Math 503 Rutgers University Fall 2020 Prof. Alex Kontorovich

Recall:

$$\left(-\frac{\zeta'}{\zeta}(s)\right) = \frac{\log 2}{2^s} + \frac{\log 3}{3^s} + \frac{\log 2}{4^s} + \frac{\log 5}{5^s} + \frac{\log 7}{7^s} + \frac{\log 2}{8^s} + \ldots$$

$$= \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}, \quad \Lambda(n) = \begin{cases} \log p, & n = p^\ell \\ 0, & \text{else.} \end{cases}$$

$t > 0$.

$\Lambda_f(t)$

f even, "smooth", "nice".

Poisson: $\longrightarrow$

$$\sum_{n\geq 1} f(nt) = \underbrace{\frac{1}{t}\sum_{m\geq 1} \hat{f}\left(\frac{m}{t}\right) + \frac{1}{2t}\hat{f}(0) - \frac{f(0)}{2}}.$$

$$\tilde{f}(s)\cdot\zeta(s) \overset{\text{Re}(s)>1}{=} \int_0^\infty \left(\sum_{n\geq 1} f(nt)\right) t^s \frac{dt}{t} = \int_1^\infty + \int_0^1 (\ \ ) t^s \frac{dt}{t},$$

$t \mapsto \frac{1}{t}$

$$= \int_1^\infty \underbrace{\left(\sum_{n\geq 1} f(nt)\cdot t^s + \sum_{m\geq 1} \hat{f}(mt) t^{1-s}\right)}_{\text{entire.}} \frac{dt}{t} - \underbrace{\frac{\hat{f}(0)}{2(1-s)} - \frac{f(0)}{2s}}$$

$\uparrow$ cont.

Chose $f(x) \overset{\hat{f}}{=} e^{-\pi x^2}$, $\tilde{f}(s) = \frac{1}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$.

$$\zeta(s) = \frac{1}{\frac{1}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)} \left[\int_1^\infty \sum_{n\geq 1} e^{-\pi n^2 t^2}\cdot\left(t^s + t^{1-s}\right)\frac{dt}{t} - \frac{1}{2(1-s)} - \frac{1}{2s}\right]$$

$\boxed{s \longleftrightarrow 1-s}$

Re $s < 0$

$$\underbrace{\pi^{-s/2}}_{\neq 0}\underbrace{\Gamma\left(\frac{s}{2}\right)}_{\neq 0}\underbrace{\zeta(s)}_{?} = \underbrace{\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)}_{\neq 0}\overset{\text{Re}(1-s)>1}{\underbrace{\zeta(1-s)}_{\neq 0}}$$

Λ

Riemann Zeta Function

Λ poles at $s = 0, 1$

Zeros at $s = 0, -1, -2, \ldots$

$s = 0, 2, 4, 6 \quad (?)$

__Need:__ zeros & poles of ζ. (Euler product $\Rightarrow \zeta \neq 0$ on $\mathrm{Re}\, s > 1$!)

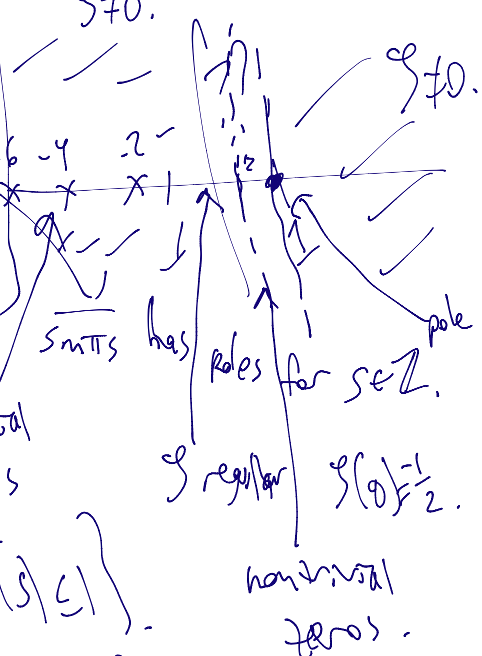
__Exercise:__ $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\underset{\sim}{\sin \pi s}}$. $(s \mapsto 1-s)$

$\underset{\sim}{0 < s \leq 1}$ simple poles at $\Rightarrow s \in \mathbb{C}$, $s = \underset{\sim}{0, -1, -2, -3, \ldots}$

simple poles at $1-s = 0, -1, -2, -3, \ldots$ $\Leftrightarrow$ $s = \underset{\sim}{1, 2, 3, 4, \ldots}$

entire $\Rightarrow$ no zeros!

