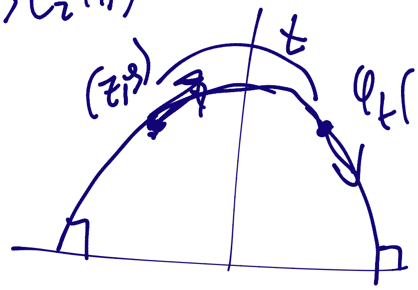
Last time: McMullen: Fix $K=\mathbb{Q}(\sqrt{6})$, $\mathcal{A}\subseteq\mathbb{N}$ finite alphabet, $\delta_{\mathcal{A}} > 1/2 \Rightarrow \{[a_0,\ldots,a_\ell]\in K \mid a_j\in\mathcal{A}\}$ grows exp'l.

Recall: $T^1\mathbb{H} \simeq G = PSL_2(\mathbb{R})$ — $g \to (gi, g\uparrow)$

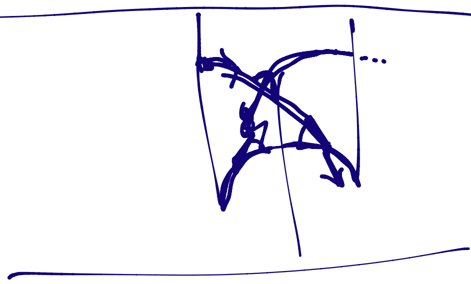
$\varphi_t \circlearrowright T^1\mathbb{H}$, φ_t = geod flow



$g(z,\zeta) = \left(\frac{az+b}{cz+d}, \frac{\zeta}{(cz+d)^2}\right)$

(Gelfand-Fomin) right-mult by $\begin{pmatrix} e^{t/2} & \\ & e^{-t/2}\end{pmatrix} =: a_t$.

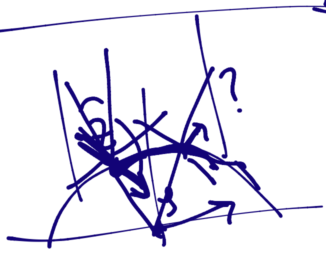
Now: $PSL_2(\mathbb{Z}) = \Gamma\backslash\mathbb{H} =: Y$



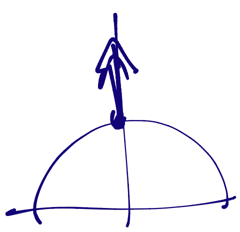
$\varphi_t \circlearrowright T^1 Y = X \cong \Gamma\backslash G$

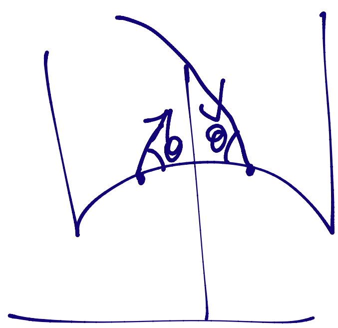
Viewpoint 1: flowing & resetting back into $\mathcal{F}$.

