Review: $Q = [A, B, C] = Ax^2 + Bxy + Cy^2$.

Which $n = Q(x,y)$? Which $n = Q(x,y)$ with $(x,y)=1$ (i.e. primitively)?

Equivalence: $Q_1 \sim Q_2 \iff \exists\, \gamma \in SL_2(\mathbb{Z})$ s.t. $Q_1 = Q_2 \circ \gamma$.

Given a form Q, $\rightsquigarrow$ $[Q]$ = "equivalence class of Q"

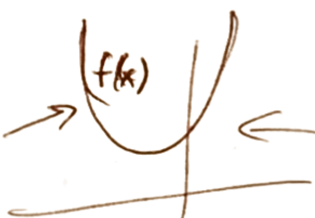
$= \{Q' \mid Q' \sim Q\}$.

Goal: Understand $[Q]$ by finding "reduced" representative.

Ex: $Q_0 = [7, -11, 5]$, $D_{Q_0} = D = 121 - 140 = -19 < 0$ → definite ⇒ only takes positive values.

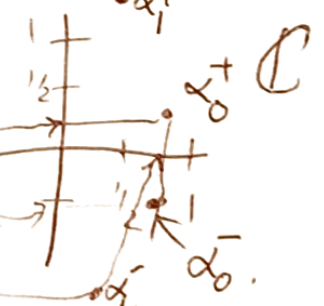
$Q_0\big|_{y=1} = 7x^2 - 11xy + 5y^2\big|_{y=1}$

$= 7x^2 - 11x + 5.$ →



← no real roots

Roots: $\alpha_Q^{\pm} = \dfrac{-B \pm \sqrt{D}}{2A} = \dfrac{11 \pm \sqrt{-19}}{14}$



Equivalent form: $Q_1 = Q_0 \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, → $Q_1(x,y) = Q_0\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)$

$= Q_0(x, x+y) = 7x^2 - 11(x)(x+y) + 5(x+y)^2$.

$= x^2 - xy + 5y^2$, $Q_1 = [1, -1, 5]$.

What are roots of Q_1? $\alpha_1^{\pm} = \frac{1 \pm \sqrt{-19}}{2}$

"Q_1 is "more reduced" than Q_0". Can we reduce Q_1 farther?

Thm If $Q_1 \sim Q_2 \Rightarrow D_{Q_1} = D_{Q_2}$.

Exercise: Do this directly from algebra:

$$\left\{\begin{array}{l} A_1 = A ac \\ B_1 = \\ C_1 = \end{array}\right\} \quad B_1^2 - 4A_1C_1 =$$

"Better" pf: **Obs:** If $Q = [A, B, C] = Ax^2 + Bxy + Cy^2$;

Its half-Hessian (Gram) matrix: $M = \frac{1}{2}\left(\partial_i \partial_j\right)$

$$= \frac{1}{2}\begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \quad \text{symmetric matrix.}$$

$$\operatorname{Det} M = AC - \frac{1}{4}B^2 = -\frac{1}{4}\left(B^2 - 4AC\right) = -\frac{1}{4} \cdot D_Q.$$

Exercise: $Q(x, y) = (x\ y) M \begin{pmatrix} x \\ y \end{pmatrix} = (x\ y) \underbrace{\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$.

$$(x\ y) \cdot \begin{pmatrix} Ax + \frac{B}{2}y \\ \frac{B}{2}x + Cy \end{pmatrix} = Q(x, y)$$

Rmk: $7x^2 - 5xz + 3yw + w^2 - z^2 - 2zw$

$$\Big(\underbrace{\underbrace{(x\ y\ z\ w)}_{1\times 4} \underbrace{\begin{pmatrix} 7 & 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \\ -\frac{5}{2} & 0 & -1 & -1 \\ 0 & \frac{3}{2} & -1 & 1 \end{pmatrix}}_{4\times 4}} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}_{4\times 1} \Big).$$

(2)

So if $Q_2 = Q_1 \circ \gamma$ (i.e. $Q_1 \sim Q_2$)

Then $\cancel{(x,y)}\binom{x}{y}^t M_2 \binom{x}{y} = Q_2(x,y) = Q_1\left(\left[\gamma\binom{x}{y}\right]^t\right)$.

$(x,y) M_2 \binom{x}{y} = \left[\gamma\binom{x}{y}\right]^t M_1 \left[\gamma\binom{x}{y}\right] = (x,y)\gamma^t M_1 \gamma \binom{x}{y}$.

<u>Ex:</u> $Q_0 = [7,-11,5] \sim Q_1 = [1,-1,5]$

$Q_0(x,y) = (x,y)\begin{bmatrix} 7 & -\frac{11}{2} \\ -\frac{11}{2} & 5 \end{bmatrix}(x,y)^t$ $\qquad \boxed{(AB)^t = B^t A^t}$

$Q_1(x,y) = (x,y)\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 5 \end{bmatrix}(x,y)^t$

$Q_1(x,y) = Q_0\left(\left[\begin{pmatrix}1&0\\1&1\end{pmatrix}\binom{x}{y}\right]^t\right) = Q_0\left((x,y)\begin{pmatrix}1&0\\1&1\end{pmatrix}^t\right)$.