$\boxed{\Rightarrow \Gamma \neq 0.}$



__Def:__ "critical strip" $= \{ s \mid 0 \leq \mathrm{Re}(s) \leq 1 \}$.

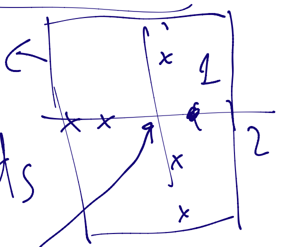
"critical line" $= \{ s = \frac{1}{2} + it,\ t \in \mathbb{R} \}$

__Explicit Formula:__

$$\sum_{n \geq 1} \Lambda(n) \varphi\left(\frac{n}{x}\right) \overset{\text{def.}}{=} \frac{1}{2\pi i} \int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right) \underbrace{\tilde{\varphi}(s)}_{\text{entire}} x^s \, ds$$

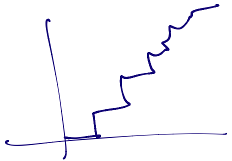
$$= \tilde{\varphi}(1) x^1 - \sum_{\rho} \tilde{\varphi}(\rho) x^{\rho}$$

pull contours



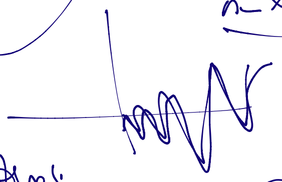
Take $\varphi(x) = \mathbb{1}_{0 < x < 1}$, $\tilde{\varphi}(s) = \frac{1}{s}$. ← not entire.

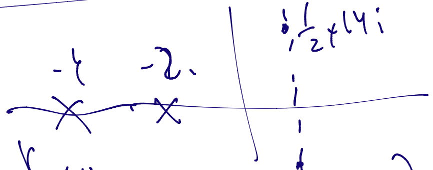
$$\sum_{n \leq x} \Lambda(n) = x^1 \cdot \frac{1}{1} - \sum_{\rho} \frac{x^{\rho}}{\rho} - \underbrace{\frac{\zeta'}{\zeta}(0) \cdot x^0}_{\log 2\pi}.$$



$$\sum_{n\le x}\Lambda(n) - (x-\log 2\pi) = -\sum_{\rho}\frac{x^{\rho}}{\rho}.$$

group ρ & 1-ρ.



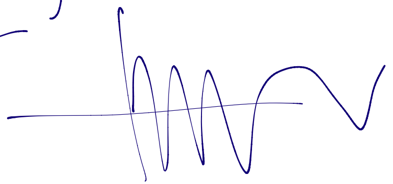


trivial zeros:

$$\frac{x^{-2}}{-2}+\frac{x^{-4}}{-4}+\frac{x^{-6}}{-6}+\dots$$

$$=\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)$$

$$x^{1/2} \equiv \left(\frac{x^{\cancel{\frac{1}{2}}+14i}}{\frac{1}{2}+14i}+\frac{x^{\cancel{\frac{1}{2}}-14i}}{\frac{1}{2}+14i}\right) \approx \cancel{x}\cdot\sin(\log \cancel{x})$$



Exercise: ① $\sum_{n<x}\Lambda(n) = \sum_{p<x}\log p + O(x^{1/2+\varepsilon})$.

② $\psi(x) = \sum_{p<x}\log p = x + o(x) \Longrightarrow \pi(x) = \sum_{p<x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$.

$\uparrow$ Li(x)..

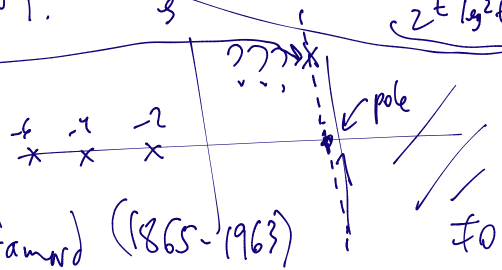
Hint: (partial summation): $\pi(x) = \int_2^x \frac{d\psi(t)}{\log t} = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}\,dt$

Need: (for PNT $\sum_{n<x}\Lambda(n) = x + o(x)$).

Pf: ① ✓

$\mathrm{Re}(\rho) < 1$ i.e. $\zeta(1+it) \ne 0$.

② $\Rightarrow \sum_{n<x}\Lambda(n) \sim x.$



1896 Hadamard (1865-1963)

de la Vallée Poussin. (1866-1962).

