Last time: primitive closed geod on $T^1(\Gamma\backslash\mathbb{H})$ $\cong \Gamma\backslash G$, $G = PSL_2(\mathbb{R})$, $\Gamma = PSL_2(\mathbb{Z})$

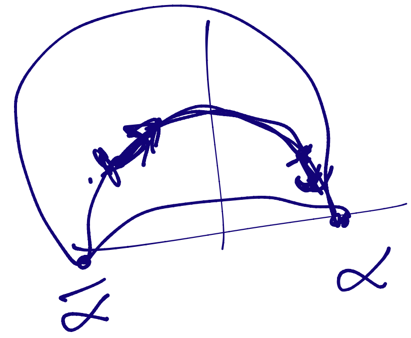
classes of indefinite binary quad forms $\leftrightarrow$ primitive conj classes of hyperbolic elements $\{M\} \subset \Gamma$.

$Q = \{A, B, C\}$, $A > 0$, $\gcd(A,B,C) = 1$.

$D = B^2 - 4AC > 0$, $\neq \square$

$g a_\ell = M \cdot g$, $g \in SL_2(\mathbb{R})$

$M\alpha = \alpha$.



$$\alpha_Q = \frac{-B \pm \sqrt{D}}{2A} \longleftrightarrow \alpha_M = \frac{-(d-a) + \sqrt{tr^2 - 4}}{2c},$$

Game: Given M find Q & vice versa.

$M \to Q$: Given $M \in PSL_2(\mathbb{Z})$, hyperbolic, primitive. Guess $B_0 = d-a$, $D_0 = (a+d)^2 - 4$, $A_0 = c$,

$$D_0 = B_0^2 - 4A_0C_0, \quad C_0 = \frac{B_0^2 - D_0}{4A_0} = \frac{(d-a)^2 - ((a+d)^2 - 4)}{4c} = \frac{4 - 4ad}{4c}$$

$C_0 = -b$.

Let $\delta = \gcd(A_0, B_0, C_0) = \gcd(c, d-a, -b)$. Then $Q = \frac{\mathrm{sgn}(tr_M)}{\delta}\{A_0, B_0, C_0\}$

$Q \to M$: Given $Q = \{A, B, C\}$, Need:
$$\begin{cases} tr^2 - 4 = Ds^2 \\ d - a = Bs \\ c = As. \end{cases}$$

$\to$ Solve (t, s) in $t^2 - Ds^2 = 4$.

Take fundamental solution.

Know $t = a+d$, $Bs = d-a$ $\Big\}$ $d = \frac{t+Bs}{2}$, $a = \frac{t-Bs}{2}$, $c = As$.

$$1 = ad - bc = \left(\frac{t-Bs}{2}\right)\left(\frac{t+Bs}{2}\right) - bAs = \frac{t^2 - B^2s^2}{4} - bAs.$$

$$bAs = \frac{\cancel{4} + s^2(\cancel{B^2} - 4AC) - \cancel{B^2 s^2}}{4} - \cancel{1} = -ACs^2$$

$b = -Cs$.

$$M = \begin{pmatrix} \frac{t-Bs}{2} & -Cs \\ As & \frac{t+Bs}{2} \end{pmatrix}.$$

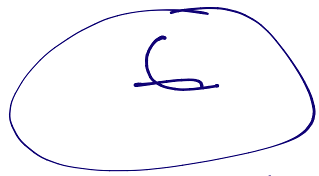
Exercise: this M is primitive.

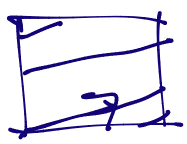
Exercise: $M(Q\circ\gamma) = \gamma\cdot M\cdot\gamma^{-1}$ & $Q(\gamma M\gamma^{-1}) = Q\circ\gamma$.

Exercise: Who found these first? Tell me please!

Compare: on sphere $K>0$: every geod closed

on torus $K=0$

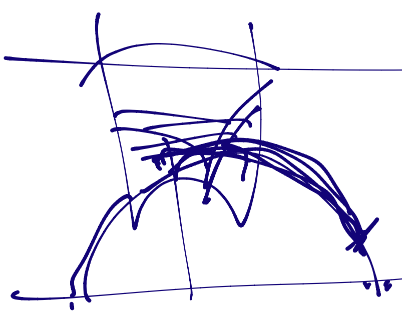




- rat'l slopes ⇝ closed orbits
- irrat dense orbits

$\mathbb{H}$, $K<0$

- closed trajectories approx any.

