Last time: $\Delta = -y^2(\partial_{xx}+\partial_{yy})$, $E(z,s) = \sum (\operatorname{Im}\gamma z)^s$, $z \in \mathbb{H}$, $\gamma \in \Gamma_\infty\backslash\Gamma$, Res $s=1$.

$\Gamma = SL_2(\mathbb{Z})$, $\Gamma_\infty = \left\{\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\}$. $Q' \sim Q = \{A,B,C\}$, $\Gamma\backslash\mathbb{H}$, $\alpha_Q = \frac{-B+\sqrt{|D|}\,i}{2A}$

Observe: $E(\alpha_Q, s) = E(\alpha_{Q'}, s)$.

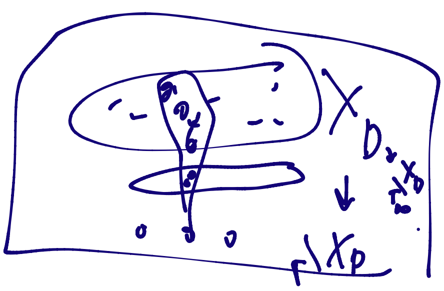
$$\boxed{D = B^2 - 4AC < 0}$$

$$\sum_{(c,d)=1} \left(\frac{\sqrt{|D|}}{2A}\right)^s \frac{1}{\left(\frac{1}{A}Q(c,-d)\right)^s} = \left(\frac{\sqrt{|D|}}{2}\right)^s \underbrace{\sum_{(c,d)=1} \frac{1}{Q(c,d)^s}}_{\text{Epstein Zeta function.}}$$

Let $X_D = \{Q = \{A,B,C\},\ D = B^2-4AC,\ (A,B,C)=1\}$, $\Gamma \circlearrowright$, $\mathcal{C}_D = \Gamma\backslash X_D$. $h(D) = |\mathcal{C}_D|$.

Look at:

$$\sum_{[Q]\in\mathcal{C}_D} E(\alpha_{[Q]}, s) = \left(\frac{\sqrt{|D|}}{2}\right)^s \underbrace{\sum_{Q\in\Gamma\backslash X_D}\ \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}}_{\text{unfold.}} \frac{1}{Q((0,1)\gamma)^s}$$



$(c,d) = (0,1)\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$$= \left(\frac{\sqrt{|D|}}{2}\right)^s \sum_{Q \in \Gamma_\infty \backslash X_D} \frac{1}{Q(0,1)^s}$$

$Q = \{A, B, C\}.$

$Q(0,1) = C.$

$$Q\Big(\underbrace{(x,y)\begin{pmatrix}1 & \ell \\ 0 & 1\end{pmatrix}}_{(x,\ \ell x+y)}\Big) = Ax^2 + Bx(\ell x+y) + C(\ell x+y)^2.$$

$$= (A + B\ell + C\ell^2)x^2 + (B + 2C\ell)xy + Cy^2.$$

So $\Gamma_\infty \backslash X_D = \{(A,B,C) : B^2 - 4AC = D,\ B \in \mathbb{Z}/2C\}.$

$$\left(\frac{\sqrt{|D|}}{2}\right)^s \sum_{C \geq 1} \frac{1}{C^s} \cdot R(C)$$

where $\boxed{L(1, \chi_{\bmod q})}$

$\boxed{D = -q,\ q \equiv 3\ (4)}.$

$$R(C) = \#\{B \bmod 2C : \underbrace{\exists A,\ B^2 - 4AC = D}\}.$$

$$\Updownarrow$$

$$B^2 \equiv D \pmod{4C}.$$

<u>Exercise</u>: $R(C) = \sum_{\ell \mid C} \chi(\ell).$

E.g.: If $C=p$, $\#\{B \bmod C \,;\, B^2\equiv D \pmod C\}$
$= 1+\chi(p)$.

$$\prod_{p\mid C}(1+\chi(p)) = \sum_{\ell\mid C}\chi(\ell).$$

$$\sum_{Q\in\mathcal{C}_D}^{+} E(\alpha_Q,s) = \left(\frac{\sqrt{|D|}}{2}\right)^s \sum_{C\geq 1}\frac{1}{C^s}\underbrace{\sum_{\ell\mid C}\chi(\ell)}_{(1*\chi)(C)}.$$

$D\equiv 1\,(4)$ $\underline{D<0}$;

Dedekind zeta function.