Riemann 1859

↳ 1994: Goldfeld-Hoffstein-Lieman
(Hoffstein-Lockhart)

uses trig trick

robust. → $3+4\cos\theta+\cos 2\theta \geq 0$,
$2(1+\cos\theta)^2$.

$$\zeta(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}, \quad \log\zeta(s) = \sum_p \underbrace{\sum_{k\geq 1} \frac{1}{kp^{ks}}}_{-\log\left(1-\frac{1}{p^s}\right)}. \quad (\operatorname{Re} s > 1).$$

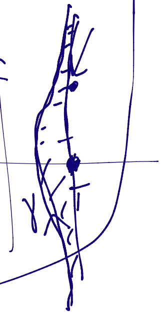
$s>1$ Fix $t>0$.

$$F_t(s) = \zeta^3(s)\,\zeta^2(s+it)\,\zeta^2(s-it)\,\zeta(s+2it)\,\zeta(s-2it). = \boxed{\zeta^3(s)\,|\zeta(s+it)|^4\,|\zeta(s+2it)|^2} \geq 0.$$

Look at
$$\log F_t(s) = \sum_p \sum_{k\geq 1} \frac{1}{kp^{ks}} \Big[\underbrace{3+2p^{-kit}+2p^{kit}+p^{2kit}+p^{-2kit}}_{|1+p^{ikt}+p^{-ikt}|^2.}\Big] \geq 0.$$

$$|1+p^{it}+p^{-it}|^2 = 1+p^{2it}+p^{-2it}+2p^{it}+2p^{-it}+2$$

$\Rightarrow$ $F_t(s) \geq 1$ for all $s>1$. (& for all t).



Suppose $\zeta(1+iT)=0$. as $s\to 1^+$, $\zeta(s+iT)\to 0$.

$$\Rightarrow \underbrace{\zeta^3(s)}_{\substack{\text{pole of order 3}\\ \sim\frac{1}{(1-s)^3}}} \underbrace{|\zeta(s+iT)|^4}_{\substack{\uparrow\\ \text{zero of}\\ \text{order} \geq 4}} \underbrace{|\zeta(s+2iT)|^2}_{\text{regular}} \longrightarrow 0 \quad \times.$$

$\boxed{\zeta(s)\neq 0 \text{ on } \sigma > .99 \Rightarrow \text{FM.}}$

$|t|>1$, $\forall\, s$ in: $\sigma > 1-\frac{c}{\log t}$.

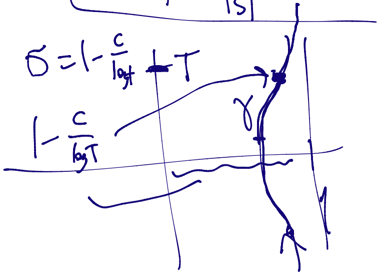
Fact: (Hadamard's Factorization) $\left|-\frac{\zeta'}{\zeta}(\sigma+it)\right| \ll \log t$

PNT:
$$\sum_n \Lambda(n)\,\varphi\!\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right)\tilde{\varphi}(s)\,X^s\,ds.$$

Pull contour
$$= X^1\cdot\tilde{\varphi}(1) + \frac{1}{2\pi i}\int_\gamma \left(-\frac{\zeta'}{\zeta}(s)\right)\tilde{\varphi}(s)\,X^s\,ds.$$

$$\boxed{\tilde{\varphi}(s) = \int_0^\infty \varphi(t)\,t^s\,\frac{dt}{t} = \int_0^\infty \varphi''(t)\frac{t^{s+1}}{s(s+1)}\,dt}$$

$$|\tilde{\varphi}(s)| \le \frac{C}{|s|^2}.$$



$$|\cdot| \le C\cdot\left[\int_0^T \underbrace{\log t \cdot \frac{1}{t^2}\, X^{1-\frac{c}{\log T}}\, dt}_{} + \int_T^\infty \underbrace{\log t\, \frac{1}{t^2}\, X^1\, dt}_{}\right]$$

$$|X^s| < X^{\sigma} < X^{1-\frac{c}{\log T}}$$

$$\ll X^{1-\frac{c}{\log T}} + X\cdot\frac{1}{T}\cdot\left(\cancel{T^{\varepsilon}}\right).$$

$$\sum_n \Lambda(n)\,\varphi\!\left(\frac{n}{X}\right) = X\cdot\tilde{\varphi}(1) + O_\varphi\!\left(X^{1-\frac{c}{\log T}} + \frac{X}{T}\right)$$

optimal choice of T: $X^{1-\frac{c}{\log T}} = \frac{X}{T}$

$$T = X^{\frac{c}{\log T}}$$

$$\log T = \frac{c}{\log T}\cdot\log X.$$

$$\log T = \sqrt{c\log X},\quad T = e^{\sqrt{c\log X}}$$

$$\boxed{\sum_n \Lambda(n)\,\varphi\!\left(\frac{n}{X}\right) = X\cdot\tilde{\varphi}(1) + O_\varphi\!\left(X e^{-\sqrt{c\log X}}\right)}$$

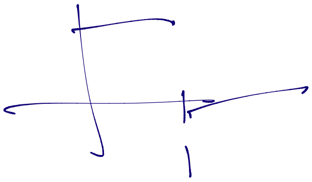
$$\frac{X}{e^{\sqrt{c\log X}}}$$

Compare: $X^{0.99} = X^{1-\underbrace{0.01}_{}\not\to 0}$

$$= \frac{X}{X^{0.01}} = \frac{X}{e^{0.01\log X}} \leftarrow \text{much better than}$$

$$\frac{x}{\log^2 x} = \frac{x}{e^{\underbrace{2\log\log x}}} \leftarrow \text{much worse!}$$

want $\varphi(x) = \mathbb{1}_{\{0<x<1\}}$



$\tilde{\varphi}(s) = \frac{1}{s}$. not $\le \frac{1}{|s|^2}$.

