Recall: $Q=[A,B,C]$, $\rightsquigarrow$ $\alpha_Q = \dfrac{-B+\sqrt{D}}{2A} \in \mathbb{H} \subset \mathbb{C}$.

$D = B^2-4AC<0$ (Q is definite). $\qquad$ "root" of $Q(x,1)=0$.

Thm: If $Q_2 = Q_1 \circ \gamma$ (that is, $Q_2 \sim Q_1$, $\gamma \in SL_2(\mathbb{Z})$), $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

then $\alpha_{Q_2} = \underbrace{\underbrace{\gamma^{-1}} \circ \alpha_{Q_1}}_{\text{fractional linear transformation.}}$

Pf: Start with $\gamma^{-1} \circ \alpha_{Q_1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \circ \left(\dfrac{-B_1+\sqrt{D}}{2A_1} \right)$

$$= \frac{\left(d\left(\frac{-B_1+\sqrt{D}}{\cancel{2A_1}}\right) - 2bA_1\right)\left(2aA_1 + cB_1 + c\sqrt{D}\right)}{\left(-c\left(\frac{-B_1+\sqrt{D}}{\cancel{2A_1}}\right) + 2aA_1\right)\left(\underbrace{2aA_1 + cB_1} + \underbrace{c\sqrt{D}}\right)}$$

(Galois)

$$= \frac{\left\{ \begin{array}{l} -dB_12aA_1 - dB_1^2c - 2bA_1^2a - 2bA_1cB_1 \\ \quad + dcD + \sqrt{D}\left[d \cdot 2aA_1 + \cancel{d \cdot cB_1} \cancel{- cdB_1} - 2bcA_1\right] \end{array} \right.}{\left(4a^2A_1^2 + 4acA_1B_1 + c^2B_1^2\right) - c^2D}.$$

the numerator $= \left(\begin{array}{l} -4abA_1^2 \quad -2A_1B_1(ad+bc) \quad + 2A_1\sqrt{D}\left[ad-bc\right] \\ \cancel{-cdB_1^2} + dc\left(\cancel{B_1^2} - 4A_1C_1\right) \end{array} \right)$

numerator: $-4abA_1^2 - 2A_1B_1(ad+bc) - 4cdA_1C_1$
$+2A_1\sqrt{D}$

numerator: $2A_1\left(-\underbrace{\left(2abA_1 + (ad+bc)B_1 + 2cdC_1\right)}_{B_2} + \sqrt{D}\right)$,

denominator: $4a^2A_1^2 + 4acA_1B_1 + \cancel{c^2B_1^2} - c^2\left(\cancel{B_1^2} - 4A_1C_1\right)$,

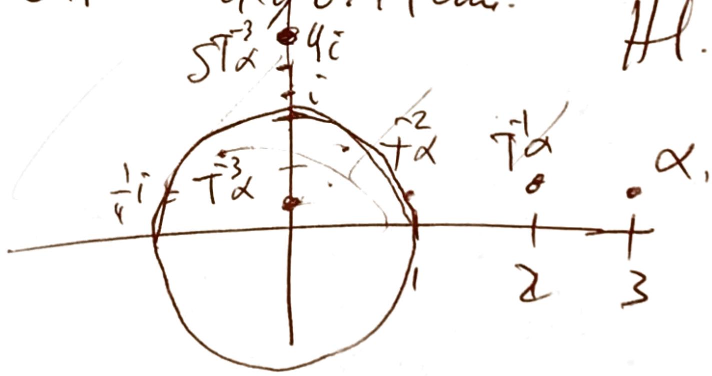
$$\boxed{Q_2 = Q_1 \circ \gamma} = 2A_1\Big[(2)\underbrace{\left[a^2A_1 + acB_1 + c^2C_1\right]}_{A_2}\Big].$$

$$\alpha_{Q_2} = \frac{num}{den} = \frac{-B_2 + \sqrt{D}}{2A_2} = \gamma^{-1} \circ \alpha_{Q_1}$$

Geometric reduction algorithm.

$\alpha_1 \in \mathbb{H}$

$3 + \frac{1}{4}i$



Note: $T = \begin{pmatrix}1&1\\0&1\end{pmatrix} \in SL_2(\mathbb{Z})$, $\quad T\circ\alpha = \begin{pmatrix}1&1\\0&1\end{pmatrix}\circ\alpha = \left(\frac{1\cdot\alpha+1}{0\cdot\alpha+1}\right) = \alpha+1$

$S = \begin{pmatrix}0&1\\-1&0\end{pmatrix} \in SL_2(\mathbb{Z})$, $\quad S\circ\alpha = \frac{0\cdot\alpha+1}{(-1)\cdot\alpha+0} = \frac{-1}{\alpha} = \frac{-\bar{\alpha}}{|\alpha|^2}$.

