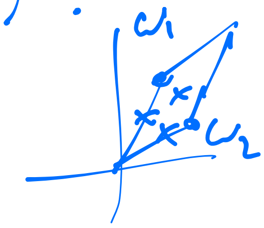
Last time:

$$\wp_\Lambda(z) = \sum_{\lambda \in \Lambda} \left[\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right]$$

$\Lambda = \langle \omega_1, \omega_2 \rangle$, $\quad \dfrac{\omega_2}{\omega_1} \notin \mathbb{R}$

$$= \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \downarrow \quad . \quad \text{elliptic}$$

& • all elliptic functions wrt Λ are $R(\wp, \wp')$.



• $\wp'(z)^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$

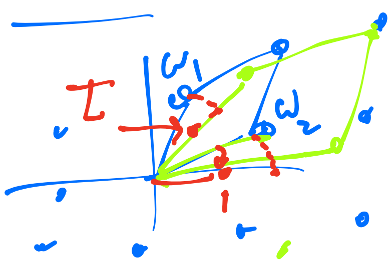
$\wp\left(\frac{\omega_1}{2}\right) = e_1$, $\wp\left(\frac{\omega_2}{2}\right) = e_2$, $\wp\left(\frac{\omega_1+\omega_2}{2}\right) = e_3$.

$(\omega_1, \omega_2) \mapsto$ invertible $\mathbb{Z}$-linear combination, i.e.

$$(a\omega_1 + b\omega_2,\ c\omega_1 + d\omega_2)^t = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\gamma} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$\Lambda = \langle \omega_1, \omega_2 \rangle = \langle \gamma(\omega_1, \omega_2) \rangle$. $\forall \gamma \in \Gamma = GL_2(\mathbb{Z})$.

Note: $u \in \mathbb{C}^\times$, $\quad \wp_{\Lambda u}(uz) = \frac{1}{u^2} \cdot \wp_\Lambda(z)$.

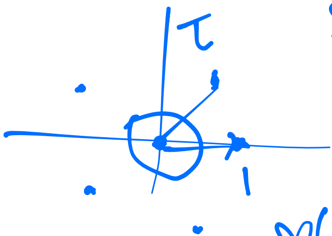


Thinking of two lattices as having the same "shape" if one is a rescaling & rotation of the other, we can reduce study of Λ to: $(\omega_1, \omega_2) \mapsto (\underbrace{\omega_1/\omega_2}_{\tau \in \mathbb{H}}, 1)$ so $\wp_\Lambda(z) \rightsquigarrow \wp_\tau(z)$

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{\lambda \in \langle \tau, 1 \rangle^*} \left[\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right]$$

$\boxed{\text{Come back to: space of lattices up to homothety}}$

Let's study Taylor series of $\wp_\Lambda(z)$ near $z=0$.



say $\forall \lambda \in \Lambda^*$, $\left|\frac{z}{\lambda}\right| \le 1/2$. ?

$$\wp(-z) = \wp(z) = \frac{1}{z^2} + \cancel{\frac{a_{-1}}{z}} + \cancel{a_0} + \cancel{a_1 z} + a_2 z^2 + \dots$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*}\left[\frac{1}{\lambda^2\left(\frac{z}{\lambda}-\frac{\lambda}{\lambda}\right)^2} - \frac{1}{\lambda^2}\right],$$

$$\frac{1}{(1-\alpha)} = 1+\alpha+\alpha^2+\alpha^3+\dots$$

$$-(1-\alpha)^{-2}(-1) = 1+2\alpha+3\alpha^2+4\alpha^3+\dots$$

$$= \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\frac{1}{\lambda^2}\left[\sum_{\ell \ge \cancel{0}\ 1}(\ell+1)\left(\frac{z}{\lambda}\right)^\ell \cancel{-1}\right]$$

$$\frac{1}{(1-\alpha)^2} = \sum_{\ell\ge 0}(\ell+1)\alpha^\ell, \quad |\alpha| \le 1/2,$$

conv abs & unif.

$$\wp_\Lambda(z) = \frac{1}{z^2} + \frac{0}{z} + 0\cdot z^0 + \sum_{\ell\ge 1} z^\ell(\ell+1)\cdot \underbrace{\sum_{\lambda\in\Lambda_\tau^*}\frac{1}{\lambda^{\ell+2}}}_{E_{\ell+2}(\tau)}$$

$\Lambda_\tau = \langle \tau, 1\rangle$

"Eisenstein series" of weight k: $E_k(\tau) = \sum_{\lambda\in\Lambda^*}\frac{1}{\lambda^k}$ $\boxed{\tau\in\mathbb{H}}$

$$= \underbrace{\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{1}{(m\tau+n)^k}}.$$

Series converges absolutely for $k>2$, i.e. $k\ge 3$.

Converges abs & unif on $\operatorname{Im}\tau \ge \delta > 0$.

If k odd, $E_k(\tau)\equiv 0$.

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{k\ge 1} z^{2k}(2k+1)E_{2k+2}(\tau)$$

So $E_k(\tau)$ is hol'c in $\tau\in\mathbb{H}$.

