Last time: $\Lambda = \langle 1, \tau \rangle$, $\tau \mapsto \tau+1$

$$E_k(\tau) = \sum_{\ell \in \Lambda^*} \frac{1}{(z+\ell)^k}.$$

~ "modular form" of wt k.

$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL_2(\mathbb{Z})$, $E_k(\gamma\tau) = E_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k} E_k(\tau)$

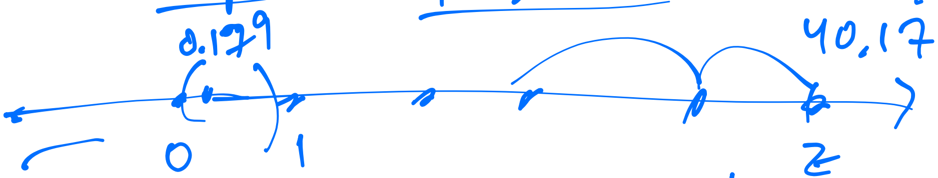
Fund dom: $\Gamma = \mathbb{Z}$, $G = \mathbb{R}$, $\mathcal{D}$ = fund dom for $\Gamma \curvearrowright G$.

$\longrightarrow G = \mathbb{R}$, $\mathcal{D} = (-1/2, 1/2]$ or $(0, 1]$

- if $z, w \in \mathcal{D}$ & $z = w + n \Rightarrow z = w$ & $n = 0$. (unique / disjoint)
- $\bigsqcup_{n \in \mathbb{Z}} \mathcal{D} + n = G = \mathbb{R}$. (covering). $\mathcal{D}$ = tile. "reduction alg".

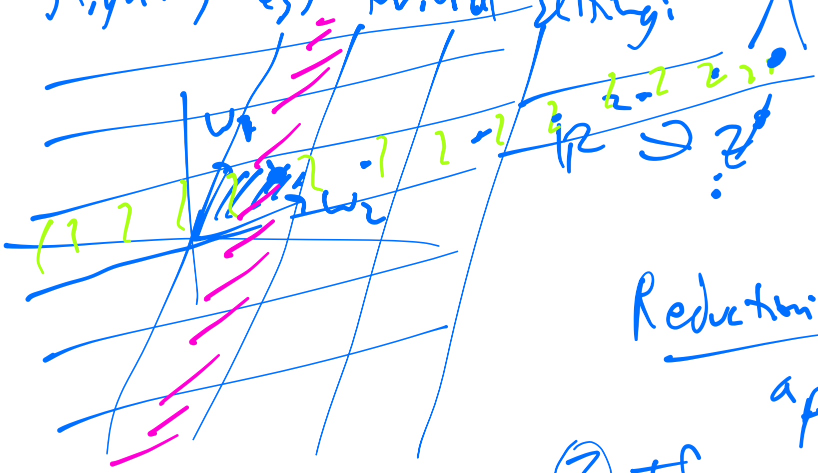
$\Rightarrow$ given any $z \in \mathbb{R}$, reduction algorithm returns "simplest" representative for coset $\underbrace{z + \Gamma}$, $= z + \mathbb{Z}$.

$\Downarrow$ $z_0 \in \mathcal{D}$, $z \mapsto z+1$ (T)



alg: if $z > 1$, apply T^{-1}, if $z < 0$, apply T. repeat until halt. $0 \le z \le 1$.

Slightly less trivial setting: $\Lambda = \langle \omega_1, \omega_2 \rangle \curvearrowright \mathbb{R}^2$.



$\mathcal{D}$: .

Reduction Algorithm: ① if outside, apply $\pm\omega_1$ until inside.

② If outside, apply $\pm\omega_2$.

Λ is abelian.

Case at hand: $\Gamma = SL_2(\mathbb{Z})$ non abelian $\curvearrowright X = \mathbb{H}$.

$z \in \mathbb{H}$. Want Reduction alg to a fund dom D for $\Gamma \curvearrowright \mathbb{H}$?

