Last time: $\mathcal{A} \subset \mathbb{N}$ (finite?) alphabet,

$R_{\mathcal{A}} = \left\{ \frac{b}{d} = [0; a_1, \dots, a_\ell] \mid a_j \in \mathcal{A} \right\}$, $D_{\mathcal{A}} = \left\{ d : \exists (b,d)=1, \frac{b}{d} \in R_{\mathcal{A}} \right\}$.

Def: n is admissible (for $\mathcal{A}$) if $n \in D_{\mathcal{A}} \pmod q$, $\forall q \geq 1$.

Def: $\Gamma_{\mathcal{A}} = \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} : a \in \mathcal{A} \right\rangle^{+} \cap SL_2$. $(\mathcal{A} = \{1,2\})$ all are admissible.

$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & * \\ b & * \end{pmatrix}$.

Hensley / B-K Conj: If $\delta_{\mathcal{A}} > 1/2$ & n is admissible (finite primes obstructions) & $n \gg_{\mathcal{A}} 1$ $\leftarrow$ (∞ prime obstructions) Then $n \in D_{\mathcal{A}}$.

(Huang: if $\delta_{\mathcal{A}} > 0.7$, all n are admissible.)

Thm B-K: If $\delta_{\mathcal{A}} > 0.999$, $\frac{1}{N} \# D_{\mathcal{A}} \cap [1,N] \to 1$, as $N \to \infty$.

Idea 1: Circle Method: make a "representation number" = multiplicity $= m_{\mathcal{A}}(n)$.

$$R_N(n) = \sum_{\gamma \in \Gamma_{\mathcal{A}} \cap B_N} \mathbb{1}_{\{n = \langle v\gamma, v \rangle\}}. \qquad v = (1,0) = e_1$$

$\langle v\gamma, v\rangle = d_\gamma$, $\gamma = \begin{pmatrix} d & * \\ * & * \end{pmatrix}$. $(n \asymp N)$ $n < N$.

$R_N(n) \neq 0 \Rightarrow n \in \mathcal{D}_A.$

$$R_N(n) = \int_0^1 \hat{R}_N(\theta)\, e(-n\theta)\, d\theta, \text{ where}$$

$$\hat{R}_N(\theta) = \sum_{n \in \mathbb{Z}} R_N(n)\, e(n\theta) = \sum_{\gamma \in \Gamma_A \cap B_N} e(\theta \langle v\gamma, v\rangle)$$

$$= \sum_{\gamma \in \Gamma_A \cap B_N} \sum_{\cancel{n \in \mathbb{Z}}} \mathbb{1}_{n = \langle v\gamma, v\rangle} \cdot \cancel{e(n\theta)} \cdot \underbrace{e(\theta\langle v\gamma, v\rangle)}.$$

Key idea of circle method: Capture $\int_0^1 = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}$

(Hardy-Ramanujan Partition Function)

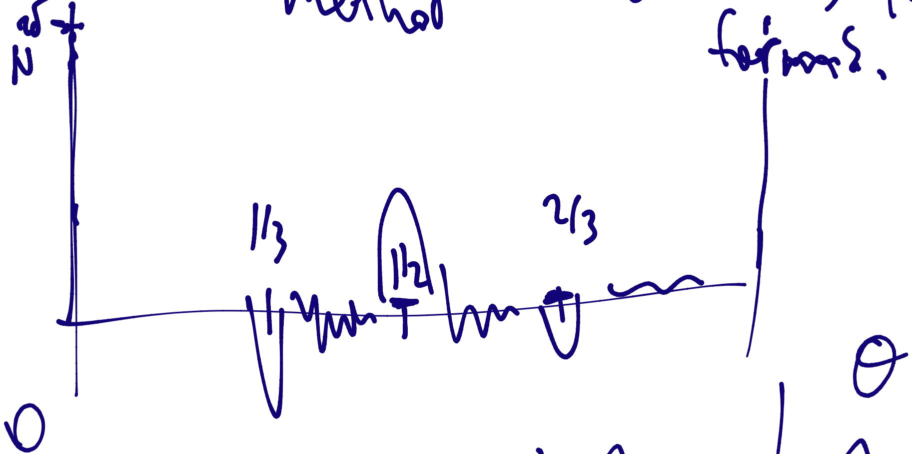
Major arcs "signal"

minor arcs. "noise"

$p(5) = 1+1+1+1+1$ $\quad p(n)$
$\quad 2+1+1+1$
$\quad 2+2+1 \quad 3+2$
$\quad 3+1+1$
$\quad 4+1$
$= 6$

Hardy-Littlewood / Kloosterman's method — Waring's Problem / quaternary quadratic forms.