Replace $\langle \tau, 1\rangle$ by $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$

$$\gamma\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}a\tau+b\\ c\tau+d\end{pmatrix} \sim \begin{pmatrix}\frac{a\tau+b}{c\tau+d}\\ 1\end{pmatrix}$$

What happens to $E_k(\tau)$ when $\tau \mapsto \frac{a\tau+b}{c\tau+d}$?

$$E_k\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{\left(m\left(\frac{a\tau+b}{c\tau+d}\right)+n\right)^k} \qquad \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix} = \begin{pmatrix}u\\ v\end{pmatrix}$$

$$= (c\tau+d)^k \cdot \sum_{(m,n)\in\mathbb{Z}^{2*}} \frac{1}{\left(m(a\tau+b)+n(c\tau+d)\right)^k}$$

$$= (c\tau+d)^k \sum_{(m,n)\in\mathbb{Z}^{2*}} \frac{1}{\left(\underbrace{(ma+nc)}_{u}\tau + \underbrace{(mb+nd)}_{v}\right)^k} = (c\tau+d)^k \cdot E_k(\tau)$$

Caveats, $\frac{a\tau+b}{c\tau+d} \in \mathbb{H} \iff \det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = +1$.

Thm: $\forall \gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$, $E_k(\gamma\circ\tau) = (c\tau+d)^k E_k(\tau)$,

i.e. E_k is a modular form of weight k.

Again, $(\wp(z)) = \left(\frac{1}{z^2} + z^2\cdot 3E_4 + z^4\cdot 5\cdot E_6 + \cdots\right)$

$$\wp^2 = \left(\frac{1}{z^4} + \frac{0}{z^2} + 2\cdot 3\cdot E_4\cdot z^0 + 10E_6\cdot z^2 + \cdots\right)$$

$$\wp^3 = \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 \cdot z^0 + O(z^2).$$

$$\wp' = \frac{-2}{z^3} + 6E_4 z + 20E_6 z^3 + \cdots$$

$$(\wp')^2 = \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 \cdot z^0 + \cdots$$

$$-4\wp^3 = \frac{-4}{z^6} + \frac{-36E_4}{z^2} + {-60E_6} \cdot z^0 + \cdots$$

$$\boxed{\wp'^2 - 4\wp^3 + 60E_4\wp} = \cancel{-60E_4 \cdot \frac{1}{z^2}} \underbrace{- 140E_6}_{\text{as } z\to 0} + O(z^2).$$

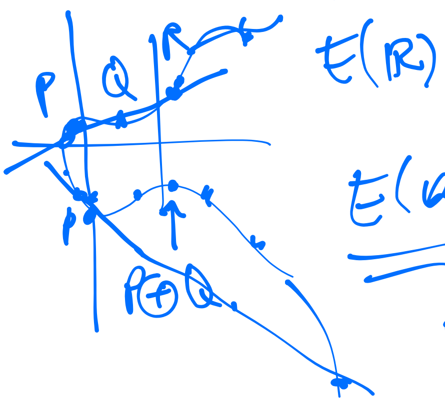
elliptic order 0 $\Rightarrow$ Const.

Thm: $(\wp')^2 = 4\wp^3 - 60E_4\wp - 140E_6.$

$$= 4(\wp-e_1)(\wp-e_2)(\wp-e_3).$$

$$= 4\wp^3 - 4\underbrace{(e_1+e_2+e_3)}_{0}\wp^2 + 4\underbrace{(e_1e_2+e_2e_3+e_3e_1)}_{-15E_4}\wp - 4\underbrace{e_1e_2e_3}_{35E_6}.$$

Given elliptic curve $\underbrace{E: y^2 = x^3 + Ax + B}$, $A, B \in \mathbb{Q}$



$\underline{E(\mathbb{Q})}$ = fin. gen. abelian gp. $= \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}^{r}$ (alg rank r)

Look at $\# E(\mathbb{F}_p) = N_p = p + 1 - a_p$

Thm: $|a_p| \le 2\sqrt{p}$. (Riemann Hypothesis for Elliptic Curves)

Shimura–Taniyama–Weil: "Ramanujan conj".

$$E_k(\underbrace{\tau+1}_{\begin{pmatrix}1&1\\0&1\end{pmatrix}\circ\tau}) = \cancel{(0\tau+1)^k}\, E_k(\overset{x+iy,\ y>0}{\tau}) = \sum_{n\in\mathbb{Z}} a_n(y)\, e^{2\pi i n x}$$

E_k is hol'c in $\tau \Rightarrow$ Cauchy–Riemann $\Rightarrow a_n(y) = a_n e^{-2\pi n y}$

$$E_k(\tau) = \sum_{n\in\mathbb{Z}} a_n \cdot e^{2\pi i n\tau}, \quad a_n = 0 \text{ for } n<0,$$

$$= \sum_{n\geq 0} a_n \cdot e^{2\pi i n \tau}$$

a_n can be worked out explicitly.

For any elliptic curve, $\exists$ F modular form s.t. Fourier coefficients $a_p(F) = a_p(E)$. $\boxed{\text{Hecke} \Rightarrow a_p \to a_n.}$

$$E \to \underbrace{a_p(E)}_{} \to a_n(E), \quad \underbrace{\sum a_n e^{2\pi i n \tau}}_{}$$

("Langlands"). $\to$ is this modular??

Frey: If $a^p + b^p = c^p$ has solution $/\mathbb{Z}$, $p\geq 5$, Then:

$$y^2 = x(x-a^p)(x-b^p)$$

observed: won't be modular. thm (Serre, Ribet)

So $\underline{\text{TSW}} \Rightarrow \underline{\text{FLT}}$. Wiles + Taylor–Wiles.

$\to 0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, -\frac{1}{7!}, 0 \ldots$

$$\underbrace{0 + z + 0z^2 - \frac{1}{3!}z^3 + 0 + \frac{z^5}{5!} + \cdots}\ = \underline{\underline{\sin z}}$$

$z \mapsto \underline{\underline{z + 2\pi}}$