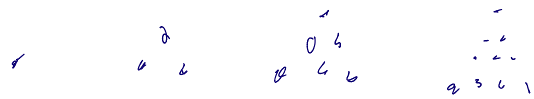
Prehistory: 1670s Leibniz → Paris

Huygens: $\sum \frac{1}{\Delta}$



1 3 6 10 $\frac{k(k+1)}{2}$

$$= \sum \frac{2}{k(k+1)} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \dots$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$= 2\left(\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\dots\right) = \boxed{2}.$$

What about $\sum \frac{1}{k^2}$? (Mengoli ~ 1650s). $= \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots < \infty$

Leibniz → Johan & Jakob Bernoulli (Basel Problem), $\approx 1.64493\dots$? $= \frac{\pi^2}{6}$.

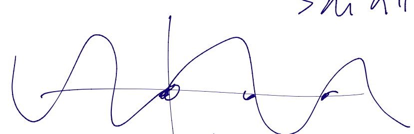
(b 1707) ← 27 yo. St Petersburg

Euler (1734) solves it, becomes instantly world famous.

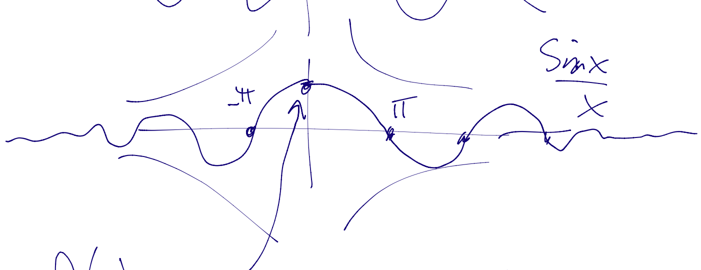
Playing inf series against inf products.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$\sin n\pi = 0$.



$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots$$



Knows: If f is a deg n poly with $f(0)=1$ & roots $a_1,\dots,a_n$,

Then (Exercise!) $f(x) = \left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\cdots\left(1-\frac{x}{a_n}\right)$.

Applies to $\frac{\sin x}{x} = \underbrace{\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{-\pi}\right)}\underbrace{\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{-2\pi}\right)}\cdots$

Can be made rigorous w/ Weierstrass/Hadamard factorization thm for entire functions 1880s.

$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

$$= 1-\frac{x^2}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots\right)+o(x^4).$$

$$\Rightarrow \frac{1}{6} = \frac{1}{\pi^2}\zeta(2) \ \checkmark. \qquad \zeta(s) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots$$

Exercise: Use same idea to compute $\zeta(4)$ & $\zeta(6)$ ...

Exercise: What is $\zeta(3)$?. Not known not quad irrational??

Apery 1978 (quarter millenium later?) Irrational!

Trying to understand ζ identities, notices

$$\left(1-\frac{1}{2^s}\right)\zeta(s) = \left(1-\frac{1}{2^s}\right)\cdot\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)$$

$$= 1+\cancel{\frac{1}{2^s}}+\frac{1}{3^s}+\cancel{\frac{1}{4^s}}+\cdots - \cancel{\frac{1}{2^s}}-\cancel{\frac{1}{4^s}}-\frac{1}{6^s}-\frac{1}{8^s}\cdots$$

$$= \sum_{n,\,2\nmid n}\frac{1}{n^s}.$$

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = \left(1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots\right)\left(1-\frac{1}{3^s}\right).$$

$$= 1+\cancel{\frac{1}{3^s}}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots-\cancel{\frac{1}{3^s}}-\frac{1}{9^s}-\frac{1}{15^s}-\frac{1}{21^s}-\cdots$$

$$= \sum_{n,\,2,3\nmid n}\frac{1}{n^s}.$$

$$\cdots\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = \sum_{n,\,\forall p,\,p\nmid n}\frac{1}{n^s} = 1$$

($\Longleftrightarrow$ Fund Thm Arithmetic)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\underline{1737} \Rightarrow \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right)$$
$$= \sum \frac{1}{n^s}.$$

Exercise: Show that

$$\pi = \sqrt{6} \cdot \frac{2}{\sqrt{3}} \cdot \frac{3}{\sqrt{8}} \cdot \frac{5}{\sqrt{24}} \cdot \frac{7}{\sqrt{48}} \cdot \frac{11}{\sqrt{120}} \cdot \frac{13}{\sqrt{168}} \cdots \frac{p}{\sqrt{p^2-1}} \cdots$$

Knows $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = \infty$, i.e. $\zeta(s) \to \infty$ as $s \to 1^+$. "Divergence of harmonic series"

(Mengoli 1650s, Oresme 1350s). $\log \infty = \infty$.

$$\infty \xleftarrow[\text{as } 1^+ \leftarrow s]{} \log \zeta(s) = \sum_{p \geq 2} -\log\left(1 - \frac{1}{p^s}\right)$$

$$= \underbrace{\sum_p \frac{1}{p^s}} + O\Big(\underbrace{\sum_p \frac{1}{p^{2s}}}_{< \sum_n \frac{1}{n^2} < \infty}\Big).$$

$$1 + x + x^2 + \dots = \frac{1}{1-x}$$
$$x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\log(1-x)$$
for $|x| \leq \frac{1}{2}$, $= x + O(x^2)$.