$K=\mathbb{Q}(\sqrt{D})$; $\zeta_K(s) = \underline{\zeta(s)\cdot L(\chi,s)}$.

order

imaginary quad.

$\mathbb{Z}[\sqrt{D}] \subset \mathcal{O}_K$ = ring of integers of K

$\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$.

α is alg num if $\exists$ $p\in\mathbb{Z}[x]$, $p(\alpha)=0$.

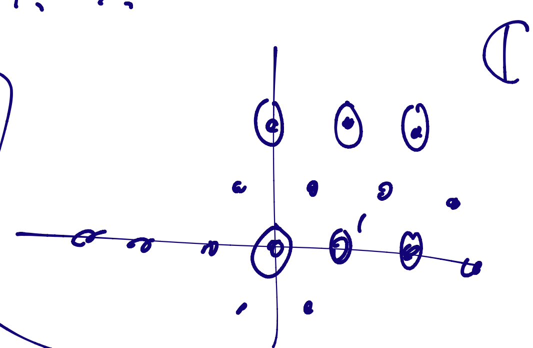
α is alg. integer if monic

$X^2 - X - \frac{D-1}{4}$, $\quad 1^2-4(1)\left(-\frac{D-1}{4}\right) = D$.

$\alpha = \frac{1+\sqrt{D}}{2}$.

$\mathfrak{a}\subset\mathcal{O}$

Ideal $(\forall\, a\in\mathfrak{a},\ x\in\mathcal{O},\ xa\in\mathfrak{a})$.



$$\zeta_K(s) = \sum_{\mathfrak{a}\neq 0}\frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}}\left(1-\frac{1}{N\mathfrak{p}^s}\right)^{-1}$$

$|\mathcal{O}/\mathfrak{a}| = N\mathfrak{a}$.

p rat'l prime — split: $(p) = \mathfrak{p}\cdot\bar{\mathfrak{p}}$, $N(\mathfrak{p}) = p$.
— inert: (p) prime. $N((p)) = p^2$.

$$= \prod_{p \text{ split}} \underbrace{\left(1-\frac{1}{p^s}\right)^{-1}}\left(1-\frac{1}{p^s}\right)^{-1} \cdot \prod_{p \text{ inert}} \underbrace{\left(1-\frac{1}{p^{2s}}\right)^{-1}}_{\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{-1}{p^s}\right)^{-1}}.$$

$$= \prod_p \left(1-\frac{1}{p^s}\right)^{-1} \cdot \prod_p \left(1-\frac{\chi(p)}{p^s}\right)^{-1}.$$

$$= \zeta(s)\, L(s,\chi).$$

Summary: $\underbrace{\zeta(2s)}_{\downarrow\ \frac{\pi^2}{6}} \underbrace{\sum_{Q\in\mathcal{C}_D} E(\alpha_Q, s)}_{\downarrow\ \frac{3}{\pi}} = \underbrace{\left(\frac{\sqrt{|D|}}{2}\right)^s \zeta(s) L(s,\chi)}$,

$\operatorname{Res}_{s=1} \frac{\sqrt{|D|}}{2}\cdot L(1,\chi)$

Need: $\operatorname{Res}_{s=1}$

Analytic Cont of Eisenstein Series

$$\zeta(2s)\underbrace{E(z,s)} = E^*(z,s) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}.$$

idea: $E^*(z+1,s) = E^*(z,s)$

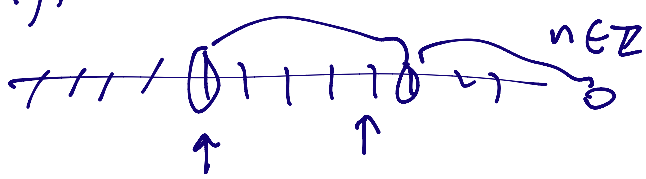
$$\sum_{k\in\mathbb{Z}}{}' a_k(y;s)\, e(kx), \quad \text{where } a_k(y;s) = \int_0^1 E^*(x+iy, s)\, e(-kx)\, dx$$

If $m=0$, $2\sum_{n\geq 1} \frac{y^s}{n^{2s}} = 2\zeta(2s)\, y^s$. (only appears if $k=0$).

SL_2 vs PSL_2.

