Crash Course in Sieving.

Dirichlet $\sum_{p\equiv a(q)} \frac{1}{p} = \infty$.

Thm (Brun 1919): $\displaystyle\sum_{p \text{ twin primes}} \frac{1}{p} < \infty$.

$p,\ p+2 = \text{prime}$.

Thm: $\pi_2(x) = \#\{p<x : p+2 \text{ prime}\} \ll \dfrac{x}{\log^2 x}$.

$$\Rightarrow \int_2^x \frac{1}{t}\, d\pi_2(t) = \frac{\pi_2(x)}{x} + O(1) + \int_2^x \frac{\pi_2(t)}{t^2}\, dt.$$

$$\ll \int^x \frac{1}{t\log^2 t}\, dt \ll 1.$$

Follow Eratosthenes: look at

$$S(x,z) := \sum_{m<x} \mathbb{1}_{(m(m+2),\, P_z)=1}, \text{ where}$$

$P_z = \prod_{p<z} p$. Take $z = x^{1/3}$.

this catches twin primes.

ie $\pi_2(x) \le x^{1/3} + S(x,z)$.

Need to understand $S(x,z)$. Legendre idea: inclusion-exclusion. Recall: $f * \mu = g \iff g * 1 = f$.

$\Rightarrow \sum_{d|n} \mu(d) = \begin{cases} 1 & n=1 \\ 0 & \text{else.} \end{cases}$

$L_f \cdot \frac{1}{\zeta} = L_g \iff L_g \cdot \zeta = L_f$.

$\zeta \cdot \frac{1}{\zeta} = 1$.

$\Rightarrow S(x,z) = \sum_{m<x} \sum_{\substack{d | m(m+2) \\ d | P_z}} \mu(d) = \sum_{d | P_z} \mu(d) \underbrace{\sum_{m<x} 1_{m(m+2) \equiv 0 (d)}}_{\to\ \omega(d)\left(\frac{x}{d} + O(1)\right)}$

$\omega(p) = \begin{cases} 1 & p=2 \\ \boxed{2} & p<z \\ 0 & p>z \end{cases}$ sieve dimension

$\omega(d) = \#\{m \pmod d : m(m+2) \equiv 0 (d)\} = \prod_{p|d} \omega(p)$. $\boxed{\text{CRT}}$

Fails! How large can d get? $P_z = \prod_{p<z} p = e^z \neq e^{x^{1/3}}$

$\log P_z = \sum_{p<z} \log p \sim z$. So d gets way too big!

Rmk: If $z = \frac{1}{2}\log x$, no problem with $d < \sqrt{x}$.

Gives us "nothing" towards sieve.

Selberg idea: (1950s) "Λ^2 sieve"

If $\lambda : \mathbb{N} \to \mathbb{R}$, & $\lambda(1) = 1$ then
$$\sum_{d|n} \mu(d) \le \Big(\sum_{d|n} \lambda(d) \Big)^2$$

If $n = 1$, LHS $= 1$, RHS $= 1$ ✓. $n > 1$, LHS $= 0$.

First looks wrong, then looks stupid/useless.

$$S(x,z) \le \sum_{m<x} \Big(\sum_{\substack{d | m(m+2) \\ d | P_z}} \lambda(d) \Big)^2 \qquad (\text{if } d < D)$$

$D^{(2)}$.

$$= \sum_{d_1,d_2} \lambda(d_1)\lambda(d_2) \sum_{\substack{m<x \\ m(m+2) \equiv 0\,(d_1) \\ m(m+2) \equiv 0\,(d_2)}} 1$$

$m(m+2) \equiv 0\,([d_1,d_2])$

But! We have (almost) complete freedom to choose λ, will take it to have "smaller" support.

$$\hookrightarrow = \sum_{d_1,d_2} \lambda(d_1)\lambda(d_2)\left[\omega([d_1,d_2])\left(\frac{x}{[d_1,d_2]} + O(1)\right)\right],$$

$$= \underline{\underline{X}} \cdot Q + \mathcal{E}, \quad \text{where}$$

$$Q = \sum_{d_1,d_2} \lambda(d_1)\lambda(d_2)\frac{\omega([d_1,d_2])}{[d_1,d_2]}. \quad \leftarrow \text{quad form in } \lambda(d).$$

$$\mathcal{E} = \sum_{d_1,d_2} \lambda(d_1)\lambda(d_2)\,\omega([d_1,d_2])\cdot O(1),$$

$\left(Q = \underline{\sum y^2}\ \cancel{+z^2}\right)$ Q

Need to minimize Q w.r.t $\lambda(d)$ s.t. $\lambda(1) = 1$.