Obs: · $\begin{pmatrix}1&1\\0&1\end{pmatrix} \in SL_2(\mathbb{Z})$, $\underbrace{\begin{pmatrix}1&1\\0&1\end{pmatrix}}_{T} \cdot z \mapsto \frac{1\cdot z+1}{0\cdot z+1} = z+1$

· $\underbrace{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}_{S} \in SL_2(\mathbb{Z})$, $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot z = \frac{-1}{z}$. $\boxed{S^2 = -1 \text{ in } SL_2(\mathbb{Z}),\ S^2 = I \text{ in } PSL_2}$

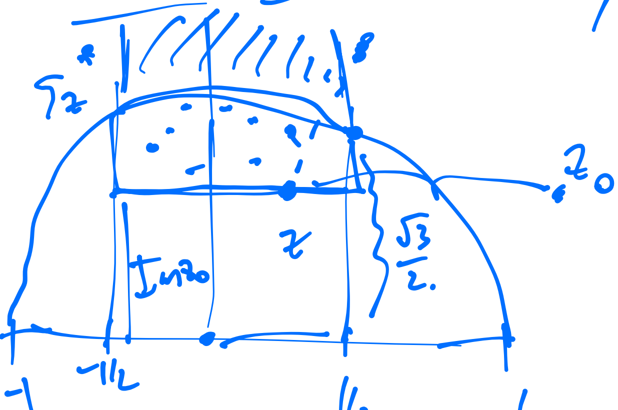
$$\operatorname{Im} \overbrace{\begin{pmatrix}a&b\\c&d\end{pmatrix}}^{g}_{\in M_{2\times2}(\mathbb{R})} \cdot z = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}\frac{(az+b)(c\bar z+d)}{(cz+d)(c\bar z+d)} = \operatorname{Im}\frac{\cancel{ac|z|^2} + adz + \cancel{bd} + bc\bar z}{|cz+d|^2}$$

$$= \frac{ady + bc(-y)}{|cz+d|^2} = \frac{\overbrace{(ad-bc)}^{\det g}\operatorname{Im} z}{|cz+d|^2}$$

If $\det g > 0$, then $\operatorname{Im} gz > 0$.

For $\gamma \in SL_2(\mathbb{R})$, $\operatorname{Im}\gamma z = \frac{\operatorname{Im} z}{|cz+d|^2}$. So $\operatorname{Im} Sz = \operatorname{Im}\frac{-1}{z} = \frac{\operatorname{Im} z}{|z|^2}$.

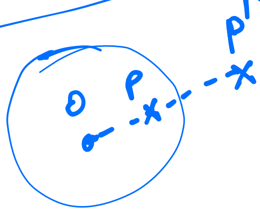
Obs: If $|z| < 1$, then $\operatorname{Im} Sz > \operatorname{Im} z$.



Alg: ① use $T^{\pm 1}$ until $|\operatorname{Re} z| \le 1/2$.

② If $|z| < 1$, apply S. Else HALT.

Geometrically, S inverts in unit circle & reflects across y-axis



$\frac{OP}{r} = \frac{r}{OP'}$, $OP \cdot OP' = r^2$.

Apply ①, Apply ②.

Claim: Alg halts. $\Rightarrow$ $\mathcal{D} = \{ z \in \mathbb{H} \mid |\mathrm{Re}\, z| < 1/2,\ |z| > 1 \}$.
is a fund dom for $SL_2(\mathbb{Z})$.

If not, $\exists$ $z_1, z_2, z_3, \dots \in \{ |\mathrm{Re}\, z| \le 1/2,\ |z| \le 1,\ |\mathrm{Im}\, z| \ge \mathrm{Im}\, z_0 \}$.

$z_1 = \gamma_1 \cdot z_0$, $z_2 = \gamma_2 \cdot z_0$, $\gamma_j := S \cdot T^{n_2} \cdot S \cdot T^{n_1}$

$\exists$ subseq convergent $|\mathrm{Im}\, \gamma_j z_0| \ge \mathrm{Im}\, z_0$, $\gamma_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