$\varphi_t \circlearrowright \Gamma g \in \Gamma\backslash G$



$\zeta \in \mathbb{C}$

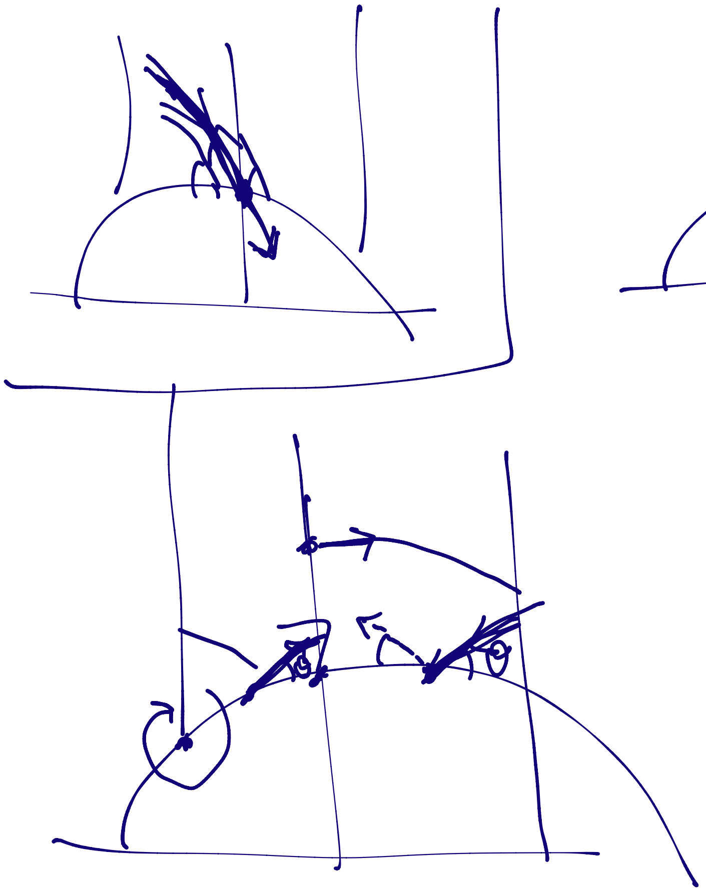


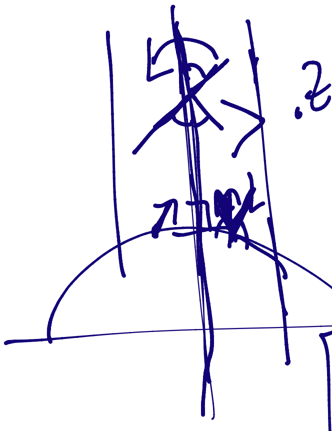


$S(z,\zeta) = \left(\frac{-1}{z}, \frac{\zeta}{z^2}\right)$

$S = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$

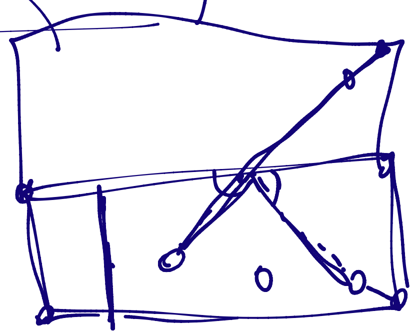
$g(i,i)$

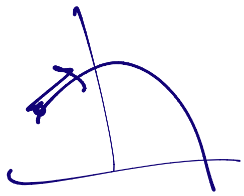


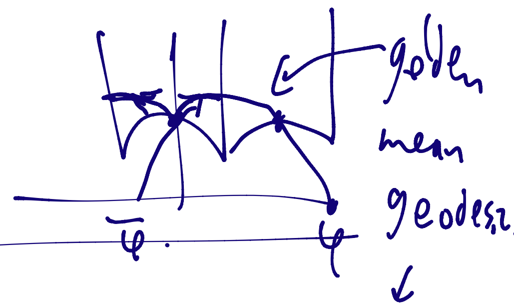


reflective $\Delta(2,3,\infty)$

$R_y : z \mapsto -\bar{z}$.





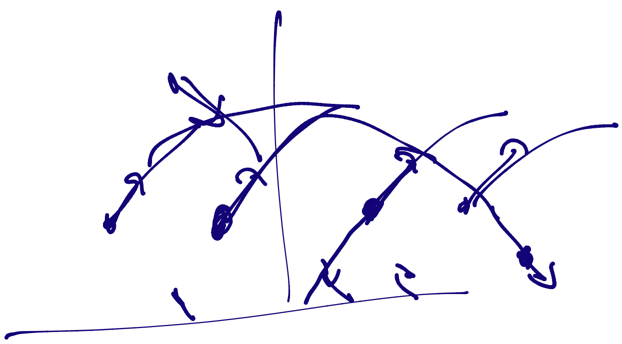


$\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Q: Can geodesic trajectory close?

$$\Gamma g a_\ell = \Gamma g$$

$$g a_\ell = \gamma g$$



If $g \simeq (z,\xi)$ is a starting pt of a closed geo,

then $\exists\, \ell > 0$, $\exists\, \gamma \in \Gamma$ s.t.

$$\gamma^2 = g a_\ell^2 g^{-1}$$

$$\Rightarrow \quad \gamma = g a_\ell g^{-1}, \quad \Rightarrow \quad \gamma = \text{hyperbolic}$$

$$\ell = 2\cosh^{-1}(\mathrm{tr}\,\gamma/2).$$

$$\mathrm{tr}\,\gamma = \mathrm{tr}\,a_\ell = e^{\ell/2} + e^{-\ell/2} = 2\cosh(\ell/2).$$

Classification of hyperbolic motions: $g \in G = SL_2(\mathbb{R})$

(hyperbolic) • g has two distinct real eigenvalues $\Leftrightarrow |\mathrm{tr}\, g| > 2$.

