Last time: Thm (Duke): As $D \to \infty$,

$$\frac{1}{h(D)} \sum_{\gamma \in \mathcal{C}_D} \frac{1}{\ell(\gamma)} \int_\gamma f \, ds \xrightarrow{D \to \infty} \frac{1}{\mathrm{vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} f$$

ELMV: Einsiedler-Lindenstrauss-Michel-Venkatesh

(Linnik)

$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$

Rmk: If $\exists$ ∞ly many fundamental D's with $h(D) = 1$,

Those curves individually space-fill.

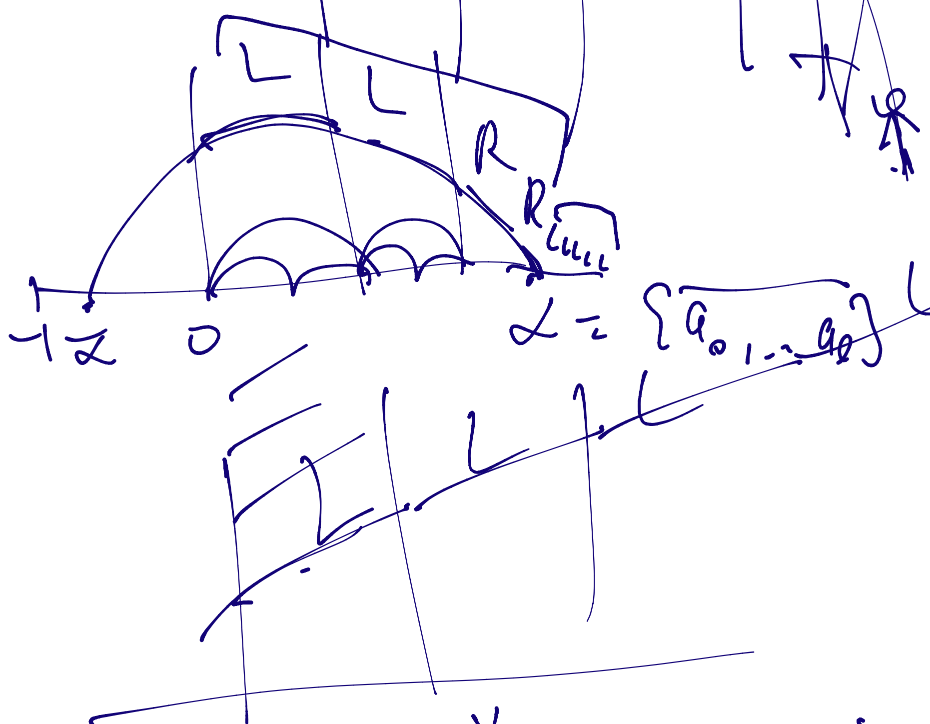
$\boxed{h(D) > D^{1/2 + o(1)} \text{ when } D = t^2 - 4.}$

People believe: As long as your limit is along D's with $h(D) < D^{1/2 - \varepsilon}$ (for some fixed $\varepsilon > 0$), then every individual geodesic still e.d.'s.

ELMV: Can there be other limits, e.g. can the support be compact?

$\exists$? $A \subset \infty$ s.t. ∞ly many closed geodesics, i.e. M $\quad Y_A = \{ z \in \mathcal{F} \cap \operatorname{Im} z < A \}$,

Easy to do! How can a geodesic go "high"?



To go high, needs a big partial quotient.

$\{$ "low-lying" "Y_A" closed geod $\} \longleftrightarrow \underbrace{\Gamma_A = \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} ; a \leq A \right\rangle}$.

ELMV: Can one create ∞ seq of "low-lying" fundamental closed geod?

$D \equiv 1\ (4) \iff D$ sq-free.

$\boxed{M\alpha = \alpha, \quad \alpha = \frac{t + \sqrt{t^2 - 4}}{2}}$

Claim: This looks "easy" to solve.

More through $D = t^2 - 4$.

$\#\Gamma_R \cap B_N = N^{2\delta_A}$

$\delta_A = H.\dim.$

$M \in \Gamma_A$, $\quad \underbrace{\mathrm{tr} M^2 - 4} = D \not{x}$.

Obs: If $\underbrace{\mathrm{tr} M^2 - 4 = \text{sq. free}} \Rightarrow \gamma(M)$ is fundamental.

Sieves allow to understand (almost primes) from distribution in progressions of sequence.

$\boxed{\text{Sieve Black Box:}}$ Suppose: $\exists \alpha > 0$ s.t.