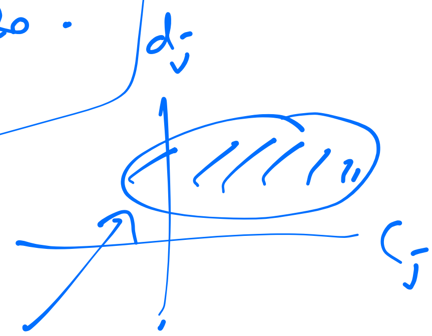
$$|\mathrm{Im}\, \gamma_j z_0| = \frac{\mathrm{Im}\, z_0}{|c_j z_0 + d_j|^2} \ge \mathrm{Im}\, z_0 .$$

$\Rightarrow$ $|c_j z_0 + d_j|^2 \le 1$

$|c_j z_0 + d_j|^2 = (c_j x_0 + d_j)^2 + (c_j y_0)^2$

For x_0, y_0 fixed,

so $\exists$ fin many $(c_j, d_j) \in \mathbb{Z}^2$ in



Claim: If γ_1 & γ_2 have same bottom row, they differ by T's.

Pf: $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix}$, $\gamma_1 \gamma_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -\beta \\ -c & \alpha \end{pmatrix} = \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}$

So alg halts, & if we know $E_k(\tau)$ for $\tau \in \mathcal{D}$, then we can extend every where by modularity.

$E_k(\underbrace{1\tau + 1}_{0\tau + 1}) = (1)^k \cdot E_k(\tau)$, $E_k\left(\frac{-1}{\tau}\right) = \tau^k E_k(\tau)$

1630s Fermat: $p \equiv 1 \ (4) \Rightarrow p = x^2 + y^2$.

1770s Lagrange: $\forall n \ge 0$, $n = x^2 + y^2 + z^2 + w^2$.

__Jacobi:__ ϑ-function: $\vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$. converges for $\tau \in \mathbb{H}$.

$$\vartheta^2(\tau) = \sum_x \sum_y e^{\pi i (x^2+y^2)\tau} = \sum_{n\geq 0} r_2(n)\, e^{\pi i n \tau},$$

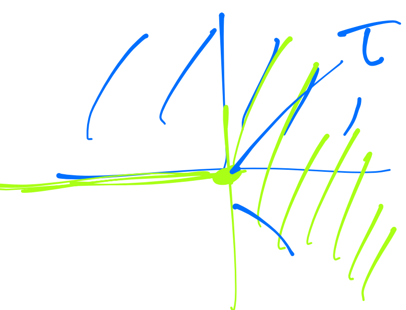
where $r_2(n) = \#\{n = x^2+y^2\}$. __Claim:__ ϑ is a $\underbrace{\text{modular form of wt } 1/2}$

__Obs 1:__ $\vartheta(\tau+2) = \vartheta(\tau)$. What about $\vartheta\left(\frac{-1}{\tau}\right)$??

__Gaussian:__ $f(x) = e^{-\pi x^2}$, $\hat{f}(\xi) = e^{-\pi \xi^2}$

$f_y(x) = f(x\cdot y)$, $\hat{f_y}(\xi) = \frac{1}{y}\hat{f}\left(\frac{\xi}{y}\right)$, $\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$.

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n\in\mathbb{Z}} f\left(n\sqrt{\tfrac{\tau}{i}}\right) = \sum_{m\in\mathbb{Z}} \sqrt{\tfrac{i}{\tau}}\, \hat{f}\left(m\sqrt{\tfrac{i}{\tau}}\right)$$

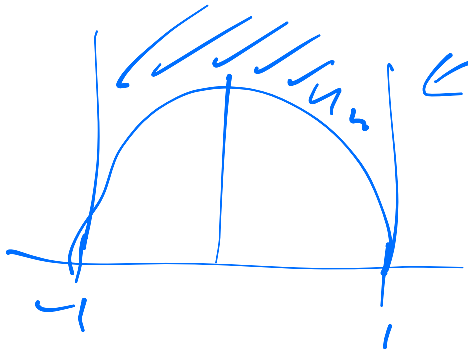


τ/i, $-\frac{\pi}{2} \leq \theta \leq \pi/2$

$$= \sqrt{\tfrac{i}{\tau}}\, \underbrace{\sum_{m\in\mathbb{Z}} e^{-\pi m^2 \frac{i}{\tau}}}_{\vartheta\left(\frac{-1}{\tau}\right)}.$$

$-\frac{\pi}{4} \leq \arg\sqrt{\frac{\tau}{i}} \leq \frac{\pi}{4}$ $\Rightarrow$ $\vartheta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\vartheta(\tau)$.

