Last time: inverse map $a \mapsto a^{-1}(p)$, random Kloosterman sum: $S_p(a,m) = \sum_{r \in \mathbb{F}_p} e_p(ar^m)$

Multiplication: $a \mapsto a \cdot b \mod p$.

Saw: "quality" of randomness depended on

$$\frac{b}{p} = \frac{1}{a + \frac{1}{a} + \frac{1}{a^2} + \cdots}$$

The Zaremba '69: $D_B(r\left(\frac{a}{p}, \frac{b \cdot a}{p}\right)_{\mod p}) \ll \frac{A}{\log p}$ (p ordinary)

But $A = A\left(\frac{b}{p}\right)$ is not a constant!
Conj Zassenhaus '72: \( \exists A \subseteq A \) s.t. \( \forall d \geq 1 \\
\exists (b,d) = 1 : \frac{b}{d} \in \langle 0, a_1, \ldots, a_k \rangle \), & all \( a_j \in A \).

Conj: \( A = 5 \). \( A + 1 \). Look at \( d = 6 \). \( 6 \mid 5 \), \( \frac{5}{6} \in \langle 0, 1, 5 \rangle \). \( \sqrt{6} \in D_4 \).

Let \( P_A = \{ \frac{b}{d} \in \langle 0, a_1, \ldots, a_k \rangle \mid a_j \in A \} \).

Let \( D_A = \{ d \geq 1 \mid \exists (b,d) = 1, \frac{b}{d} \in P_A \} \).

Conj: \( D_5 = \mathbb{N} \geq 1 \). Look at \( P_A \) graded by \( d \).
To study this, consider $L(s) = \sum_{A} \frac{\lambda_{A}(d)}{d^{s}}$. 

\[ m_{A}(d) = \text{multiplicity of } \mathbb{Z}/d\mathbb{Z} \begin{cases} 1, & d \in \mathbb{R}_{A} \end{cases} \]

Then, \[ \sum_{d \in \mathbb{Z}} m_{A}(d) = \# \mathbb{R}_{A}(N). \]

**Landau's Lemma:** Given Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_{n}}{n^{s}}$, $a_{n}>0$. Converges to some $\Re(s)>C$. If it has hol. cont. to neigh of $C$, then $\exists \alpha_{0} > 0$ s.t. $f_{\alpha_{0}}$
Converges abs for \( \text{Re}(s) > C - \varepsilon \).

Absolutely false for Dirichlet L-functions!

\( L(\pi, s) \) entire \( \frac{1}{\xi(\xi)} \)

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Exercise: Assume \( C = 0 \).

Look at Taylor series expansion of \( f \) around 1.

\[ f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (s-1)^k \]

Converges abs for \( |s-1| < 1 + \varepsilon \) for some \( \varepsilon > 0 \).

\[ f^{(k)}(s) = \sum_{n=1}^{\infty} a_n e^{-\varepsilon \text{log}_n (-\text{log}_n)^k} \]
\[ f \overset{\text{Conv. abs.}}{=} \frac{1}{\log n} \sum_{k=1}^{\infty} a_n \left[ \frac{(-\log n)^k}{n^k} \right] \]

\[ \exp \left( (1+3)(\log n) \right) = n^{1+3} \]

\[ a_n \geq \sum_{n \geq 1} \frac{a_n}{n^{1+3}} = \sum_{n \geq 1} \frac{a_n}{n^{3+\epsilon}} > 3 \left| \frac{a_n}{n^{3+\epsilon}} \right| \]

\[ f \text{ has poles on } \Re s > -3. \]
Cori: If \( f \) is a Dirichlet series with nonzero coefficients, then it has an abscissa of convergence, i.e., \( \Re(s) > \sigma \) if \( \operatorname{conv} \) ads on \( \Re(s) > \sigma \) & not on \( \Re(s) < \sigma \).

Might have converges of

\[ S(1+it+s) \]

Back to \( \sum \frac{a_n}{n^s} \),

\[ R_A(n) = \sum_{m|n} m^{it} \frac{1}{m^{it}} cn^t \]
Has some abs. of conv. at $\sigma = 2\delta_A$.

Assume $C = U \cup C = \text{Rad}^r$ approx. waves.

As $A \to \infty$ and $n(C) = 0$, $\text{dim} C = 1$. Let $\delta_A$ as $A \to \infty$.

$$\# R_{A}(N) = \frac{1}{2\pi i} \int_{\text{contour}} L_{A}(s) N^{s-1} ds$$

All contour to $\text{Re} (s) = 2\delta_A + 3$.

$$\Rightarrow \# R_{A}(N) \leq N^{2\delta_A + 3}$$

Since $\# R_{A}(N) \leq N^{2\delta_A + 3}$,

would converge absolutely.
What might we expect from this data?

What happens when the red and blue signals are hit roughly equally?

Union vs. difference. In each circle, one might.

Fmaldly

\[ R_\alpha(x) = \binom{N}{k} \text{ for } k \leq N/2 \]

\[ \sum \binom{N}{k} = 2^N \]
\[ \exists \delta > 0 \quad \forall N \in \mathbb{N} \quad (\text{large} \hspace{1cm} m \not\equiv 0 \mod A) \rightarrow \delta; \]

Hensley Gay "92": \[ D_2 \geq N_{\text{771}.} \]

The (Bazgan - K '14): \[ \exists A < 50 \quad (A = 5) \]

st. 100% of nodes are in \( D_A \),

\[ \frac{1}{N} \hspace{1cm} \# D_A \cap S_1(N) \rightarrow 1. \]

\[ \# D_A \cap S_1(N) \approx N^{1/3} \cdot (N) \]

\[ \sum \# D_A \cap S_1(N) = N + O(1). \]