Exercise: give explicit constant for $|x| \leq \frac{1}{2}$.

$$\Rightarrow \boxed{\sum_p \frac{1}{p} = \infty} \Rightarrow \exists \infty\text{ly many primes. (Euclid 300 BCE)}$$

Birth of Analytic Number Theory. (Not until Dirichlet that power of this idea flowers).

Exercise: ① $\zeta(s) = \frac{1}{s-1} + O(1)$ as $s \to 1$.

② Use these ideas to find $\sum_{p<x} \frac{1}{p} = ??$ $\quad \sum_{n<x} \frac{1}{n} = \log x + O(1).$

Doesn't give up fight against $\zeta(3)$ pushes further

In July 25, 1748 annular solar eclipse

In Ukraine, Euler may have been inspired to write

1749 paper $\left[\begin{array}{l} \odot = 1 - 2^s + 3^s - 4^s + 5^s - \cdots \\ \mathcal{D} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} \end{array}\right.$

evaluates for $s=1$, 100 terms give -50, 101 terms give $+51$, claims limiting value is $\frac{1}{4}$.

People haven't discovered analytic continuation.

$$\underbrace{1+x+x^2+\cdots}_{F(x)} = \underbrace{\frac{1}{1-x}}_{=G(x)}, \quad x=2, \quad \underbrace{1+2+4+8+\cdots}_{\text{"}F(x)\text{"}} = \underset{\text{"}G(x)\text{"}}{-1}$$

For $x=2$, $1-2+4-8+16-32\cdots = \frac{1}{3}$.

Evaluates $\odot$ for integer (& half-integer) values of s

$\mathcal{D}$ — — — — — — — — values of s.

Notices a pattern: $$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}\cdots}{1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}\cdots} = \frac{-(n-1)!\,(2^n-1)}{(2^{n-1}-1)\pi^n}\cos\frac{n\pi}{2}.$$

Conj: holds for "all" values of n.

<u>Exercise:</u> Convert into standard functional equation for ζ.

Really wants: $n=3$. $\frac{0}{\zeta(3)} = 0$. No information, sadly.

Found (trivial) zeros of ζ! $\zeta(-2n)=0$.

Are the only real zeros of ζ. We might think

$$(s-1)\zeta(s) \simeq \underbrace{\left(1-\frac{s}{-2}\right)\left(1-\frac{s}{-4}\right)\left(1-\frac{s}{-6}\right)\left(1-\frac{s}{-8}\right)\cdots}_{\text{product does not converge (need Hadamard factors).}} ??$$

Missing nontrivial zeros, which Euler can't see because he's only letting $s \in \mathbb{R}$! Uncovered by Riemann century later.

Exercise (in Euler): Show: $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots = \frac{\pi^3}{32}$.

Q: Why doesn't this tell us $\zeta(3)$???

Riemann Memoir 1859 gives analytic cont, functional equation & explains how to prove Gauss's Conj (PNT) that $\#\{p<x\} \sim Li(x) = \int_2^x \frac{dt}{\log t}$: "density of primes near x $\sim \frac{1}{\log x}$" from knowing location of zeros of ζ. ($\Rightarrow$ $\zeta(1+it) \neq 0$).

Using explicit exp, Γ, ..., red herrings

Tate 1950 thesis: real workhorse: $\boxed{\text{Poisson Summation}}$.

Thm (Poisson Summation): $f: \mathbb{R} \to \mathbb{R}$ "nice" $\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$.

"std" pf: (where $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e(-x\xi)\, dx$, $\underbrace{e(z) = e^{2\pi i z}}$)

automorphize f

Let $F(x) := \sum_{n \in \mathbb{Z}} f(x+n)$ on $\mathbb{R}/\mathbb{Z}$. use Fourier on circle:

$$F(x) = \sum_{m \in \mathbb{Z}} \hat{F}(m)\, e(mx), \text{ where } \hat{F}(m) = \int_{\mathbb{R}/\mathbb{Z}} F(x)\, e(-mx)\, dx$$

Fix some Fund Domain $\mathcal{F}$ for $\mathbb{R}/\mathbb{Z}$

"unfolding trick"
$$= \int_{[0,1) = \mathcal{F}} \sum_{n \in \mathbb{Z}} f(x+n)\, e(-mx)\, dx = \sum_{n \in \mathbb{Z}} \int_{\mathcal{F}} f(x+n)\, \underbrace{e(-mx)\, dx}$$