$\boxed{\text{Need to use smooth}}$

Note since $\exists$ zeros on $\frac{1}{2}$-line, $\forall C$,

$$\left|\sum_{n<x}\Lambda(n) - x\right| \ge C x^{1/2} \quad \text{i.o.}$$

i.e.: $\displaystyle\sum_{n<x}\Lambda(n) = x + \Omega(x^{1/2})$.

Möbius function: $\mu(n) = \begin{cases} (-1)^k, & n = \overset{\text{distinct}}{p_1 \cdots p_k} \\ 0, & \text{else } p^2 | n. \end{cases}$

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right)^{+1} = \sum \frac{\mu(n)}{n^s}$$

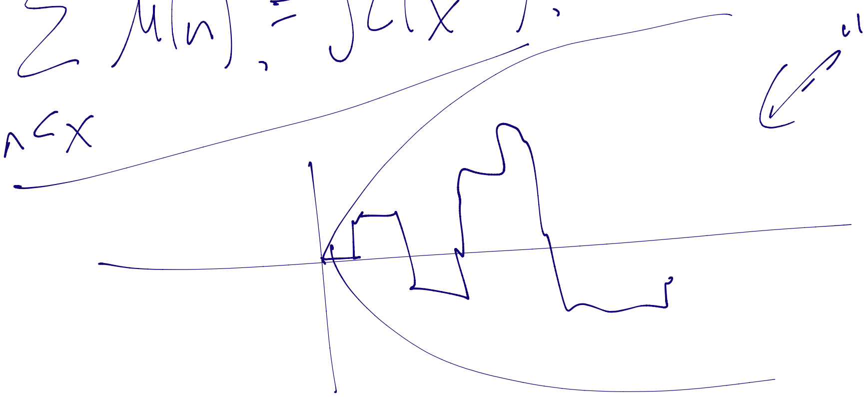
$$= \left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\cdots$$

$$= 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} + \frac{1}{10^s} + \cdots$$

$$\underline{\sum_n \mu(n)\varphi\left(\frac{n}{x}\right)} = \frac{1}{2\pi i}\int_{(2)} \frac{1}{\zeta(s)}\tilde{\varphi}(s)\, x^s\, ds \underset{\uparrow}{=} O\left(x^{1/2+\varepsilon}\right).$$

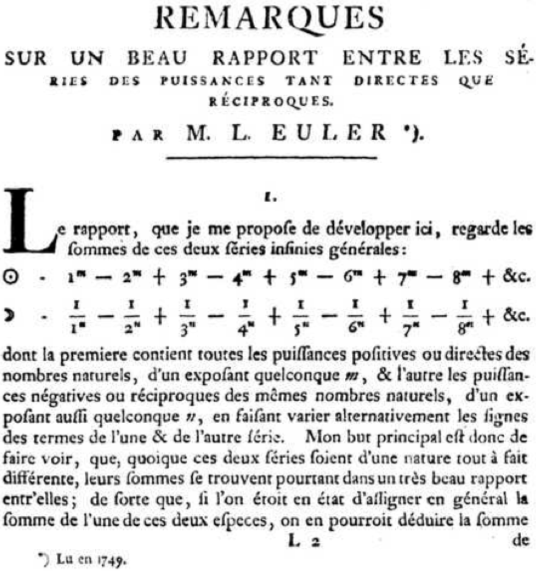
on RH

$$\sum_{n<x} \mu(n) = \Omega\left(x^{1/2}\right).$$



"Brownian motion".

The functional equation (Euler 1749...)



REMARQUES

SUR UN BEAU RAPPORT ENTRE LES SÉRIES DES PUISSANCES TANT DIRECTES QUE RÉCIPROQUES.

PAR M. L. EULER *).

1.