"sieve dimension":

- $$\sum_{\gamma \in \Gamma_A \cap B_N} \mathbb{1}_{\mathrm{tr}\gamma^2 - 4 \equiv 0 (q)} = \frac{2^c}{q} \cdot \# \Gamma_A \cap B_N + \mathcal{E}(q, N)$$

"exponent of distr"

- $$\sum_{q < Q} |\mathcal{E}(q, N)| = o(\# \Gamma_A \cap B_N), \quad \underbrace{Q = N^\alpha}$$

"level of distribution"

$\exists \beta > 0$:

$\Rightarrow$ $\mathrm{tr}\gamma^2 - 4$ can't have more than $2/\beta$ prime factors

$$\# \{ \gamma \in \Gamma_A \cap B_N \mid p \mid (\mathrm{tr}(\gamma)^2 - 4) \Rightarrow p > N^\beta \} \gg \frac{\# \Gamma_A \cap B_N}{\log^2 N}.$$

$\boxed{\text{Black Box: Expansion}}$: $\exists \theta_A > 0$ s.t. $\forall q$, $\forall \gamma_0 \in SL_2(q)$,

$$\sum_{\gamma \in \Gamma_{A} \cap B_N} \mathbb{1}_{\gamma \equiv \gamma_0 (q)} = \frac{1}{|SL_2(q)|} \# \Gamma_A \cap B_N + O\left(\frac{\# \Gamma_A \cap B_N}{N^\theta}\right)$$

only nontrivial only if $q^3 < N^\theta$.

$\boxed{\text{Exercise}}$ $|SL_2(q)| \approx q^3$, | $\# \gamma \in SL_2(q)$: $\mathrm{tr}^2 \gamma - 4 \equiv 0 (q)$

$$\sum_{\gamma \in \Gamma_{A \cap B_N}} \mathbb{1}_{\mathrm{tr}\, \gamma^2 - 4 \equiv 0 (q)} = \sum_{\gamma_0 \in SL_2(q)} \mathbb{1}_{\mathrm{tr}\, \gamma_0^2 - 4 \equiv 0 (q)} \left[\sum_{\substack{\gamma \in \Gamma_{A \cap B_N} \\ \gamma \equiv \gamma_0 (q)}} 1 \right]$$

Expansion

$$= \sum_{\gamma_0 \in SL_2(q)} \mathbb{1}_{\mathrm{tr}\, \gamma_0^2 - 4 \equiv 0 (q)} \left[\frac{1}{|SL_2(q)|} \# \Gamma_{A \cap B_N} + O\left(\# \Gamma_{A \cap B_N} \cdot N^{-\theta}\right)\right]$$

$$= \frac{2}{q} \# \Gamma_{A \cap B_N} + \underbrace{O\left(\# \Gamma_{A \cap B_N} \cdot N^{-\theta} q^2\right)}_{\mathcal{E}(q, N)}.$$

$$\sum_{q<Q} |\mathcal{E}(q,N)| \ll \#\Gamma_A \cap B_N \cdot N^{-\theta} Q^3 \overset{\text{want}}{=} o(\#\Gamma_A \cap B_N).$$

$$Q^3 < N^{\theta}, \quad Q = N^{\alpha}, \quad \boxed{\alpha < \theta/3}.$$

$\Rightarrow$ (Sieve) $\exists\, \beta$ s.t.

$$\#\underbrace{\{\gamma \in \Gamma_A \cap B_N \mid p \mid tr^2\gamma - 4 \Rightarrow p > N^{\beta}\}}_{S} \gg \frac{\#\Gamma_A \cap B_N}{\log^2 N}.$$

<u>Want</u>: $\#\{\gamma \in \Gamma_A \cap B_N \mid tr\gamma^2 - 4 = \text{sqfree}\}. \quad (\to \infty),$

$$\geq \#\{\gamma \in S \mid tr\gamma^2 - 4 = \text{sqfree}\}.$$

$$= \#S - \#\{\gamma \in S \mid \underbrace{tr\gamma^2 - 4 \text{ not sqfree}}\}.$$

$\exists\, p^2 \mid tr^2\gamma - 4 = (tr+2)(tr-2)$

$p > N^{\beta} \Rightarrow p^2 \mid tr \pm 2.$

$\Rightarrow p < N^{1/2}.$

$$\#\{\gamma \in S \mid \mathrm{tr}^2 - 4 \text{ not sq-free}\} \le \sum_{N^\beta < p < N^{1/2}} \sum_{t < N}' \sum_{\substack{\gamma \in \Gamma_A \cap B_N \\ \underbrace{\mathrm{tr}\gamma = t}}} 1.$$

$$\underbrace{\{t^2 - 4 \equiv 0 (p^2)\}}$$

$$\#\{\gamma \in SL_2(\mathbb{Z}) \cap B_N \mid \mathrm{tr}\gamma = t\} \ll \underbrace{N} N^\varepsilon$$

$$a + d = t, \quad d = t - a. \text{ det'd},$$

$$ad - bc = 1, \quad bc = \underbrace{ad - 1}.$$

$$\ll N^\varepsilon \underbrace{N} \sum_{N^\beta < p < N^{1/2}} \left(\frac{N}{p^2} \cancel{+1} \right).$$

$$\ll N^{2+\varepsilon} \int_{N^\beta}^{\infty} \frac{1}{x^2}\, dx \ll N^{2+\varepsilon - \beta}.$$

$$\Rightarrow \star \gg N^{2\delta_A} - O(N^{2-\beta}).$$

So: if $2\delta_A > 2 - \beta$, we win.