Re $\hat{R}_N(\theta)$



$\theta = \frac{a}{q} + \beta$

"small" "small"

$$\hat{R}_N(0) = \#\, \Gamma_A \cap B_N \approx N^{2\delta_A}, \quad |\hat{R}_N(\theta)| \leq \hat{R}_N(0).$$

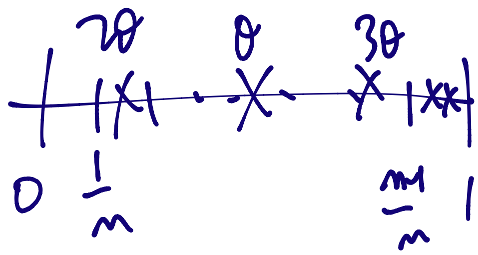
Main term "should" come from Major Arcs

Comprise tiny proportion of circle.

Most of circle (minor arcs) $\to$ noise.

Dirichlet Approx Thm: (Pigeonhole!) $\forall M \geq 1$, $\forall \theta \in \mathbb{R}$, $\exists\, q \leq M$ $\exists\, (a,q)=1$ s.t. $\left|\theta - \frac{a}{q}\right| \leq \frac{1}{qM}$.

pf: Look at $\underbrace{0, \theta \bmod 1, 2\theta, \ldots, M\theta}$ mod 1.



$M+1$ values

M bins. $\exists$ bin with two values

$\exists\, k, l < M$, m, $|\underbrace{k\theta - l\theta}_{q\theta} - m| < \frac{1}{M}$, $\left|\theta - \frac{a}{q}\right| < \frac{1}{qM}$.

Back to $R_N(n) = \int_0^1 \hat{R}_N(\theta) e(-n\theta)\, d\theta = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}$, $\mathfrak{m} := [0,1] \setminus \mathfrak{M}$

$= \mathfrak{M}_N(n) + \mathcal{E}_N(n)$.

Conj: Later: $(n<N)$ $\mathfrak{M}_N(n) \sim \frac{2\delta}{n} \#\{\Gamma_A \cap B_n\} \cdot \zeta(2) \cdot \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$.

Γ_A in L^∞ ball B_n, $\begin{pmatrix} d & \ast \\ \alpha & x \end{pmatrix}$ largest element

$\mathrm{Mult}(n) = \partial\,\Gamma_A$ at n.

$\# \Gamma_A \cap B_N = c \cdot n^{2\delta}$

Proves: $\mathcal{M}_N(n) \gg \dfrac{N^{2\delta}}{N} \cdot \underbrace{\mathfrak{S}(n)}_{\text{"singular series"}} \gg \dfrac{N^{2\delta}}{N}$. for admissible n.

$\neq 0 \iff n$ admissible

If you could show that $|\mathcal{E}_N(n)| = o\left(\dfrac{N^{2\delta}}{N}\right)_{N \to \infty}$ that's whole conjecture! Claim: For density-1, enough to do this "on average":

$$\Rightarrow \sum_{n<N} |\mathcal{E}_N(n)|^2 = o\left(\frac{N^{4\delta}}{N^2} \cdot N\right)$$

(If $\delta \to 1$).

$\mathcal{M}_N(n) = -\mathcal{E}_N(n)$

pf (Claim): look at $\#\{ n \le N \mid n \text{ admissible but } R_N(n) = 0 \}$

$$\le \sum_{\substack{n<N \\ \text{adm}}} \mathbb{1}\{ |\mathcal{E}_N(n)| \ge |\mathcal{M}_N(n)| \} \le \sum_{n<N} \mathbb{1}\left\{ |\mathcal{E}_N(n)| \gg \frac{N^{2\delta}}{N} \right\}.$$

$$\le \sum_{n<N} \cancel{\mathbb{1}\left\{ |\mathcal{E}_N(n)| > \frac{N^{2\delta}}{N} \right\}} \cdot \left(\frac{|\mathcal{E}_N(n)|}{N^{2\delta-1}} \right)^2 = \frac{N^2}{N^{4\delta}} \sum_{n<N} |\mathcal{E}_N(n)|^2$$