$\Leftrightarrow$ g has two distinct real fixed pts.

$gx = x$.

$$\frac{ax+b}{cx+d} = x$$

$$cx^2 + (d-a)x - b = 0.$$

$$x = \frac{a-d \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

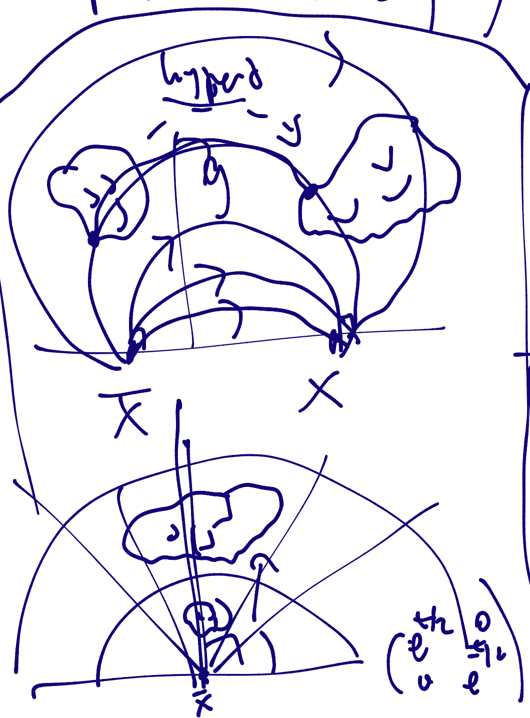
$$= \frac{-(d-a) \pm \sqrt{t^2-4}}{2c}.$$

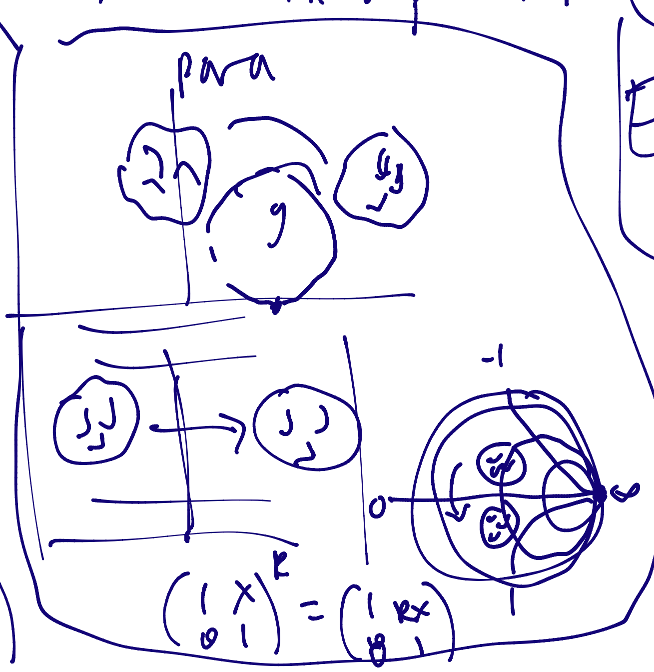
(parabolic) • $\mathrm{tr} = \pm 2$, g has double real root, g has one real fixed pt.

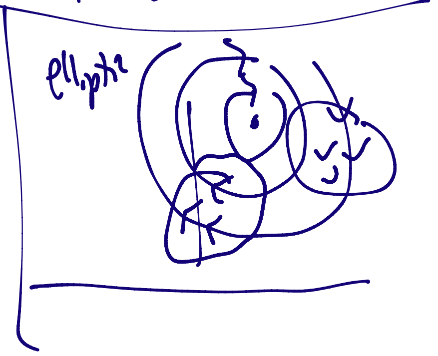
(elliptic) • $|\mathrm{tr}| < 2$, g has one fixed pt in $\mathbb{H}$.

Exercise: $\det(g - \lambda I) = 0$.

$\lambda = \ . \ -$

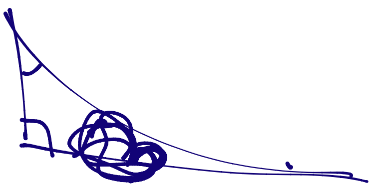






If we had started at $\xi_g^{\varepsilon r}$, $\xi_{g q_\ell} = \xi_{\underbrace{\gamma \xi^{-1}}} \xi_g$

primitive closed geodesics on $X = T^1 Y \cong \Gamma\backslash G$ $\longleftrightarrow$ primitive conj classes of hyperbolic matrices.