To make analysis easier, lets diagonalize.

Say $(d_1,d_2) = c$, $d_1 = a\cdot c$, $d_2 = b\cdot c$. $(a,b) = 1$.

$[d_1,d_2] = abc$.

$$Q = \sum_a \sum_b \sum_c \lambda(ac)\,\lambda(bc)\,\frac{\cancel{\omega(abc)}}{abc} \qquad \omega(abc) = \omega(a)\,\omega(b)\,\omega(c).$$

$$[(a,b)=1] = \sum_{\substack{l \mid a \\ l \mid b}} \mu(l),$$

$$= \sum_l \mu(l) \sum_c \frac{\omega(c)}{c} \left[\sum_{a \equiv 0(l)} \frac{\omega(a)}{a} \sum_{b \equiv 0(l)} \frac{\omega(b)}{b}\, \lambda(ac)\,\lambda(bc) \right]$$

$$= \sum_l \mu(l) \sum_c \frac{\omega(c)}{c} \left[\sum_{a \equiv 0(l)} \frac{\omega(a)}{a}\, \lambda(ac) \right]^2 .$$

$$= \sum_l \mu(l) \sum_c \frac{c}{\omega(c)} \left[\sum_{ac \equiv 0(lc)} \frac{\omega(ac)}{ac}\, \lambda(ac) \right]^2 .$$

Let $h = lc$.

$$= \sum_h \underbrace{\left[\sum_{c \mid h} \mu\!\left(\frac{h}{c}\right) \frac{c}{\omega(c)} \right]}_{\alpha(h)} \underbrace{\left[\sum_{f \equiv 0(h)} \frac{\omega(f)}{f}\, \lambda(f) \right]^2}_{y(h)} .$$

$y(h) \leftarrow$ linear change vars from λ.

$$Q = \sum_n \alpha(n) \cdot y(n)^2 \cdot \text{Diagonalized.}$$

Want to minimize Q s.t. $\underbrace{\lambda(1) = 1}_{\text{need this on } y\text{'s}}$.

Exercise: If $f : \mathbb{N} \to \mathbb{C}$ finite supp on sqfrees

Let
$$g(n) := \sum_{a \equiv 0(n)} f(a). \text{ Then } \uparrow \nearrow,$$

&
$$f(n) = \mu(n) \sum_{a \equiv 0(n)} \mu(a) g(a).$$

(Mobius inversion not on divisors but on progressions).

(<u>Hint</u>: insert def of g, reverse orders, change vars)

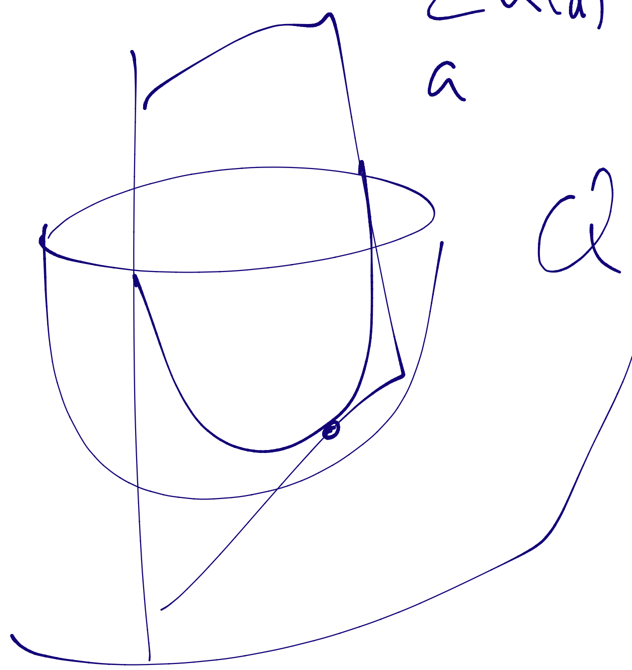
Since $y(n) = \sum_{f \equiv 0(n)} \frac{w(f)}{f} \lambda(f)$

$\Rightarrow \frac{w(n)}{n} \lambda(n) = \mu(n) \sum_{a \equiv 0(n)} \mu(a) \cdot y(a).$

