Recall: $Q = [A,B,C]$, quad form, $Q = Ax^2 + Bxy + Cy^2$.

D_Q = discriminant $= B^2 - 4AC$.

- Q is degenerate if $D_Q = 0$. $(Q \sim x^2)$.
- Q is reducible if $D_Q = d^2$ $(Q \sim x \cdot y)$.
- Q is definite (takes only one sign) if $D_Q < 0$. $\Rightarrow$ (not degen, not red.).
- Q is indefinite if $D_Q > 0$ (i.e. both + & − values).

Think of quadratic polynomials:

$\exists$ real solutions $\Leftrightarrow$ $D_Q > 0$.

Basic Question: Which n are represented by Q?

i.e. $\exists ?\ x,y \in \mathbb{Z}$ s.t. $Q(x,y) = n$?

Ex. $Q = 7x^2 - 11xy + 5y^2$ ~~Indefinite~~ Definite. Positive

$D_Q = -19$ $(= 121 - 4 \cdot 5 \cdot 7)$.

$Q(1,0) = 7$ (In general, $Q(1,0) = A$).

What #s n are we seeing? $n = 7$, $5 = Q(0,1)$,

[Aside $0 = Q(0,0)$ always rep], $Q(1,1) = 1$, $4 = Q(2,2)$

So: get all squares since $Q(kx, ky) = k^2 \cdot Q(x,y)$

Better question:

Which n are primitively represented, i.e.

$n = Q(x,y)$ & $(x,y) = 1$.

Is 4 represented primitively? ($Q(4,2) = 4$ is an imprimitive representation.)

$7 = Q(2,3) = \underbrace{7\cdot 4}_{28} - \underbrace{11\cdot 2\cdot 3}_{66} + \underbrace{5\cdot 9}_{45} = 7$ ✓.

$= Q(x,y)$ 5

7

$y \backslash x$	1	2	3	4	5	6
1	1	11	35	73	125	191
2	5	X	17	X	85	X
3	19	7	X	25	55	X
4	43	X	11	X	35	X
5	77	43	23	17	X	47
6	121	X	X	X	25	X

$Q(4,6) = 2^2 \cdot Q(2,3) = 28.$

$Q(1,2) = 7 - 11\cdot 2 + 20 = 5.$

$Q = 7x^2 - 11xy + 5y^2$

Exercise 1: ~~Q only takes odd values Primitively~~

→ Primitive values of Q are all odd. Ans.

Exercise 2: What values does Q represent primitively mod 3? Ans: Q mod 3 = 1, 2, NOT 0!

⟹ No multiples of 3 will be primitively represented & No multiples of 2.

Def: A quad form $Q = [A, B, C]$ is **primitive**: if $\gcd(A, B, C) = 1$.

Ex: **Imprimitive form** $Q = 2x^2 + 4xy - 6y^2$.

$$= 2\left[x^2 + 2xy - 3y^2\right].$$

So values taken by Q are simply twice values of $x^2 + 2xy - 3y^2$

Ex: Is $Q = \cancel{2x^2+3xy+6y^2}$ primitive?

$$6x^2 + 10xy - 15y^2 \quad \boxed{\text{YES}}$$

So primitive form $\not\Rightarrow$ pairwise primitive coefficients

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A **primitive representation** of $n = Q(x, y)$ is one with $(x, y) = 1$

Exercise 3: What values does Q take primitively mod 5?

$Q \bmod 5 = 0, 1, 2, 3, 4$

Heuristic in a $p \times p$ box, we have about roughly p^2 values to check, & they are all numbers mod p.

So for larger & larger p, odds are better & better that you'll find *all* values.

"Dual" idea: Instead of keep. Q & changing x & y, Fix x & y & change Q.

$Q_0 = 7x^2 - 11xy + 5y^2$, $Q_0(1,0) = 7 = A$.

Instead of changing (x,y) from $(1,0)$ to $(1,1)$, changing Q_0 to ________, $(??)$ evaluated at $(1,0)$.

Idea: In $x^2 + y^2 = Q_1$, replace x by $x+y$.

$$\longrightarrow Q_1(x+y,y) = (x+y)^2 + y^2 = x^2 + 2xy + 2y^2 = Q_2(x,y).$$

So Q_2 primitive represents exactly the same values as Q_1.

Ex: $Q_1(3,4) = 25$. Find x & y s.t.

$Q_2(x,y) = 25$.

$Q_2(x,y) = Q_1(x+y,y)$

Solution: $Q_2(-1,4) = 25$
$= 1 - 2\cdot 4 + 2\cdot 4^2$.

Daniel solved: $\begin{cases} x+y = 3 \\ y = 4 \end{cases} \Rightarrow \begin{matrix} x = -1 \\ y = 4 \end{matrix}$

Point: Q_1 & Q_2 differ by $\mathbb{Z}$-invertible linear change of variables

Ex: Find (x,y) s.t. $Q_2(x,y) = 13 = 2^2+3^2$
$= Q_1(2,3)$.

$$Q_2(x,y) = Q_1(\underbrace{x+y}_{=2}, \underbrace{y}_{=3})$$ So $(x,y) = (-1, 3)$.

So $Q_2(-1,3) = 13$.

Recall: $GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a,b,c,d \in \mathbb{R}. \; \det = ad-bc \neq 0 \right\}$.

$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{Z}. \; ad-bc = \pm 1 \right\}$.

$SL_2(\mathbb{Z}) = \{ \gamma \in GL_2 \mid \det = +1 \}$.

Exercise: If $Q_1 = [A, B, C]$ & $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. What is

$$Q_2(x,y) = Q_1 \circ \gamma = Q_1(ax+by, cx+dy).$$

If $Q_2 = [A_2, B_2, C_2]$. Work out

$A_2 = ?$

$B_2 = ?$ In terms of A, B, C

$C_2 = ?$ & a, b, c, d.
