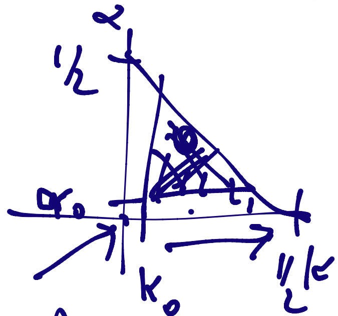
Last time:
$$\sup_{\theta \in W_{Q,K}} |\hat{R}_N(\theta)| < \frac{N^{2\delta} \cdot N^{2(1-\delta)}}{K \cdot Q}.$$

$Q = N^{\alpha}$, $K = N^{\kappa}$

$$W_{Q,K} = \left\{ \theta = \frac{a}{q} + \beta \;\middle|\; q \asymp Q,\ (a,q)=1,\ |\beta| \lesssim \frac{K}{N} \right\}$$



$$\int_{W_{Q,K}} |\hat{R}_N|^2 < \frac{N^{4\delta + \varepsilon}\, \cancel{Q^2} K}{K^2 \cancel{Q^2} N}$$

$XY = N$

Major Arcs $\Rightarrow$ $M_N(n) \gg N^{2\delta - 1}$

$\boxed{\text{Need: } \sum_n |\mathcal{E}|^2 = \int |\hat{R}|^2 = o(N^{4\delta - 2} \cdot N)}$

$v = (1,0)$

$$\hat{R}_N(\theta) = \sum_{\gamma_1 \in \Gamma_X} \sum_{\gamma_2 \in \Gamma_Y} e(\theta \langle v, \gamma_1 \gamma_2 v \rangle).$$

Try 1st refinement:

$$\int_{|\beta| \lesssim \frac{K}{N}} \underbrace{\sum_{q \asymp Q} \underbrace{\sum_{a(q)}{}' |\hat{R}_N(\theta)|}}\ \cdot \sup_{\theta \in W_{Q,K}} |\hat{R}_N(\theta)|$$

$\frac{K}{N} Q\, N^{2\delta}$ $\sum_a |\hat{R}_N|$

KQ $Q\, N^{2\delta}$

Need to save Q^{1+}.

Fix q, β.

$$(*) = \sum_{a(q)}{}' \underset{\substack{\uparrow \\ 1\cdot 1 = 1}}{\zeta_a} \cdot \underbrace{\sum_{\gamma_1 \in \Gamma_X}} \sum_{\gamma_2 \in \Gamma_Y} e\left(\theta \frac{\langle \gamma_1^{+} v, \gamma_2 v \rangle}{\langle v, \gamma_1 \gamma_2 v \rangle}\right)$$

$$\zeta_a = \frac{\hat{R}_N(\theta)}{|\hat{R}_N(\theta)|}.$$

C-S $\leq \left(\sum_{\gamma_1 \in \Gamma_X} 1^2 \right)^{1/2} \left(\sum_{\gamma_1 \in \Gamma_X} \left| \sum_{a(q)}{}' \zeta_a \sum_{\gamma_2 \in \Gamma_Y} e(\ldots) \right|^2 \right)^{1/2}.$

$\leftarrow$ need to save Q^{2+}

$$\le X^{\delta}\Big[\sum_{a,a'}\sum_{\gamma_2,\gamma_2'}\Big|\cancel{\sum\sum_{a,a'}}\ \overset{?,0}{\sum_{y\in\mathbb{Z}^2}} V\Big(\frac{y}{X}\Big)e\big(\theta\langle y,\gamma_2 v\rangle-\theta'\langle y,\gamma_2' v\rangle\big)\Big|^2\Big]^{1/2}$$

$$X^2\cancel{\sum_{k\in\mathbb{Z}^2}}\int_{y\in\mathbb{R}^2}V\Big(\frac{y}{\cancel{X}}\Big)e\big(\langle y,X(\theta\gamma_2 v-\theta'\gamma_2' v)\rangle\big)e(Xk\cdot y)\,dy \qquad \boxed{\operatorname{supp}\hat V\subset[-1,1]}.$$

$y\mapsto yX.$

$$\hat V\big(X[(\theta\gamma_2-\theta'\gamma_2')v_2-k]\big).$$

$$\ne 0 \Rightarrow \|(\theta\gamma_2-\theta'\gamma_2)v\|<\frac{1}{X}.$$

$$\Big\|\frac{a}{q}\gamma_2 v-\frac{a'}{q}\gamma_2' v\Big\|\le\|\theta\gamma_2 v-\theta'\gamma_2 v\|+|\beta(\gamma_2 v-\gamma_2' v)|$$

$<\frac{1}{q}$ if $\ne 0$. $\qquad \le \frac{1}{X}+\frac{K}{N}\cdot Y<\frac{K}{X}.$

$<\frac{1}{Q}$

$$QK\le N^{1/2}=X. \Rightarrow a\gamma_2 v\equiv a'\gamma_2' v\ (q).$$

$$\circledast\le X^{\delta}X\Big[\sum_{a,a'}\sum_{\gamma_2}\Big(\sum_{\substack{\cancel{\gamma_2'}\\ z'\in\mathbb{Z}^2\cap B_Y}}\mathbb{1}_{a\gamma_2 v\equiv a'z'(q)}\ \mathbb{1}_{|\gamma_2 v-z'|<\frac{N}{XK}}\Big)^2\Big]^{1/2}$$

$$\cancel{Q}^2\,Y^{2\delta}\Big(\Big(\frac{Y}{K}\Big)^2\frac{1}{\cancel{Q}^2}+Q^2\Big)^{1/2}$$

$$\lesssim X^\delta X Y_Q^\delta Y\left(\frac{Y}{QK} + \frac{Q}{Y}\right). = \underline{\underline{N^{2\delta}}}\, N^{(1+\delta)}\, \underline{\underline{Q}}\left(\frac{1}{QK} + \frac{1}{Y}\right).$$

Need $\underline{\underline{N^{2\delta}}}\, \bar{Q}$.

