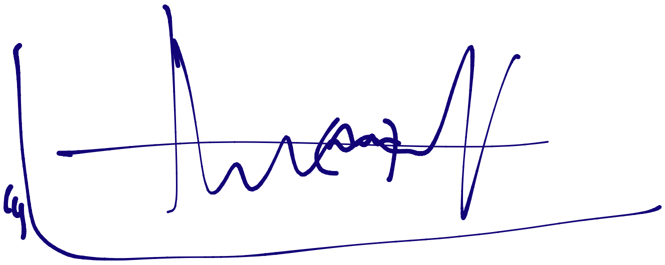
Last time: $\hat{R}_N(\theta) = \underbrace{\sum_{\gamma_1\in\Gamma_{N^{1/4}}} \sum_{\gamma_2\in\Gamma_{N^{1/4}}} \sum_{\gamma_3\in\Gamma_{N^{1/2}}}}_{\text{multilinear}} e(\theta\langle v,\gamma_1\gamma_2\gamma_3 w\rangle)$

Grouped $\overset{X}{\overset{\shortparallel}{N^{1/2}}}, \overset{Y}{\overset{\shortparallel}{N^{1/2}}}$, $\quad \sup_{\theta\in W_{Q,K}} |\hat{R}_N(\theta)| < \frac{N^{2\delta+}}{QK}$.

$W_{Q,K} = \left\{ \frac{a}{q}+\beta \;\middle|\; q\asymp Q,\ a(q)',\ |\beta|\asymp \frac{K}{N} \right\}$.



$\left(\sum_{q\asymp Q}\sum_{a(q)}' |\hat{R}_N(\theta)|\right)^2 \ll X^{2\delta} X^2$ $\quad$ C-S, $X=N^{3/4}$

$\boxed{\text{Need to save } Q^{2+}}$

$$\sum_{\substack{z,z'\in\mathbb{Z}^2\cap B_Y \\ \gamma_2 v,\ \gamma_2' v \text{ prim.}}} \sum_{q,q'\asymp Q} \sum_{a(q)}' \sum_{a'(q')}' \mathbb{1}_{\underbrace{\|\theta z-\theta' z'\|}_{\text{nearest } \mathbb{Z}^2}<\frac{1}{X}}$$

Tried: $\frac{1}{Q^2} < \frac{1}{qq'} < \left\|\frac{a}{q}z+\frac{a'}{q'}z'\right\| < \frac{K}{X}$ if $\ne 0$. got no contradiction

$\frac{1}{q} < \left\|\frac{a}{q}\mathfrak{Z}\right\| < \frac{K}{X}\cdot Y = \frac{K}{N^{1/2}}$. (if $\ne 0$) $\qquad \mathfrak{Z} = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix}$

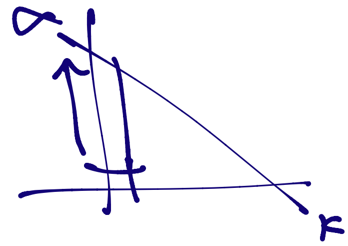
$Y=N^{1/4}$, $X=N^{3/4}$.

get contradiction, $\Rightarrow \mathfrak{Z}\equiv 0(q)$, $\Rightarrow \mathfrak{Z}\equiv 0(q') \Rightarrow \mathfrak{Z}\equiv 0([q,q'])$

$\Rightarrow \mathfrak{q} \mid z \Rightarrow q \mid \mathfrak{q} \Rightarrow q' \mid \mathfrak{q}$ saves Q^2.

as long as $z \neq 0$.

Break $\displaystyle\sum_{z_1, z'} = \sum_{\substack{z_1, z' \\ z \neq 0}} + \sum_{\substack{z_1, z' \\ z = 0}}$.



<u>Case $z \neq 0$:</u> $\mathfrak{q} \mid z \Rightarrow \mathfrak{q} \le z \ll Y^2$ $\quad \boxed{Y \sim N^{1/4}}$

$\mathfrak{q} \ll Q^2 \Rightarrow \mathfrak{q}^2 \ll Y^2 Q^2$, $\mathfrak{q} \ll YQ$.

$Q < N^{1/2} = M$

$QK < N^{1/2}$

$= \frac{X}{Y}$.

Back to

$$\frac{1}{YQ} < \frac{1}{\mathfrak{q}} < \left\| \frac{a z_1}{q} - \frac{a' z_1'}{q'} \right\| < \frac{K}{X}$$

if $z \neq 0$.

$\Rightarrow \frac{K}{X} < \frac{1}{YQ}$.

$$\Rightarrow \left(\frac{a z_1}{q} \equiv \frac{a' z_1'}{q'} \bmod 1 \right) \mathfrak{q}.$$

Let $\tilde{q} = (q, q')$, write $q = \tilde{q}\, q_1$
$\quad q' = \tilde{q} \cdot q_1'$, $\quad (q_1, q_1') = 1$.

$$\mathfrak{q} = \underbrace{q_1 \tilde{q}}_{q}\, q_1' \quad \left(\tilde{q}\, q_1' = q'\right)$$

$\Rightarrow$ clear denominators $a z_1 q_1' \equiv a' z_1' q_1 \pmod{\mathfrak{q}}$.