Le rapport, que je me propoſe de développer ici, regarde les ſommes de ces deux ſéries infinies générales:

⊙ . $1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \&c.$

☽ . $\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \&c.$

dont la premiere contient toutes les puiſſances poſitives ou directes des nombres naturels, d'un expoſant quelconque m, & l'autre les puiſſances négatives ou réciproques des mêmes nombres naturels, d'un expoſant auſſi quelconque n, en faiſant varier alternativement les ſignes des termes de l'une & de l'autre ſérie. Mon but principal eſt donc de faire voir, que, quoique ces deux ſéries ſoient d'une nature tout à fait différente, leurs ſommes ſe trouvent pourtant dans un très beau rapport entr'elles; de ſorte que, ſi l'on étoit en état d'aſſigner en général la ſomme de l'une de ces deux eſpeces, on en pourroit déduire la ſomme

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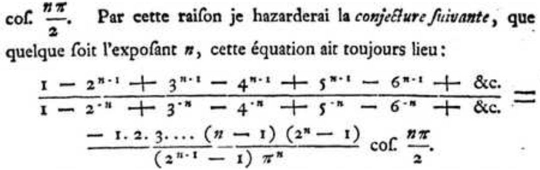
*) Lu en 1749.

The functional equation (Euler 1749...)

$$\frac{(1-2^s)\zeta(1-s)}{(1-2^{1-s})\zeta(s)} = \frac{-\Gamma(s)(2^s-1)}{(2^{s-1}-1)\pi^s}\cos(s\pi/2)$$

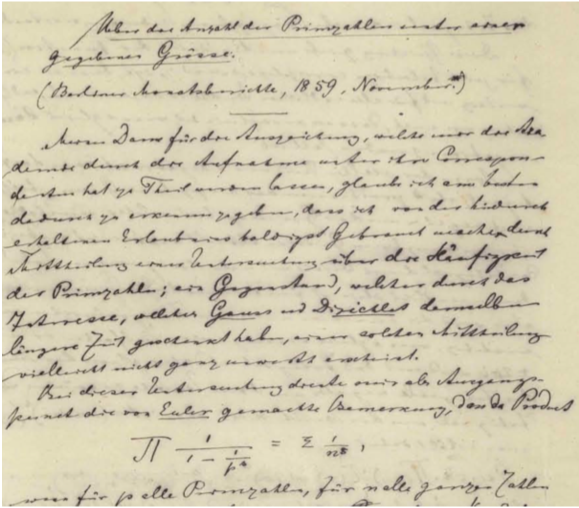


coſ. $\frac{n\pi}{2}$. Par cette raiſon je hazarderai la *conjecture ſuivante*, que quelque ſoit l'expoſant n, cette équation ait toujours lieu :

$$\frac{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + 5^{n-1} - 6^{n-1} + \&c.}{1 - 2^{-n} + 3^{-n} - 4^{-n} + 5^{-n} - 6^{-n} + \&c.} =$$

$$\frac{-1.2.3\ldots(n-1)(2^n-1)}{(2^{n-1}-1)\pi^n} \text{ coſ. } \frac{n\pi}{2}.$$

The functional equation (Riemann 1859)



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.*)

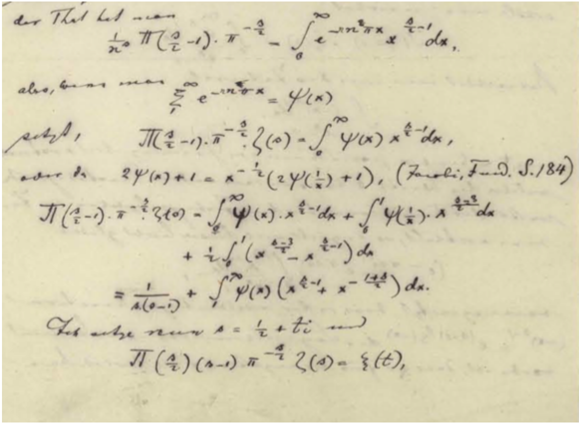
Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen

Analytic continuation (Riemann 1859)



der That hat man

$$\frac{1}{n^s}\Pi\left(\frac{s}{2}-1\right)\cdot\pi^{-\frac{s}{2}}=\int_0^\infty e^{-nn\pi x}x^{\frac{s}{2}-1}dx,$$

also, wenn man

$$\sum_1^\infty e^{-nn\pi x}=\psi(x)$$

setzt,

$$\Pi\left(\frac{s}{2}-1\right)\cdot\pi^{-\frac{s}{2}}\zeta(s)=\int_0^\infty\psi(x)\,x^{\frac{s}{2}-1}dx,$$

oder da

$$2\psi(x)+1=x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right)+1\right),$$

(Jacobi, Fund. S. 184)

$$\Pi\left(\frac{s}{2}-1\right)\cdot\pi^{-\frac{s}{2}}\zeta(s)=\int_1^\infty\psi(x)\cdot x^{\frac{s}{2}-1}dx+\int_0^1\psi\left(\frac{1}{x}\right)\cdot x^{\frac{s-3}{2}}dx$$

