u}$

$$= \frac{1}{2}\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right) = \zeta_\infty(s).$$

$u = \pi t^2$, $t = \left(\frac{u}{\pi}\right)^{1/2}$

$$\frac{du}{u} = \frac{2\pi t\, dt}{\pi t^2} = 2\frac{dt}{t}$$

So far: $\tilde{\vartheta}_f(s) = \xi(s)\cdot\tilde{f}(s) = \int_0^\infty = \underbrace{\int_1^\infty}_{} + \underbrace{\int_0^1}_{}$

$\mathrm{Re}\, s > 1$

$= \int_1^\infty \underbrace{\sum_{n\geq 1}}_{} f(nt)\, t^s \frac{dt}{t}$ = entire function of s.

$\int_0^1 \Big[\underbrace{\frac{1}{t}\vartheta_{\hat{f}}\left(\frac{1}{t}\right)}_{} + \underbrace{\frac{1}{2t}\hat{f}(0) - \frac{1}{2}f(0)\Big] t^s \frac{dt}{t}}_{}.$

$= \frac{\hat{f}(0)}{2}\frac{1}{s-1} - \frac{f(0)}{2}\frac{1}{s}$

$= \int_0^1 \vartheta_{\hat{f}}\left(\frac{1}{t}\right) t^{s-1}\frac{dt}{t}$

$= \int_1^\infty \vartheta_{\hat{f}}(t)\, t^{1-s}\frac{dt}{t}.$

$t \mapsto \frac{1}{t}$

$\xi(s)\tilde{f}(s) = \underbrace{\frac{-\hat{f}(0)}{2(1-s)} - \frac{f(0)}{2\cdot s}}_{\text{poles at } s=0,1} + \underbrace{\int_1^\infty \left(\vartheta_f(t)t^s + \vartheta_{\hat{f}}(t)\cdot t^{1-s}\right)\frac{dt}{t}}_{\text{entire!}}.$

mer'c cont.

If $f = \hat{f}$, $\xi(s)$.

$\tilde{\vartheta}_f(s) = \tilde{\vartheta}_f(1-s)$

Note: $\sum \frac{1}{n^s}$ is not the Riemann zeta function.

$$\underbrace{\xi(s) = \frac{1}{\tilde{f}(s)}\left[\frac{-\hat{f}(0)}{2(1-s)} - \frac{f(0)}{2\cdot s} + \int_1^\infty \left(\vartheta_f(t)t^s + \vartheta_{\hat{f}}(s)t^{1-s}\right)\frac{dt}{t}\right]}_{}$$

The Riemann Zeta function

CRAZY: Does not depend on choice of f !!!..

(Neither does $\sum f(n) - \sum \hat{f}(m)$!).

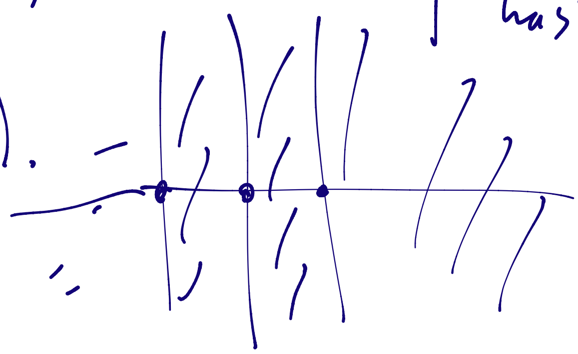
Specialize to $f(t) = e^{-\pi t^2}$, $\tilde{f}(s) = \frac{1}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$.

has zeros! only in $0 \le \mathrm{Re}\, s \le 1$.

$$\underbrace{\zeta(s)}_{\text{regular at } 0.}\ \frac{1}{2}\overset{\neq 0}{\pi^{-s/2}}\ \overset{\neq 0}{\underbrace{\Gamma\left(\tfrac{s}{2}\right)}_{\text{Simple pole at } 0.}} = \underbrace{-\frac{1}{2s} - \frac{1}{2(1-s)}}_{\text{Pole at } s=1,\,0} + \int_1^\infty \theta_f(t)\left[t^s + t^{1-s}\right]\frac{dt}{t}.$$

Exercises: $s\Gamma(s) = \Gamma(s+1)$.