So ϑ is a "wt $1/2$" mod form on $\Gamma = \langle T^2, S\rangle$



$\longleftarrow \mathcal{D}$ = fund dom for Γ

So $\vartheta^2(\tau+2)=\vartheta^2(\tau)$, $\vartheta^2\left(-\frac{1}{\tau}\right)=\frac{\tau}{i}\vartheta^2(\tau)$.

$$f(x)=\frac{1}{\cosh(x\pi)} \qquad \hat f(\xi)=\frac{1}{\cosh(\xi\pi)}.$$

$$\Psi(\tau):=\sum_{n\in\mathbb{Z}}\frac{1}{\cos(\pi n\tau)}=\sum_{n\in\mathbb{Z}}\frac{2}{e^{i\pi n\tau}+e^{-\pi i n\tau}}$$

converges for $\tau\in\mathbb{H}$.

$$=\sum_{n\in\mathbb{Z}} f\left(\frac{n\tau}{i}\right)=\sum_{m\in\mathbb{Z}}\frac{i}{\tau}\hat f\left(\frac{mi}{\tau}\right)$$

$$=\frac{1}{i\tau}\cdot\underbrace{\sum_{m\in\mathbb{Z}}\frac{1}{\cosh\left(\frac{\pi m}{\tau}\right)}}_{\Psi\left(-\frac{1}{\tau}\right)}.$$

$$\boxed{\Psi\left(-\frac{1}{\tau}\right)=\frac{\tau}{i}\Psi(\tau).}$$

$$\Psi(\tau+2)=\Psi(\tau).$$

$F(\tau):=\Psi(\tau)/\vartheta^2(\tau)$ is automorphic i.e. invariant under Γ

$\rightarrow F$ is const. For $\tau=it$, $t\to\infty$

$\rightarrow F=1$.

So $$\vartheta^2(\tau)=\sum_{n\geq 0}r_2(n)\,e^{\pi i n\tau}=\Psi(\tau)$$

Let $q = e^{\pi i \tau}$, $|q| < 1$. $\qquad = \sum_{n \in \mathbb{Z}} \frac{2}{e^{\pi i n \tau} + e^{-\pi i n \tau}}$

$$\Rightarrow \sum_{n \geq 0} \underline{r_2(n)}\, q^n = \sum_{n \in \mathbb{Z}} \frac{2}{q^n + q^{-n}} = 1 + 4 \sum_{n \geq 1} \frac{q^n (1 - q^{2n})}{(q^{2n} + 1)(1 - q^{2n})}$$

$$= 1 + 4 \sum_{n \geq 1} \frac{q^n}{1 - q^{4n}} - \frac{q^{3n}}{1 - q^{4n}}$$

$$= 1 + 4 \sum_{n \geq 1} \left(\overbrace{q^n \cdot \sum_{l \geq 0} q^{4nl}}^{= q^{(4l+1)n}} - \overbrace{q^{3n} \cdot \sum_{l \geq 0} q^{4nl}}^{= q^{(4l+3)n}} \right).$$

$$= 1 + 4 \sum_{m \geq 1} q^m \left[\underbrace{\sum_{4l+1 \mid m} 1}_{d_1(m)} - \underbrace{\sum_{4l+3 \mid m} 1}_{d_3(m)} \right].$$

$$\boxed{\underline{\text{Thm!}} \quad n \geq 1 \quad \underline{r_2(n)} = 4\big(d_1(n) - d_3(n)\big)}.$$

$n = 45 = 5 \cdot 9$, divisors: $1, 3, 5, 9, 15, 45$

$\uparrow \ \downarrow \ \uparrow \ \uparrow \ \downarrow \ \uparrow$

$r_2(45) = \overset{\pm,\pm}{4} \cdot (\underline{4 - 2})$.

$d_1 = 4$, $d_2 = 2$.

$45 = (2 \cdot 3)^2 + (1 \cdot 3)^2 = 36 + 9, = 9 + 36.$