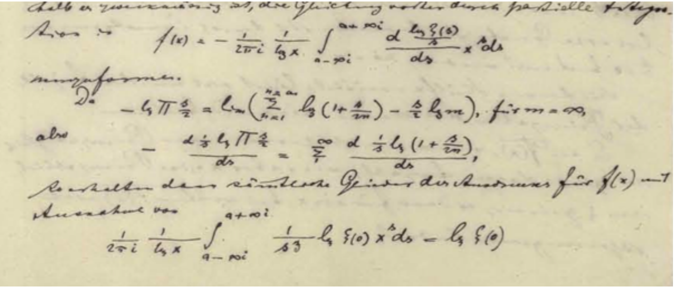
$$+\frac{1}{2}\int_0^1\left(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1}\right)dx$$

$$=\frac{1}{s(s-1)}+\int_1^\infty\psi(x)\left(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}}\right)dx.$$

Ich setze nun $s=\frac{1}{2}+ti$ und

$$\Pi\left(\frac{s}{2}\right)(s-1)\,\pi^{-\frac{s}{2}}\zeta(s)=\xi(t),$$

Contour pull (Riemann 1859)



halb er zweiwerthig ist, die Gleichung vorher durch partielle Integration in

$$f(x) = -\frac{1}{2\pi i}\frac{1}{\lg x}\int_{a-\infty i}^{a+\infty i}\frac{d\,\frac{\lg \xi(s)}{s}}{ds}x^s\,ds$$

umzuformen.

Da

$$-\lg \Pi\frac{s}{2} = \lim\left(\sum_{n=1}^{n=m}\lg\left(1+\frac{s}{2n}\right)-\frac{s}{2}\lg m\right), \text{ für } m=\infty,$$

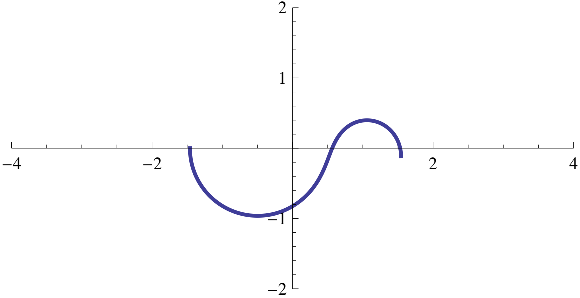
also

$$-\frac{d\,\frac{1}{s}\lg\Pi\frac{s}{2}}{ds} = \sum_1^\infty\frac{d\,\frac{1}{s}\lg\left(1+\frac{s}{2n}\right)}{ds},$$

so erhalten dann sämmtliche Glieder des Ausdrucks für $f(x)$ mit Ausnahme von

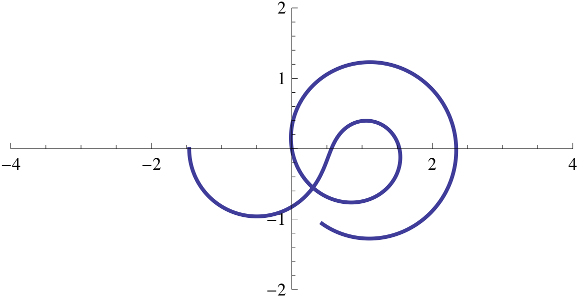
$$\frac{1}{2\pi i}\frac{1}{\lg x}\int_{a-\infty i}^{a+\infty i}\frac{1}{s^2}\lg\xi(0)\,x^s\,ds = \lg\xi(0)$$