If $\underbrace{m\neq 0}_{\text{contrib}}$, can sum over $m\geq 1$ with factor of 2.

$$\int_0^1 2\sum_{m\geq 1}{}' \sum_{n\in\mathbb{Z}} \frac{y^s}{\left((mx+n)^2+(my)^2\right)^s}\, e(-kx)\, dx.$$



$$= 2y^s \sum_{m\geq 1}{}' \frac{1}{m^{2s}} \sum_{n\in\mathbb{Z}} \int_0^1 \frac{1}{\left(\left(x+\frac{n}{m}\right)^2+y^2\right)^s}\, e(-kx)\, dx$$

$$\sum_{n\in\mathbb{Z}} = \sum_{n_0(m)} \sum_{n\equiv n_0(m)}$$

$x\mapsto x-\frac{n}{m}$.

$$\boxed{e_m(x) = e\left(\frac{x}{m}\right)}$$

$$= 2y^s \sum_{m\geq 1} \frac{1}{m^{2s}} \sum_{n_0(m)} e_m(kn_0) \sum_{n\equiv n_0(m)} \int_{n/m}^{1+\frac{n}{m}} \frac{1}{(x^2+y^2)^s}\, e(-kx)\, dx$$

$$= 2y^s \left[\sum_{m \geq 1} \frac{1}{m^{2s}} \quad m \cdot 1_{\{m|k\}} \right] \cdot \left[\int_{\mathbb{R}} \frac{1}{(x^2+y^2)^s} e(-kx)\, dx \right]$$

modular archimedean.

$\sigma_s(\ell) = \sum_{m|\ell} m^s$.

$\sigma_{1-2s}(|k|)$. $\leftarrow$ if $k \neq 0$, if $k=0$, $\zeta(2s-1)$,

archimedean: multiplied by $\zeta(2s)$, to clear modular vars

Try multiplying by $\Gamma(s) = \zeta_\infty(2s)$. π^{-s}

$$\Gamma(s) \cdot \text{arch} = \int_0^\infty e^{-u} u^s \frac{du}{u} \cdot \int_{\mathbb{R}} \frac{1}{(x^2+y^2)^s} e(-kx)\, dx.$$

$$= \int_{\mathbb{R}} e(-kx) \int_0^\infty e^{-u} \left(\frac{u}{x^2+y^2} \right)^s \frac{du}{u} \quad dx.$$

$u \mapsto u(x^2+y^2)$.

$$= \int_{\mathbb{R}} e(-kx) \int_0^\infty e^{-ux^2} e^{-uy^2} u^s \frac{du}{u} dx.$$

$$= \int_0^\infty e^{-uy^2} u^s \int_{\mathbb{R}} e^{-ux^2} e(-kx)\, dx \cdot \frac{du}{u}.$$

$$x \mapsto \frac{x}{\sqrt{u}} \sqrt{\pi}$$

$$= \int_0^\infty e^{-uy^2} u^{s-1/2} \underbrace{\int_{\mathbb{R}} e^{-\pi x^2} e\left(-k\sqrt{\tfrac{\pi}{u}}\, x\right) dx}_{e^{-\pi\left(k\sqrt{\frac{\pi}{u}}\right)^2}} \sqrt{\pi}\, \frac{du}{u}.$$

$$= \sqrt{\pi} \int_0^\infty e^{-uy^2 - \pi^2 k^2/u} u^{s-1/2} \frac{du}{u}. \quad \text{If } k=0, \to \Gamma.$$

$$u \mapsto u\frac{|k|\pi}{y}$$

$$= \sqrt{\pi} \underbrace{\int_0^\infty e^{-\frac{|k|\pi y}{\cancel{y}}(u+1/u)} u^{s-1/2} \frac{du}{u}}_{} \cdot \left(\frac{|k|\pi}{y}\right)^{s-1/2}.$$

$$\underbrace{\pi^{-s}\Gamma(s)\zeta(2s)}_{\xi(s).} E(z,s) = \sum_k a_k(y,s)\, e(kx).$$

$$k=0:\quad 2\xi(2s)\, y^s + \underbrace{\text{second factor}}$$

$$k\neq 0:\quad 2\,\cancel{y^s}\,\underbrace{|k|^{s-1/2}\,\sigma_{1-2s}(|k|)}\; y^{1/2-\cancel{s}} \underbrace{\int_0^\infty e^{-\frac{|k|\pi y}{\cancel{2}}(u+1/u)}\, u^{s-1/2}\,\frac{du}{u}}.$$