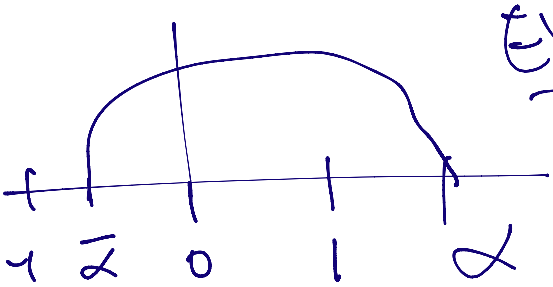


$\alpha = \{a_0, a_1, \ldots a_\ell, \ldots\}$.

Thm (Gauss): $h(D) < \infty$ $\#\mathcal{C}_D$, $D > 0$, $D \neq \square$.

Pf: Def: Q is reduced if $\alpha_Q > 1$ & $-1 < \bar{\alpha}_Q < 0$.

$-[0, \overline{a_\ell, \dots, a_1}]$



Exercise: $\Updownarrow$ $\alpha = [\overline{a_0, \dots, a_\ell}] \sim [\overline{a_2, a_3, \dots, a_0, a_1}]$

"Eating away digits" by $\underbrace{\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}}$ $\sim [\overline{a_{\ell-1}, a_\ell, a_0, \dots}]$

$\Rightarrow$ Every class has a reduced rep.

Unlike $D<0$, reduced $\not\Rightarrow$ (almost) unique. Here, lots equivalent reduced forms.

Need to show # reduced forms $< \infty$.

$$1 < \alpha_Q = \frac{-B+\sqrt{D}}{2A}, \quad -1 < \frac{-B-\sqrt{D}}{2A}, \quad \frac{-B-\sqrt{D}}{2A} < 0.$$

$\#\mathcal{C}_D \leq \#$ red forms $< \infty$. $\Rightarrow$ $B < \sqrt{D}$, ... Exercise.

(Gauss) Conj: If $D < 0$, $h(D) \to \infty$.

Proved: Deuring–Heilbronn Principle (1920s), Landau–Siegel zero... (1930s)

'76 Goldfeld + Gross–Zagier '84: effective solution.

Conj: If $D > 0$, $\overline{D \text{ fnd}}$, $h(D) = 1$ i.o.

Proved by Dirichlet for a family of nonfnd discriminant.

$$D \text{ fund} \iff \begin{cases} D \equiv 1 \ (4) & D \text{ sqfree.} \\ D \equiv 0 \ (4), & D/4 \equiv 2,3 \ (4), \text{ sq free.} \end{cases}$$

$\boxed{\text{Dirichlet Class Number Formula}}$

$D<0$: $L(1,\chi_D) \overset{\checkmark}{=} \pi \frac{h(D)}{\sqrt{D}} \approx 1.$

$D>0$: $L(1,\chi_D) = \frac{h(D) \log \varepsilon_D}{\sqrt{D}}$, $\left(\varepsilon_D = \frac{t + \sqrt{D} s}{2} \right).$

For large D, $L(1,\chi_D) \overset{?}{\approx} 1$. If $h(D) = 1$.

$\frac{\log \varepsilon_D}{\sqrt{D}} = 1$, i.e. $\sqrt{D} = \log \varepsilon_D$, $\varepsilon_D = e^{\sqrt{D}}$.

Missing: Principle for creating huge fundamental solutions to Pell eqn.

Do know how to ensure that D has $\sqrt{D+4}$. Small ε_D: $D = t^2 - 4$, $t^2 - D = 4$. (t '' 11).

$\mathcal{E}_D \approx \sqrt{D}$., If that's the case, then

$$1 \asymp \frac{h_D \log \mathcal{E}_D}{\sqrt{D}} = \frac{h_D \cdot \log \sqrt{D}}{\sqrt{D}} \Rightarrow h_D = D^{\frac{1}{2}+o(1)}.$$

(Cohen-Lenstra heuristics ...).

Thm (Duke '88): Look at closed geodesics of discriminant D. (i.e. closed geod → hyp conj class → class bin quad forms D $= \mathcal{C}_D$).

As $D \to \infty$, {closed geod} equidistribute,

i.e. $\displaystyle \frac{1}{h_D} \sum_{\gamma \in \mathcal{C}_D} \frac{1}{\ell(\gamma)} \int_\gamma \mathbb{1}_A \xrightarrow{D \to \infty} \frac{1}{\mathrm{vol}(\Gamma\backslash H)} \int_{\Gamma\backslash H} \mathbb{1}_A$.

Also, $D \to -\infty$, $\displaystyle \frac{1}{h_D} \sum_{\alpha \in \mathcal{C}_D} \mathbb{1}_{\alpha \in A}$