Plot of $\zeta(1/2 + it)$



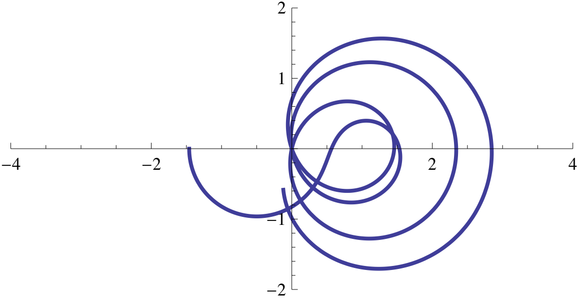
$0 < t < 10$

Plot of $\zeta(1/2 + it)$



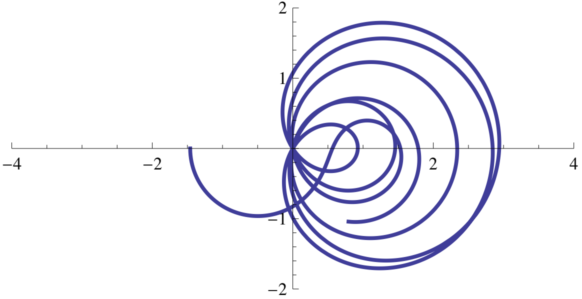
$0 < t < 20$

Plot of $\zeta(1/2 + it)$



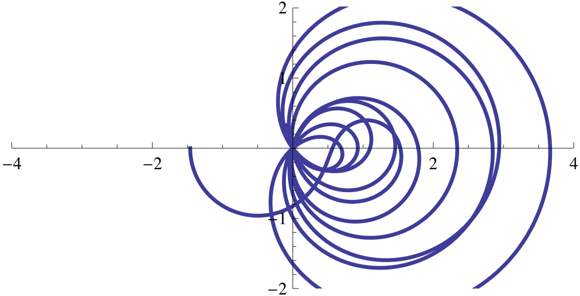
$$0 < t < 30$$

Plot of $\zeta(1/2 + it)$



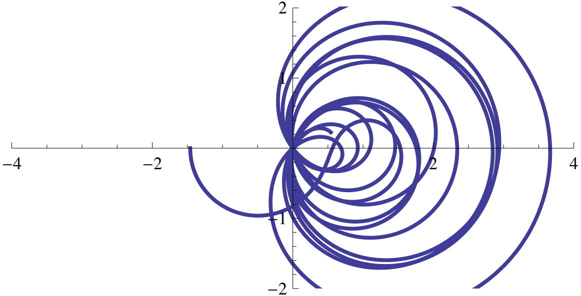
$0 < t < 40$

Plot of $\zeta(1/2 + it)$



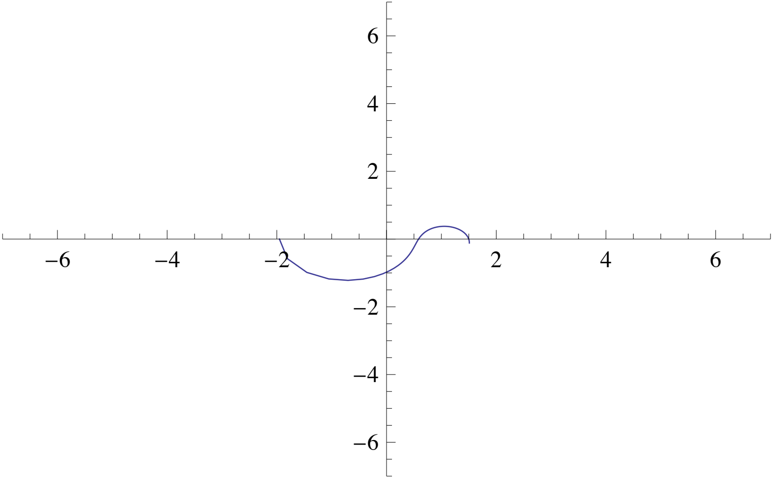
$0 < t < 50$

Plot of $\zeta(1/2 + it)$



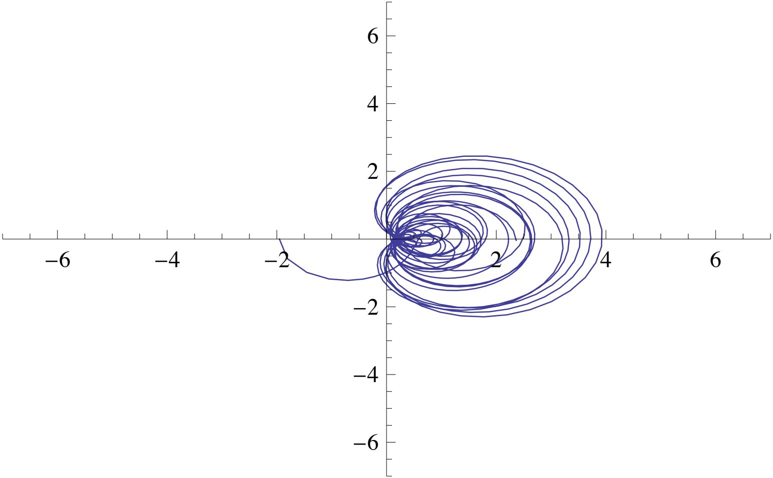
$0 < t < 60$

Plots of $\zeta(3/5 + it)$



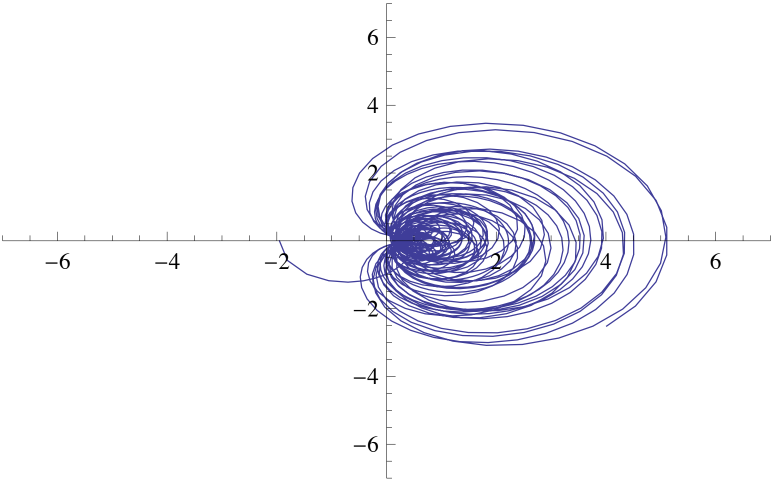