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^s\, \frac{dt}{t} \qquad \mathrm{Re}\, s > 0$$



Γ has simple poles at $s = 0, -1, -2, \ldots$

(Euler)

Exercise: $\underbrace{\Gamma(s)\Gamma(1-s)}_{\text{simple poles at } \mathbb{Z}.} = \underbrace{\frac{\pi}{\sin \pi s}}_{\text{has no zeros!}}$

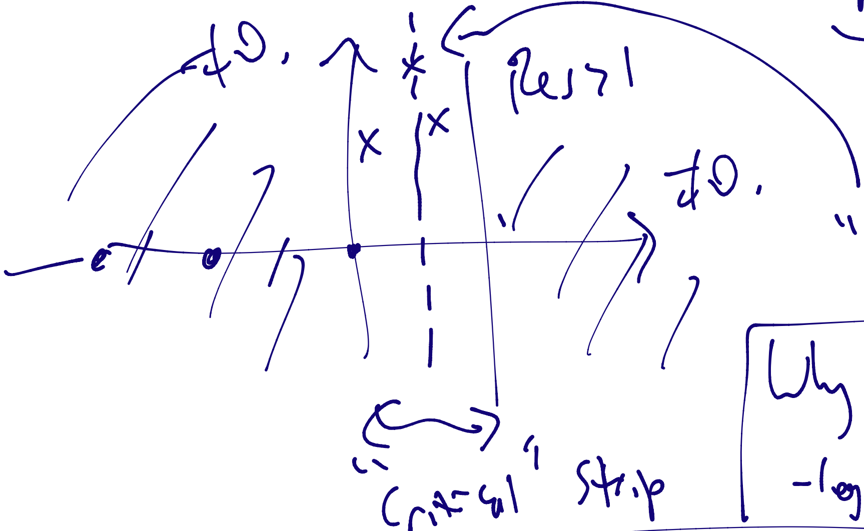
$$\sin s = \frac{e^{is} - e^{-is}}{2i}.$$

Entire. has simple zeros

$\Rightarrow \Gamma(s)$ no zeros! $\Rightarrow$ no new poles for ζ by diving Γ.

ζ has simple pole at 1, regular at 0, & $\zeta(-2) = 0 = \zeta(-4) = \cdots$ (Euler).

Recall Euler: $\zeta(s) = \underbrace{\overset{\mathrm{Re}\, s>1}{\prod_p} \left(\frac{1}{1-p^{-s}}\right)}_{\text{abs conv product } \neq 0.}$ $\neq 0$ (Exercise)



"critical line".

Why zeros, why primes?

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Riemann: "log" $\zeta(s) = \sum_p -\log(1-p^{-s}) = \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots\right)$

(Re $s>1$).

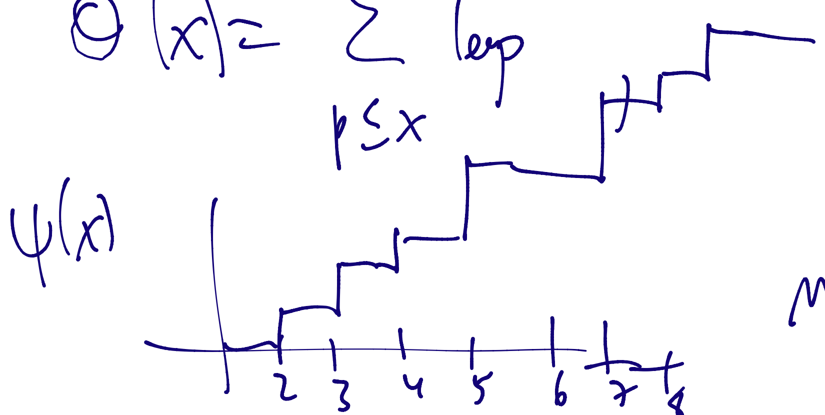
$$\frac{\zeta'}{\zeta}(s) = -\sum_p \log p \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right).$$

k^{th}: $\frac{d}{ds}\left(\frac{1}{k} \cdot e^{-ks\log p}\right) = \frac{1}{\cancel{k}} e^{-ks\log p}(-\cancel{k}\log p)$

von Mangoldt

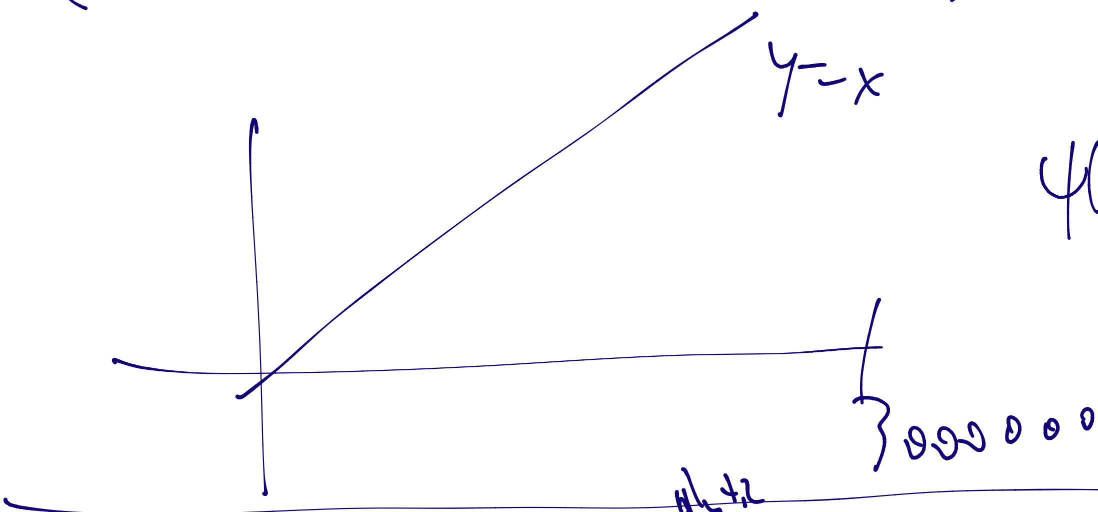
$$= -\sum_n \frac{\Lambda(n)}{n^s}, \quad \Lambda(n) = \begin{cases} \log p, & n = p^k \\ 0 & \text{oth.} \end{cases}$$