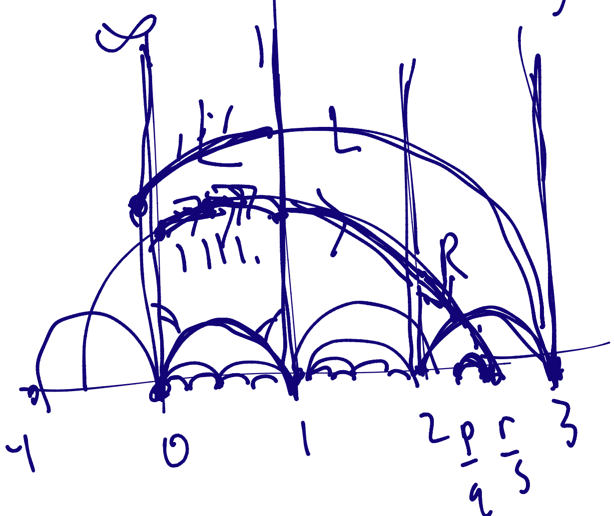
$$\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix} = \underline{\underline{\begin{pmatrix}5&3\\3&2\end{pmatrix}}}$$

Geodesic flow gives us a cutting sequence.

$$\underbrace{T^{-1}, S, T^{-1}, (T^{-1})^5, S, \ldots}$$

Series:

Farey tessellation: orbit under Γ of $(0,\infty)$



Exercise:

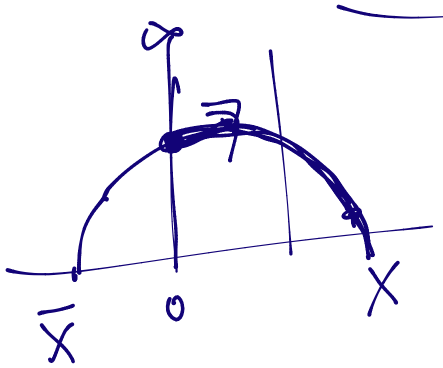
$\frac{p}{q} \frown \frac{r}{s} \iff \left|\det\begin{pmatrix}p&r\\q&s\end{pmatrix}\right| = 1.$

LLRLLLRR......

inside each ideal triangle (oriented) geodesics cuts triangle, look to which direction has 1 vertex as opposed to 2.

$\boxed{\gamma, \quad g\gamma g^{-1} = q_\ell \qquad \ell = \text{det'd.}}$

Exercise: If $[\gamma]$ prim hyper conj class with corr closed geod



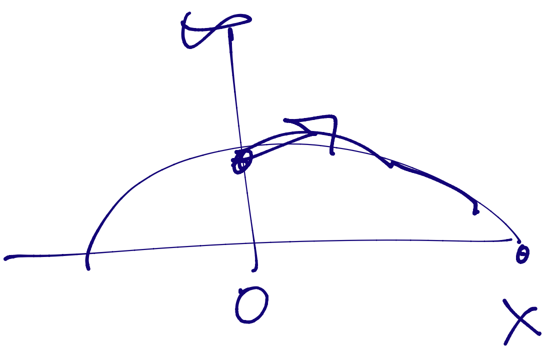
$$\underbrace{LLL}_{3}\ \underbrace{RR}_{2}\ \underbrace{LL}_{2}\ \underbrace{R}_{1}\ \underbrace{LL}_{2}$$

$$\boxed{g g_\ell = \gamma g.}$$

$x = [3, 2, \overline{2, 1, 2}, \dots]$. $\quad \gamma x = x$

$$\begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}2&1\\1&0\end{pmatrix}$$

Exercise: Why is $M = \begin{pmatrix}5&1\\3&2\end{pmatrix}$ not primitive?



$$\mathrm{Fix}(M) = \mathrm{Fix}(\gamma^2) = \mathrm{Fix}(\gamma).$$

$$x = [\overline{2,1,2,2,1,2}].$$

$$= [\overline{2,1,2}] \to \begin{pmatrix}2&1\\1&1\end{pmatrix}.$$

$$S = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$$