$S\left(\frac{1}{4}i\right) = \frac{-1}{\frac{1}{4}i} = 4i$

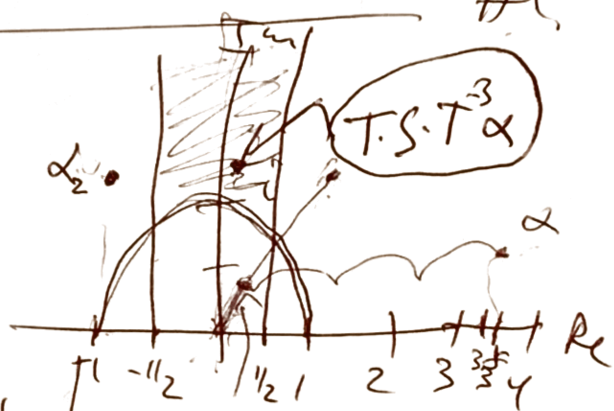
(2) $|S\circ\alpha| = \frac{1}{|\alpha|}$

Reduction algorithm:

Step 1: Use T (or $T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$)

to make $|\underbrace{\mathrm{Re}(\alpha)}_{\text{Real Part}}| < \frac{1}{2}$.

Step 2: Use S to get outside unit circle.



Ex: $\alpha = 3\frac{1}{3} + \frac{1}{2}i$.

$\boxed{\alpha_1 = T^{-3} \circ \alpha = \frac{1}{3} + \frac{1}{2}i.}$

$|\alpha_1|^2 = \frac{1}{9} + \frac{1}{4} < 1.$

$$\alpha_2 = S \circ \alpha_1 = \frac{(-1)\left(\frac{1}{3} - \frac{1}{2}i\right)}{\left(\frac{1}{3} + \frac{1}{2}i\right)\left(\frac{1}{3} - \frac{1}{2}i\right)} = \frac{-\frac{1}{3} + \frac{1}{2}i}{\frac{13}{36}}$$

$$= \left(-\frac{1}{3} + \frac{1}{2}i\right)\left(\frac{36}{13}\right) = \frac{-36}{39} + \frac{18}{13}i$$

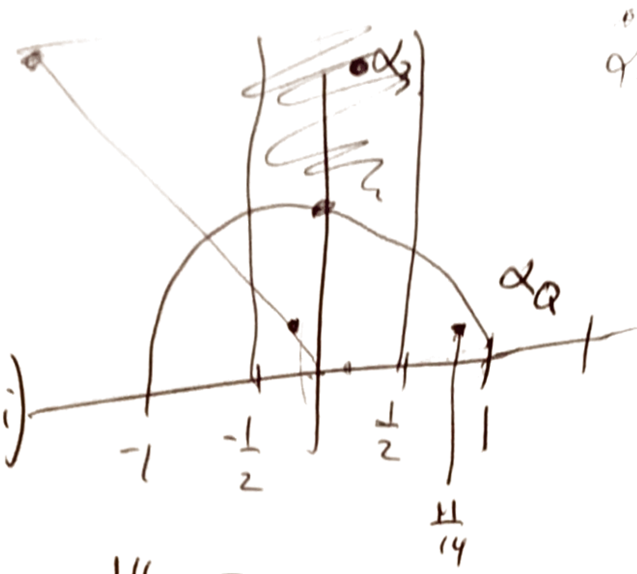
$$T \circ \alpha_2 = \frac{3}{39} + \frac{18}{13}i$$

Idea: Reduce Q by reducing α!

$Q = [7, -11, 5]$, $D = -19$.

$$\alpha_Q = \frac{11 + \sqrt{19}\, i}{14}$$

$$\alpha_1 = T^{-1} \circ \alpha = \frac{-3}{14} + \frac{\sqrt{19}}{14} i$$



$$\alpha_2 = S \circ \alpha_1 = \frac{-1\left(-\frac{3}{14} - \frac{\sqrt{19}}{14} i\right)}{\left(\frac{-3}{14} + \frac{\sqrt{19}}{14} i\right)\left(-\frac{3}{14} - \frac{\sqrt{19}}{14} i\right)}$$

$$= \frac{\left(\frac{3}{14} + \frac{\sqrt{19}}{14} i\right)\left(\frac{196}{19}\right)}{\cancel{\frac{9}{196} + \frac{19}{196}}} = \frac{42}{19} + \frac{14}{19} \sqrt{19}\, i,$$

$$\alpha_3 = T^{-2} \circ \alpha_2 = \frac{4}{19} + \frac{14}{19} \sqrt{19} \cdot i$$

$$\alpha_3 = \underbrace{\left(T^{-2} \cdot S \cdot T^{-1}\right)}_{} \circ \alpha = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}}_{\gamma^{-1}} \circ \alpha.$$

$$\underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{} \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad Q_1(a,c) \quad Q_1(b,d),$$

$$Q_2 = \underset{\shortparallel}{Q_1} \circ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. = [A_2, B_2, C_2]$$

$$[7, -11, 5] \qquad = \Big[1, \; 1, \; \frac{7 - 22 + 20}{5}\Big]$$

$$= 7x^2 - 11xy + 5y^2$$

$$2ab A_1 + (ad + bc) B_1 + 2cd C_1$$

$$2 \cdot 7 + (2+1)(-11) + 2 \cdot 2 \cdot 5$$

Quiz 1: let $Q = [14, 3, 2]$.

(a) Find α,

(b) Reduce $\alpha \in \mathbb{H}$ (upper half plane) $\subset \mathbb{C}$.

(c) Use (b) to reduce Q.

Quiz 2: let $Q = [4, 3, 2]$.

(a)
(b) Same
(c)