$\parallel$

$(x,y)\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 5 \end{bmatrix}\binom{x}{y} = (x,y)\begin{pmatrix}1&0\\1&1\end{pmatrix}^t \begin{bmatrix} 7 & -\frac{11}{2} \\ -\frac{11}{2} & 5 \end{bmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\binom{x}{y}$

$\Rightarrow$ as matrices, $M_2 = \gamma^t M_1 \gamma$, $\qquad SL_2(\mathbb{Z})$

$\Rightarrow \det M_2 = \det(\gamma^t M_1 \gamma) = \det(\gamma^t)\cdot\det(M_1)\cdot\underbrace{\det(\gamma)}_{1}$,

$\Rightarrow \quad -\frac{1}{4}\cdot D_2 = \det M_2 = \det M_1 = -\frac{1}{4} D_1$.

$\Rightarrow \quad Q_1 \sim Q_2 \Rightarrow D_{Q_1} = D_{Q_2}$.

<u>Question:</u> Does $D_{Q_1} = D_{Q_2} \Rightarrow \underbrace{Q_1 \sim Q_2}_{\Downarrow \; [Q_1]=[Q_2]}$? (No, but sometimes YES)

Q: Does $Q_1 \sim Q_2 \Rightarrow$ eigenvalues of M_1 = those of M_2?

If $M = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$ what are eigenvalues of M?

$$0 = \det(M - \lambda I) = \left|\begin{pmatrix} A-\lambda & B/2 \\ B/2 & C-\lambda \end{pmatrix}\right| = AC - (A+C)\lambda + \lambda^2 - \frac{B^2}{4}$$

$$= \lambda^2 - \mathrm{tr}M \cdot \lambda + \det M, \Rightarrow \lambda = \frac{\mathrm{tr}M \pm \sqrt{\mathrm{tr}^2 M - 4\det M}}{2}.$$

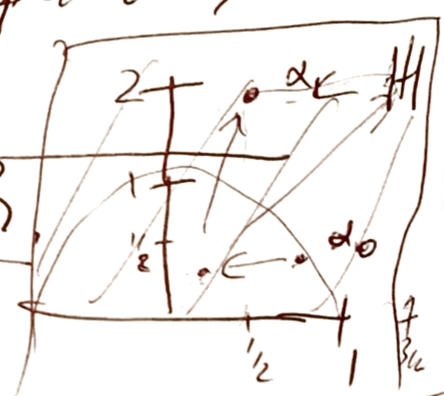
No since $A+C$ ($= \mathrm{tr}M$) changes. (neither are eigenvectors).

Back to reduction: What happens to roots $\alpha_Q^{\pm}$ geometrically when Q is replaced by an equivalent form?

$\alpha_0 \in \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

" $\frac{11 + \sqrt{19}i}{14}$

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$



$\frac{1+\sqrt{19}i}{2} = \alpha_1 \rightsquigarrow Q_1 = Q_0 \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Want to understand $\sim$ on quad forms, $Q_1 = Q_0 \circ \gamma$, in terms of what it does to α vs α_0.

$Q_0 = [A_0, B_0, C_0]$, $Q_1 = [A_1, B_1, C_1]$ $(D<0)$.

$Q_1 = Q_0 \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\alpha_0 = \frac{-B_0 + \sqrt{|D|}\cdot i}{2A_0}$$

$$\alpha_1 = \frac{-B_1 + \sqrt{|D|}\, i}{2A_1}$$

$Q_1(x,y) = Q_0(ax+by, cx+dy)$

$A_1 = Q_0(a,c) = A_0 a^2 + B_0 ac + C_0 c^2$.

$B_1 = A_0(2ab) + B_0(ad+bc) + C_0 \cdot 2cd$.

$$\alpha_1 = \frac{-[A_0(2ab) + B_0(ad+bc) + C_0\cdot 2cd] + \sqrt{|D|}\, i}{2[A_0 a^2 + B_0 ac + C_0 c^2]},$$

$$\overset{?}{=} \frac{a\cdot\alpha_0 + b}{c\cdot\alpha_0 + d} = \gamma \circ \alpha_0.$$ "fractional linear transformation"

Check: $\alpha_1 = \dfrac{1+\sqrt{19}i}{2}$, $\alpha_0 = \dfrac{11+\sqrt{19}i}{14}$. $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$,

$$\gamma \circ \alpha_0 = \frac{1\cdot\alpha_0 + 0}{1\cdot\alpha_0 + 1} = \frac{\frac{11+\sqrt{19}i}{14}}{\frac{11+\sqrt{19}i}{14} + 1}.$$ (won't work!),

Try! $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \circ \alpha_0 = \dfrac{0\cdot\alpha_0 - 1}{1\cdot\alpha_0 - 1} = \dfrac{-1}{\frac{11+\sqrt{19}i}{14} - 1}$

$$= \frac{-1}{\frac{-3+\sqrt{19}i}{14}} = -1\left(\frac{14}{-3+\sqrt{19}i}\right)\left(\frac{-3-\sqrt{19}i}{-3-\sqrt{19}i}\right),$$

$$= \frac{14(3+\sqrt{19}i)}{9+19} = \frac{14(3+\sqrt{19}i)}{\cancel{28}\ 14\cdot 2} = \frac{3+\sqrt{19}i}{2}.$$