2nd refinement: $\underline{\text{Fix } \beta}$. just not enough!

$$\sum_{q\sim Q} \sum_{a(q)}{}' \, \xi_{q,a} \underbrace{\sum_{\gamma_1\in\Gamma_X}} \sum_{\gamma_2\in\Gamma_Y} e(\theta\langle v, \gamma_1\gamma_2 v\rangle).$$

$$\int_{|\beta|<\frac{K}{N}} \underbrace{\sup \hat{R}_N} \cdot \underbrace{\sum_q\sum_a |\hat{R}|}.$$

$$\frac{K}{N} \cdot \frac{N^{2\delta}}{KQ} \cdot Q^2 N^{2\delta}.$$

Need to save Q^{1+}.

$$\overset{\text{C-S}}{\lesssim} X^\delta\left[\sum_{y\in\mathbb{Z}^2} \Upsilon\left(\frac{y}{X}\right) \sum_{q,q'} \sum_{a(q)} \sum_{a'(q')}{}^{\xi\bar{\xi}} \sum_{\gamma_2} \sum_{\gamma_2'} e(\langle y, \theta\gamma_2 v - \theta'\gamma_2' v\rangle).\right]^{1/2}$$

$$\leq X^\delta X\left[\sum_{q,q'} \sum_{a(q)} \sum_{a'(q')} \sum_{\gamma_2,\gamma_2'} \mathbb{1}_{\|\theta\gamma_2 v - \theta'\gamma_2' v\| < \frac{1}{X}}\right]^{1/2}.$$

$$\frac{1}{qq'} < \left\|\frac{a}{q}\underset{\frac{2}{z}}{\gamma_2 v} - \frac{a'}{q'}\underset{\frac{2}{z'}}{\gamma_2' v}\right\| \leq \underbrace{\|\theta\gamma_2 v - \theta'\gamma_2' v\|}_{\frac{1}{X}} + \underbrace{|\beta(\gamma_2-\gamma_2')v|}_{< \frac{K}{N}\cdot Y} < \frac{K}{X}.$$

$qq' = \frac{1}{Q^2}$ could be as small as $\frac{1}{N}$.

No contradiction!

$Q < N^{1/2}$. $QK < N^{1/2}$.

Key idea: make crucial use of multilinearity.

Write $z = \gamma_2 v \in \mathbb{Z}^2$, $z' = \gamma_2' v$.

$\gcd(z_1, z_2) = 1$ $\qquad (z_1', z_2')$.

$$\left\|\frac{a}{q} z_1 z_2' - \frac{a'}{q'} z_1' z_2'\right\| + \left\|\frac{a}{q} z_2 z_1' - \frac{a'}{q'} z_2' z_1'\right\|$$

$$\geq \left\|\frac{a}{q} Z\right\|, \quad Z = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} = z_1 z_2' - z_1' z_2 \in \mathbb{Z}.$$

$$\leq |z_2'| \, \|\theta z_1 - \theta' z_1'\| + |z_2'| \, |\beta z_1 - \beta z_1'| < \frac{Y}{X} + \underbrace{\frac{Y^2 K}{N}}_{< \frac{K}{N^{1/2}}}$$

$$+ |z_1'| \, \|\theta z_2 - \theta' z_2'\| + |z_1'| \, |\beta| \, |z_2 - z_2'|.$$

if $\neq 0$, $\geq \frac{1}{q} > \frac{1}{Q}$. $\qquad KQ < N^{1/2}$, $\frac{K}{N^{1/2}} < \frac{1}{Q}$.

What if $\frac{Y^2}{N} = \frac{1}{N^{1/2}}$, i.e. $Y = N^{1/4}$, $X = N^{3/4}$.

Change $\hat{R}_N(\theta) = \sum_{\gamma_1 \in N^{1/4}} \sum_{\gamma_2 \in N^{1/4}} \sum_{\gamma_3 \in N^{1/2}} e(\theta \langle v, \gamma_1 \gamma_2 \gamma_3 v \rangle)$.

?

Truth: $$\hat{R}_N(\theta) = \underbrace{\sum_{\approx 1}}\ \underbrace{\cdots\ \sum_{N^{1/16}}}\ \overbrace{\underbrace{\sum_{N^{1/8}} \sum_{N^{1/4}} \sum_{\gamma \in N^{1/2}}}}^{\log\log N.}$$

Summarize: $$\hat{R}_N(\theta) = \sum_{N^{1/2}} \sum_{N^{1/2}} = \sum_{\underline{N^{1/4}}} \sum_{\underline{N^{3/4}}}.$$

$$Z = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} \equiv 0\,(q), \quad \& \equiv 0\,(q'), \quad \Rightarrow \equiv 0\,(\underset{\underset{[q,q'].}{\parallel}}{\sigma})$$

$$\left(\sum_{q \sim Q} \sum_{a(q)}' |\hat{R}_N|\right) \le X^{\delta} X \left[\sum_{q,q'} \sum_{a,a'} \sum_{\substack{z \\ \parallel \\ \gamma_2 v}} \sum_{\substack{z' \\ \parallel \\ \gamma_2' v}} \mathbb{1}_{\substack{Z \equiv 0(\sigma) \\ \|\theta z - \theta' z'\| < \frac{1}{X}.}}\right]^{1/2}$$