$x \mapsto x - n$, Haar measure

$$= \underbrace{\sum_{n \in \mathbb{Z}} \int_{\mathcal{F}+n}} \underbrace{f(x)\, e(-mx)\, dx}_{\text{no } n\text{'s}} = \int_{\mathbb{R}} f(x)\, e(-mx)\, dx = \hat{f}(m).$$

set $x = 0$,
$$\sum_{n \in \mathbb{Z}} f(n+0) = F(0) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \cdot 1$$

Baby trace formula pf: given "test function" f,

Create point-pair invariant $k_f : \mathbb{R}^2 \to \mathbb{R}$,
$$k_f(x,y) = k_f(x+u, y+u) = \boxed{f(x-y)}.$$

make an automorphic kernel

$$K_f(x,y) := \sum_{n \in \mathbb{Z}} \underbrace{k_f(x+n, y)}_{k_f(x,\, y-n)} \text{ on } \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

$\mathbb{R} = G \supset \mathbb{Z} = \Gamma$ discrete subgp.

Make (Hilbert-Schmidt) Integral operator out of Kernel: $g \in \mathcal{H} = L^2(\mathbb{R}/\mathbb{Z})$.

$$(\mathcal{I} g)(x) = \int_X g(y) \cdot \underbrace{K_f(x,y)}\, dy. \qquad G/\Gamma = X.$$

$$\mathrm{Tr}\, \mathcal{I} = \int_X K_f(x,x)\, dx.$$

$x \to$, $y \downarrow$, $K(x,y)$

$\mathrm{Tr}\, M = \sum M_{ii} = \sum \lambda_i$ (eigenvalues).

We will evaluate trace in $\underline{2 \text{ different ways}}$; $\underline{\text{geometrically}}$ & spectrally.

Fix $\mathcal{F}$ f.d. for G/Γ

$$\int_{\mathcal{F}} \sum_{n \in \Gamma} K_f(x+n, x)\, dx. = \sum_{n \in \Gamma} \int_{\mathcal{F}} \underbrace{K_f(x+n,x)}_{= K_f(n,0).} dx = \sum_{n \in \Gamma} \underbrace{K_f(n,0)}_{f(n-0)} \cdot \underbrace{\mathrm{vol}(X)}_{= 1.}$$

$$= \sum_{n \in \mathbb{Z}} f(n).$$

$\mathcal{H} = L^2(G/\Gamma) \circlearrowleft \Delta = d_{xx}$ (self-adjoint), Spect thm: Can diagonalize Δ, $\exists\, e_n$ basis $\oplus\, \mathbb{C} e_n = \mathcal{H}$, & $\Delta e_n = \lambda_n e_n$.

$e_n(x) = e(nx)$.

$\lambda_n = 4\pi^2 n^2$.

Δ neg-definite.

For spectral side, $K_f(x,y)$ expand in y-variable.

For fixed x,

$$K_f(x,y) = \sum_{m \in \mathrm{Spec}\,\Delta} \hat{K_f}(x,m)\, e_m(y),$$ where

$$\hat{K_f}(x,m) = \underbrace{\int_X K_f(x,y) \overline{e_m(y)}\, dy}_{= \text{proj onto } \mathbb{C} e_m.} = (\mathcal{I} \overline{e_m})(y)$$

unfold

$$\int_{\mathcal{F}} \sum_{n \in \mathbb{Z}} K_f(x+n, y)\, \overline{e_m(y)}\, dy = \sum_{n \in \mathbb{Z}} \int_{\mathcal{F}} \underbrace{K_f(x+n,y)}_{K_f(x, y-n)} \overline{e_m(y)}\, dy.$$

$y \mapsto y+n$

Haar measure

$$= \int_{G = \mathbb{R}} K_f(x,y) \overline{e_m(y)}\, dy.$$

$$= \int_{\mathbb{R}} f(\underbrace{x-y}) \, e(-m y) \, dy = \int_{\mathbb{R}} f(z) \underbrace{e(-m(x-z))} dz = e(-mx) \hat{f}(-m)$$

$z = x-y$, $y = \overline{x-z}$

$dz = -dy$

Character $\underbrace{e(-mx)}$; $e(mz)$.

$$\sum_{n\in\mathbb{Z}} f(n) = \mathrm{Tr} = \int_X K_f(x,x)\,dx = \sum_{m\in\mathbb{Z}} \hat{f}(-m) \underbrace{\int_X \underbrace{e(-mx)\, e(mx)}_{1} \, dx}_{\mathrm{vol}(X)=1}.$$

$$= \sum_{m\in\mathbb{Z}} \hat{f}(m).$$