$\Rightarrow 1 = \frac{1}{1} \lambda(1) = 1 \sum_a \mu(a) y(a)$

$\Rightarrow$ minimize $\underset{\shortparallel}{Q}$ subject to: $\boxed{\sum_{a | P_z} \mu(a) y(a) = 1}$

$\sum_a \alpha(a) y(a)^2.$



Q

Lagrange multiplers.

$$\mathcal{L}(y, \Theta) = \sum_a \alpha(a) y(a)^2 - \Theta\left(\sum_a \mu(a) y(a) - 1\right)$$

$\frac{\partial \mathcal{L}}{\partial y(a)} = 0$ & $\frac{\partial \mathcal{L}}{\partial \Theta} = 0.$

$$\frac{\partial \mathcal{L}}{\partial y(a)} = 2\alpha(a)\, y(a) - \theta \mu(a) = 0.$$

$$\Rightarrow \quad y(a) = \underbrace{\frac{\theta \mu(a)}{2\alpha(a)}}_{\downarrow} = \frac{\cancel{2}\,\mu(a)}{A \cdot \cancel{2} \cdot \alpha(a)}.$$

$$\& \quad \theta \sum_a \cancel{\mu(a)} \left[\frac{\cancel{\mu(a)}}{2\alpha(a)} \right] = 1.$$

$$\text{Let } A = \sum_{a \mid P_z} \frac{1}{\alpha(a)}$$

$$\Rightarrow \quad \theta = \frac{2}{A}.$$

$$Q = \sum \alpha(a)\, y(a)^2 = \sum \alpha(a) \left[\frac{\overset{1}{\cancel{\mu(a)}}}{A \cdot \alpha(a)} \right]^2,$$

$$= \frac{1}{A^2} \cdot \sum_a \frac{1}{\alpha(a)} = \frac{1}{A}.$$

<u>Recall:</u>

$$\alpha(h) = \sum_{c \mid h} \mu\!\left(\frac{h}{c}\right) \frac{c}{\omega(c)}$$

$$\alpha(p) = \mu(p)\cdot\frac{1}{1} + \cancel{\mu(1)}\frac{p}{\omega(p)}$$

$$= \frac{p-\omega(p)}{\omega(p)} > 0. \qquad \boxed{0 < \omega(p) \leq p.}$$

$$\boxed{\omega(p) = \begin{cases} 1 & p=2 \\ 2 & z>p>2 \end{cases}}$$

If instead $m(m+2)(m+4)$

$\omega(3)=3$ . . . . prime powers

$$A = \sum_{a \mid P_z} \frac{1}{\alpha(a)} = \prod_{p<z}\left(1+\frac{1}{\alpha(p)}\right) = \prod_{p<z}\left(1+\frac{\omega(p)}{p-\omega(p)}\right)$$

$$= \prod_{p<z}\left(\frac{p}{p-\omega(p)}\right) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right)^{-1}$$

sieve dimension.

$$= \frac{1}{2}\cdot\prod_{2<p<z}\left(1-\frac{\boxed{2}}{p}\right)^{-1}$$

$$\asymp \prod_{p<z}\left(1-\frac{1}{p}\right)^{-2} = \prod_{p<z}\left(1-\frac{2}{p}+\frac{1}{p^2}\right)^{-1}.$$

$$\prod_{p<z}\left(1-\frac{1}{p}\right)^{-1} \sim \underbrace{2e^{\gamma}}_{1.12\ldots?\neq 1.}\log z \quad \text{as } z\to\infty.$$

Mertens

$$\underbrace{\prod_{p}\left(1-\frac{1}{p^s}\right)^{-1} = \zeta(s) = \frac{1}{s-1}+\ldots}$$

Know how blows up as $s\to 1$.

$A = (\log z)^2$ ← sieve dimension

$z = x^{1/3}$

$$Q = \frac{1}{A} \ll \frac{1}{\log^2 z} \ll \frac{1}{\log^2 x}.$$

$$\pi_2(x) \ll \frac{x}{\log^2 x}.$$

$$y(a) = \frac{\mu(a)}{A\,\alpha(a)}$$

$$\lambda(n) = \mu(n)\sum_{a\equiv 0\,(n)} \mu(a)\,y(a). \quad \ldots$$

$$\prod_{\substack{p<z \\ p\equiv 1(4)}} \left(1-\frac{1}{p}\right)^{-1} \sim c\cdot(\log z)^{1/2},$$