$\Rightarrow$ look mod q_1: $\underset{\uparrow}{a} z_1 \underset{\nwarrow}{q_1'} \equiv 0 \pmod{q_1}$.
$(a,q)=1$

$\Rightarrow z_1 \equiv 0\ (q_1)$ Same $\Rightarrow$ $z_2 \equiv 0\ (q_1)$.

But $(z_1, z_2) = 1$ (z primitive!) $\Rightarrow$ $q_1 = 1$.

Same $\Rightarrow$ $q_1' = 1$. $\Rightarrow \mathfrak{q} = q = q'$! No extra savings! Already had $q \mid \mathfrak{q}$, $q' \mid \mathfrak{q}$, $\mathfrak{q} \mid Z \neq 0$.

$\Rightarrow$ $a z_1 \equiv a' z_1' \pmod{q}$. For fixed a, (z_1, z_1') determines a' mod $q/(q, z_1')$.

$\Rightarrow$ $a z_2 \equiv a' z_2' \pmod{q}$ $\Rightarrow$ a' det'd mod $q/(q, z_2')$

But $(z_1', z_2') = 1$ $\Rightarrow$ a' det'd mod q.

Saves remaining extra factor of Q!

Summarize: $\displaystyle \sum_{\substack{z, z' \\ Z \neq 0}} \cdots \le \sum_{z} \sum_{\substack{z' \\ Z \neq 0}} \sum_{\substack{\mathfrak{q} \mid Z \\ \mathfrak{q} \ll \gamma Q}}' \sum_{\substack{q \mid \mathfrak{q} \\ q' = q < Q}}' \sum_{a(q)}'' \cancel{\sum_{a'(q)}'}_{a' \text{ det'd}} 1.$

$$\lesssim Y^{2\delta} Y^{2\delta} \cdot N^{\varepsilon} \cdot N^{\varepsilon} \cdot Q \; 1$$

Saved $Q^3 > Q^{2+}$.

<u>Case $Z=0$:</u> $z_1 z_2' = z_2 z_1' \Rightarrow \dfrac{z_1}{z_2} = \dfrac{z_1'}{z_2'}$.

$\Rightarrow z' = z$. Saves $Y^{2\delta} = N^{\delta/2}$ not even $N^{1/2}$

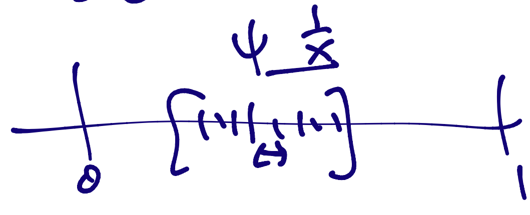
need Q^{2+} savings, could be as large as N^{1+} ... $\boxed{\delta \to 1}$.

$$\left\| \frac{a}{q} z - \frac{a'}{q'} z \right\| \le \| \theta z - \theta' z \| + \cancel{|\beta| \, |z - z|}.$$

$$\left\| \frac{a}{q} z_1 - \underbrace{\frac{a'}{q'} z_1}_{\psi} \right\| < \frac{1}{X}$$

Say fix ψ, i.e. fix z_1, a', q'. Need restrict how many possible values there are of $\frac{a}{q} z_1$ so close to ψ.

<u>Back of envelope calculation</u> to see if we have a chance. Say <u>$(z_1, q) = 1$</u>.



mesh size $\simeq \frac{1}{Q^2}$. ← distance to nearest neighbor among $\frac{a z_1}{q}$-values

values can fit: $Q^2 \cdot \frac{1}{X}$, each such fraction determines a & q. (should have had Q^2 values) Saves $N^{3/4} = X$.

For real: let $(q, z_1) = v$, $(q', z_1) = v' \mid z_1$

WLOG

Assume $v' \geq v$, write $q = v \cdot r$, $(r, z_1) = 1$, $z_1 = v \cdot b$.

$\| \frac{a \cdot b}{r} - \psi \| \leq \frac{1}{X}$. Look at $\left\{ \frac{a \cdot b}{r} \,\middle|\, \begin{array}{l} r \simeq \frac{Q}{v} \\ a \bmod q, \\ (a, r) = 1 \end{array} \right\}$ fe $\mathcal{g}$ = $(b, r) = 1$ fixed.

Mesh of size $\frac{1}{r \cdot r'} > \frac{v^2}{Q^2}$.