$\Leftarrow$ $\#\{n < N \mid \text{fail local-global}\} = o(N)$. $= o\left(\frac{N^{\delta}}{\cancel{N^{\ldots}}} \cdot \frac{\cancel{N^{\ldots}}}{\cancel{N}} \right) = o(N)$.

$$\sum_n |\mathcal{E}_N(n)|^2 = \int_{\mathfrak{m}} |\widehat{R}_N(\theta)|^2 \, d\theta.$$

$$\mathcal{E}_N(n) = \int_0 \underbrace{\mathbb{1}_{\mathfrak{m}}(\theta)\, \widehat{R}_N(\theta)\, e(-n\theta)}\, d\theta.$$

$\boxed{\text{choose } \underline{M = N^{1/2}}}$

$\underline{\text{Pigeonhole:}}$ M, $\forall \theta$ $\exists$ expression $\theta = \frac{a}{q} + \beta$, $q < M$, $|\beta| < \frac{1}{qM} \approx \frac{1}{M^2}$.

$$\widehat{R}_N(\theta) = \sum_{\gamma \in \Gamma_A \cap B_N} e(\theta \langle v\gamma, v \rangle) = \sum_{\gamma \in \Gamma_A \cap B_N} e\left(\left(\frac{a}{q} + \beta\right) \langle v\gamma, v \rangle \right)$$

$1/N$ in $\mathfrak{m}$.

$$= \sum_{\gamma_0 \in \underline{SL_2(q)}} \sum_{\underline{\substack{\gamma \in \Gamma_A \cap B_N \\ \gamma \equiv \gamma_0 (q)}}} \boxed{e_q\left(a \underbrace{\langle v, \gamma_0 v \rangle}_{\in \mathbb{Z}}\right)} \cdot \underbrace{e\left(\overbrace{\beta}^{1/N} \overbrace{\langle v, \gamma v \rangle}^{N}\right)}.$$

$\underbrace{SL_2(\mathbb{Z}/q\mathbb{Z})}_{\text{modular}}$

archimedean.

$$\sum_n |\mathcal{E}_N(n)|^2 = \int_{\mathfrak{m}} |\hat{R}_N(\theta)|^2 \, d\theta \le \sum_{\substack{Q < M = N^{1/2} \\ \text{dyadic}}}^{\# \ll \log N} \ \sum_{\substack{K < \frac{N^{1/2}}{Q} \\ \text{dyadic}}} \int_{W_{Q,K}} |\hat{R}_N(\theta)|^2 \, d\theta$$

$$\mathfrak{m} = \left\{ \theta = \frac{a}{q} + \beta \right\} \subseteq \bigcup_{\substack{Q > Q_0 \text{ or} \\ K > K_0}} W_{Q,K},$$

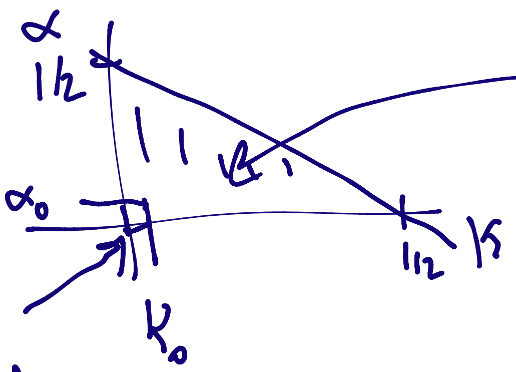
where $W_{Q,K} = \left\{ \theta = \frac{a}{q} + \beta \;\middle|\; q \sim Q,\ (a,q) = 1,\ |\beta| \sim \frac{K}{N} \right\}.$

$$\frac{K}{N} = |\beta| = \left| \frac{a}{q} - \theta \right| < \frac{1}{qM} \simeq \frac{1}{Q \cdot N^{1/2}}, \quad K < \frac{N^{1/2}}{Q}$$

$$\boxed{\begin{array}{c} f \sim g \\ \Updownarrow \\ f \ll g \ll f \end{array}}$$

$Q = N^{\alpha}$, $\alpha < 1/2$, $K = N^{\kappa}$, $\kappa + \alpha < 1/2 \Leftarrow KQ < N^{1/2}$.

$[0,1] \to$



minor arcs.

$Q_0 = N^{\alpha_0}$

$K_0 = N^{\kappa_0}$ Major arcs.