$\delta_A > 1 - \beta/2$. $\qquad \delta_A = 1 - \frac{6}{\pi^2 A} + \dots$

We know we can make $\delta_A \to 1$ by increasing A. But $\beta = \beta(\alpha)$, $\underline{\underline{\alpha = \alpha(\theta)}}$, & $\theta = \theta(\Gamma_A)$. $\Rightarrow \beta = \beta_A$.

"Beyond Expansion" Program (Bourgain-K)

To get exponents $\underline{\underline{\alpha}}$ beyond those coming from expansion θ.

<u>Idea</u>: Replace non-abelian harm analysis with <u>abelian</u>. Fix q.

$$\sum_{\gamma \in \Gamma_A \cap B_N} \mathbb{1}_{\{\Upsilon^2 - 4 \equiv 0(q)\}} = \sum_{\substack{t(q) \\ t^2-4\equiv 0(q)}} \sum_{\gamma \in \Gamma_A \cap B_N} \underbrace{\mathbb{1}_{\{\operatorname{tr}\gamma \equiv t(q)\}}}_{\frac{1}{q}\sum_{r(q)} e_q(r(\operatorname{tr}\gamma - t))},$$

$$= \sum_{\substack{t(q) \\ t^2-4 \equiv 0(q)}} \sum_{\gamma \in \Gamma_A \cap B_N} \frac{1}{q} \underbrace{\sum_{\mathfrak{q} \mid q}}_{\substack{\mathfrak{q} < Q_0 \\ \text{or } \mathfrak{q} > Q_0}} \sum_{r(\mathfrak{q})}{}' e_{\mathfrak{q}}(r(t\gamma - t)),$$

$$= \underline{\mathcal{M}} + \mathcal{E}(q, N).$$

<u>Need:</u> $\sum_{q < Q} |\mathcal{E}(q, N)| = o(\# \Gamma_A \cap B_N)$ with $Q = N^{\alpha}$ ← mult of θ_0

Need to save Q^{1+}.

$$= \underline{\sum_{q < Q}} \varsigma(q) \sum_{\substack{t(q) \\ t^2-4 \equiv 0(q)}} \sum_{\gamma \in \Gamma_A \cap B_N} \frac{1}{q} \sum_{\substack{\mathfrak{q} \mid q \\ \underline{Q > \mathfrak{q} > Q_0}}} \sum_{r(\mathfrak{q})}{}' e_{\mathfrak{q}}(r(t\gamma - t)),$$

$$= \underline{\sum_{Q_0 < \mathfrak{q} < Q}} \underbrace{\sum_{\gamma \in \Gamma_A \cap B_N}} \sum_{r(\mathfrak{q})}{}' e_{\mathfrak{q}}(r t\gamma) \cdot \sum_{\substack{q < Q \\ q \equiv 0(\mathfrak{q})}} \frac{\varsigma(q)}{q} \sum_{\substack{t(q) \\ t^2-4 \equiv 0(q)}} e_{q}(-rt)$$

C-S in γ: Need to save Q^2. $\oplus$

$e_{q_2}(\bar{r}'\pm r\gamma)$

$$\left(\sum_{q\in Q}|\mathcal{E}(q,N)|\right)^2 \ll N^{2\delta} \sum_{\cancel{\gamma\in\Gamma_A\cap B_N}} \sum_{q_1 q_1'} \sum_{r(q_2)}{}' \sum_{r'(q_2')}{}' e_{q_2}(r\pm\gamma)\, \xi_{q_1}$$

removed $\longrightarrow$ $\gamma\in SL_2(\mathbb{Z})\cap B_N$

$\sum_{\gamma_2}\sum_{\gamma_2'}\ldots$

A dependence in nonabelian harm analysis.

Diagonal: $q_2=q_2'$, $r=r'$ no cancellation most we could save is Q^2.

New Idea: create bilinear structure!

Change to $\displaystyle\sum_{\gamma_1\in\Gamma\cap B_X}\ \sum_{\gamma_2\in\Gamma\cap B_Y} \mathbb{1}_{\operatorname{tr}(\gamma_1\gamma_2)^2-4\equiv 0(q)}$.