

# Math 572 Prof. Kontorovich

## Bird's Eye View of Analytic Number Theory



**1939** Rankin-Selberg:  $L(s, f \times g) = \sum \frac{a(n) b(n)}{n^s} = \prod_p (1 - \frac{a(p)}{p^s})^{-1} (1 - \frac{b(p)}{p^s})^{-1} (1 - \frac{a(p)b(p)}{p^{2s}})^{-1}$

If  $f = g$ ,  $\sum \frac{a(n)^2}{n^s} \rightarrow L(s, f \otimes f)$

pole at  $s=1$ ,  $\Rightarrow \sum_{n \leq x} |a(n)|^2 = cx + o(x^{1/2})$ .  $\Rightarrow |a(n)|^2 \ll n^{1/2}$ .  $\Rightarrow |a(n)| \ll n^{1/4}$

Long time, people tried to create  $\sum \frac{a(n)^3}{n^s}$  has a wall of singularities! WRONG object.

Langlands 1971 The same is Euler products  $a(n)^2$  Red Herring. Right idea:

$(\alpha, \beta) \xrightarrow{\text{Sym}^3} (\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3)$  i.e.  $L(s, \text{Sym}^3 f) = \prod_p (1 - \frac{\alpha^3}{p^s})^{-1} (1 - \frac{\alpha^2\beta}{p^s})^{-1} (1 - \frac{\alpha\beta^2}{p^s})^{-1} (1 - \frac{\beta^3}{p^s})^{-1}$

$GL(2) \rightarrow GL(4)$  representation

Langlands Functoriality: any game making new L-function of deg  $n$  is the  $n$ -th L-function of some  $GL(m)$  aut rep.

**1949** Maass using Hecke characters constructs explicit Eisenstein series on  $\Gamma \backslash \mathbb{H}$

**1956** Selberg trace formula

**1970** Selberg trace formula

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**2021** Selberg trace formula

**2022** Selberg trace formula