$$\theta(x) = \sum_{p \le x} \log p \qquad \sum_{p^k \le x} \log p = \sum_{n \le x} \Lambda(n) = \psi(x)$$



Many early numbers are primes or powers of primes

(⇒ first zero of zeta will be "large" $\frac{1}{2}+14.i\ldots$).



RH ⇒

$$\psi(x) - x = O\left(x^{1/2+\varepsilon}\right)$$

Brownian motion.

Riemann: $$\frac{1}{2\pi i}\int_{(2)} \underbrace{\overbrace{\left(-\frac{\zeta'}{\zeta}(s)\right)}^{\text{analytic}}}_{\sum \frac{\Lambda(n)}{n^s}} \tilde{\varphi}(s)\, x^s\, ds. \qquad x>1.$$

φ test function on $\mathbb{R}_{>0}$

$$\tilde{\varphi}(s) = \int_0^\infty \varphi(t)\, t^s \frac{dt}{t}$$

$$= \sum_{n\geq 1} \Lambda(n) \underbrace{\frac{1}{2\pi i}\int_{(2)} \tilde{\varphi}(s) \left(\frac{n}{x}\right)^{-s} ds}_{\varphi\left(\frac{n}{x}\right).} = \underbrace{\sum_{n\geq 1} \Lambda(n)\, \varphi\left(\frac{n}{x}\right)}.$$

$\mathbb{1}$ $\frac{n}{x}<1.$

If $\varphi \cong \mathbb{1}_{(0,1)}$, $\to \tilde{\varphi}(s) = \frac{1}{s}$. ⇒ ψ Chebyshev!..

recall $$\tilde{\varphi}(s) = -\int_0^\infty \varphi'(t) \frac{t^s}{s}\, dt = \int_0^\infty \varphi''(t) \frac{t^{s+1}}{(s+1)s}\, dt$$

plenty of decay to interchange sums.

Recap: $$\frac{1}{2\pi i}\int_{(2)} \underbrace{-\frac{\zeta'}{\zeta}(s)}_{\text{poles?}}\ \underbrace{\tilde{\varphi}(s)}_{\text{entire!}}\ \underbrace{X^s}_{\text{entire}}\,ds = \sum_{n\geq 1}\Lambda(n)\,\varphi\left(\frac{n}{X}\right).$$

$X>1$

$\Updownarrow$

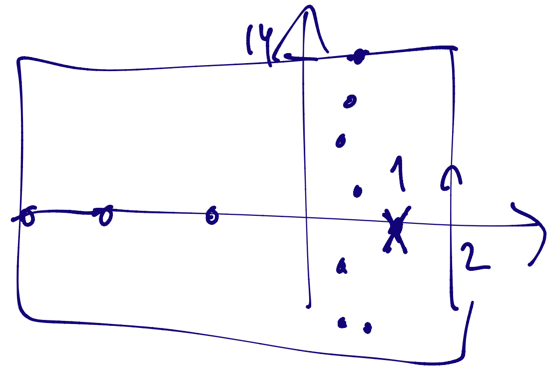
ζ has pole or $\zeta=0$.

$\mathrm{Res}\ \frac{\zeta'}{\zeta} = \pm 1$

$||$

$(s=1)$

$$\underbrace{\tilde{\varphi}(1)X^1 - \sum_{\substack{\rho \\ \zeta(\rho)=0}} \tilde{\varphi}(\rho)X^\rho.}$$



$$\Rightarrow \quad \underbrace{\sum_{n\geq 1}\Lambda(n)\,\varphi\left(\frac{n}{X}\right)}_{\text{geom side.}} = \underbrace{\tilde{\varphi}(1)X - \sum_{\rho \text{ all zeros}}\tilde{\varphi}(\rho)X^\rho}_{\text{spectral side.}}.$$

Riemann proved Poisson Summation! in primes.

Explicit Formula.

Fix chronology: von Mangoldt gave treatment in 1895. So Riemann wasn't referencing von Mangoldt, the name got attached to $\Lambda(n)$ much later.