$\#\{$such within $\frac{1}{X}$ of $\psi\} < \frac{1}{X} \cdot \frac{Q^2}{v^2} + 1$

Fix

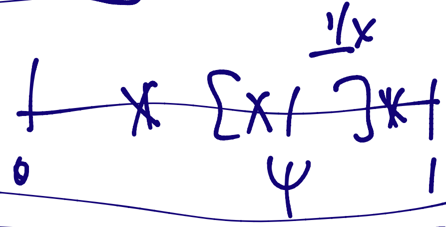
Summary:

$$\sum_{\substack{z, z' \\ z = 0}} \cdots \leq \sum_{z} \sum_{z' = z.} \sum_{v \mid z_1}' \sum_{q' \simeq Q} \sum_{a'(q'), \psi = \frac{a'}{q'} z_1}' \sum_{f \in \mathcal{g}}' \sum_{\substack{r, a \\ \text{determines } f.}} \sum_{q = r \cdot v} 1$$

$$\le Y^{2\delta}\cdot 1 \cdot N^{\varepsilon}\cdot Q \cdot Q\left(\frac{1}{X}\frac{Q^2}{\cancel{Y^{2\delta}}}+1\right)1\cdot 1$$

saves $Q^2\cdot Y^{2\delta}$.

saves $X\cdot Y^{2\delta}$.



$$\left(\sum_{q\sim Q}\sum_{a(q)}{}' |\hat{R}_N(\theta)|\right)^2 \ll \cancel{X}\, Y^{2\delta}\cdot X^2\left[Q Y^{4\delta} + Y^{2\delta}\frac{Q^4}{X} + Y^{2\delta}Q^2\right]$$

N^{ε}

$$\ll N^{\varepsilon}\cdot N^{4\delta}\underbrace{N^{\frac{3}{2}(1-\delta)}}_{N^{\varepsilon}} Q^4\left[\frac{1}{Q^3}+\frac{1}{Y^{2\delta}X}+\frac{1}{Q^2Y^{2\delta}}\right]$$

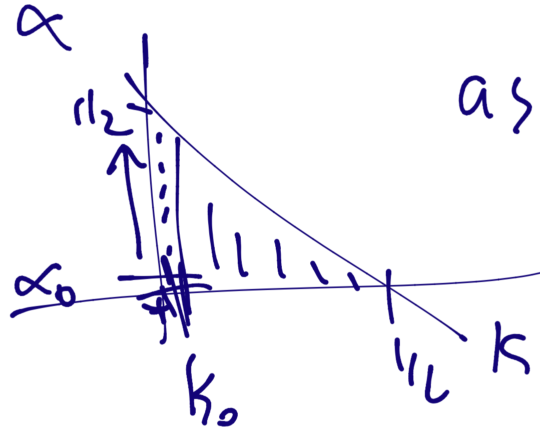
need $= o\left(\frac{N^{4\delta}}{N}\right)$.

$$\int_{W_{Q,K}} |\hat{R}_N(\theta)|^2 d\theta < \int_{|\beta|\asymp \frac{K}{N}} \sup_{\theta\in W_{Q,K}} |\hat{R}_N(\theta)| \sup_{\theta}\left(\sum_{q\sim Q}\sum_{a(q)}{}' |\hat{R}_N(\theta)|\right) d\beta.$$

$$\le \frac{N^{\varepsilon}\cancel{K}}{N}\,\frac{N^{2\delta}}{\cancel{KQ}}\,N^{2\delta}Q^{\cancel{2}}\left[\frac{1}{Q^{3/2}}+\frac{1}{Y^{\delta}X^{1/2}}+\frac{1}{QY^{\delta}}\right].$$

$$\ll \frac{N^{4\delta+\varepsilon}}{N}\left[\frac{1}{Q^{1/2}}+\frac{Q}{N^{(3+2\delta)/8}}+\frac{1}{N^{\delta/4}}\right].\quad \boxed{Q<N^{1/4}}$$

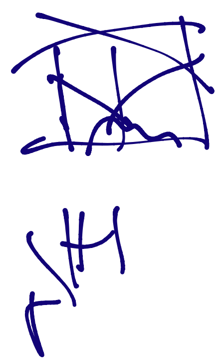
$$\ll \frac{N^{4\delta}}{N}\left[\frac{1}{Q^{1/2}} + \frac{1}{N^{(2\delta-1)/8}} + \frac{1}{N^{\delta/4}}\right].$$ ← savings.



as long as $d > Q_0 \doteq N^{\alpha_0}$

& $\delta \to 1$,

Bourgain–K '10: $\Gamma < SL_2(\mathbb{Z})$ thin group. δ near 1, no parabolics.



fix $\langle v, \Gamma w\rangle$ satisfy density 1 loc-glob.

i.e. Set of (1,1) entries of Γ

$$\frac{\#\text{ such} < N}{\#\text{ admiss} < N} \to 1.$$

Controls length of corr closed geo.

Problem: Replace $\gamma \mapsto \langle v, \gamma w\rangle$ by $\gamma \mapsto \operatorname{tr}\gamma$. ← don't know pos prop. (K–X. Zhang '22)