$0 < t < 10$

Plots of $\zeta(3/5 + it)$



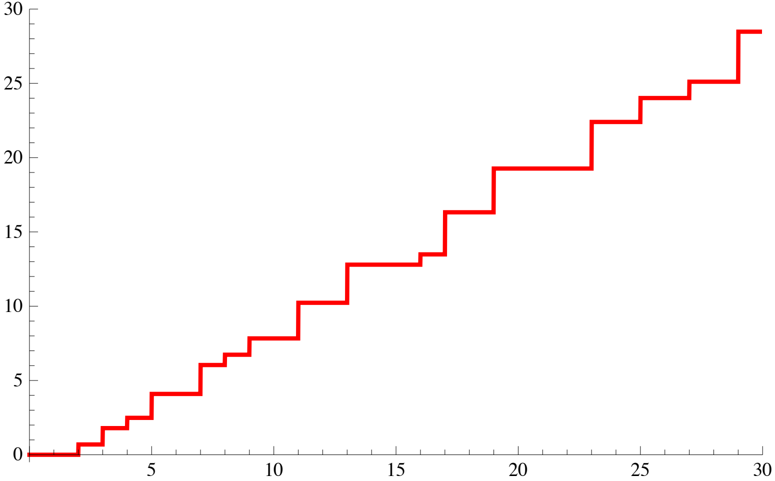
$0 < t < 100$

Plots of $\zeta(3/5 + it)$



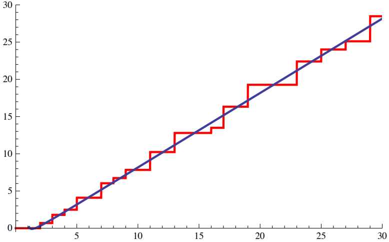
$0 < t < 200$

$$\sum_{n<X} \Lambda(n)$$

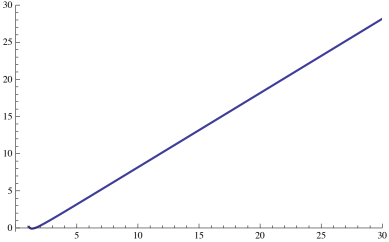


$$\sum_{n<X} \Lambda(n) = \boxed{X - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2})} - \sum_{\rho} \frac{X^{\rho}}{\rho}$$

No zeroes

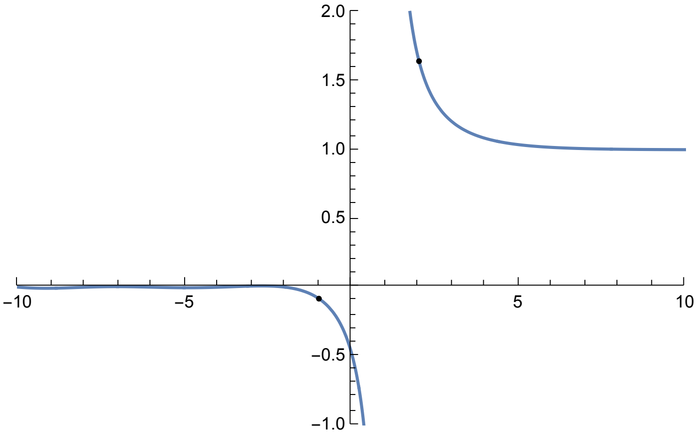


Fundamental: (PNT)



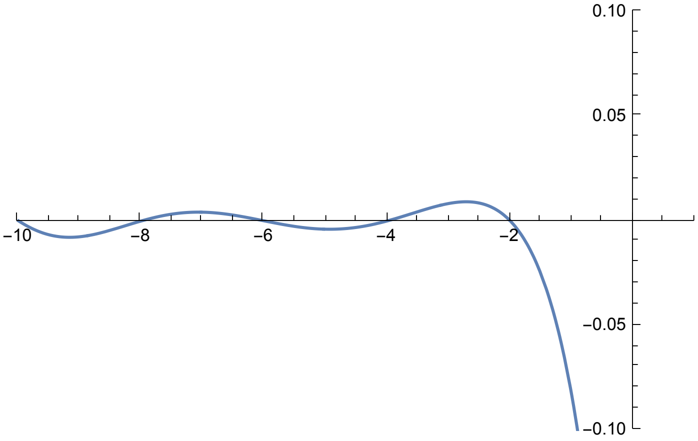
```
In[1]:= Show[Plot[Zeta[s], {s, -10, 10}, PlotRange → {{-10, 10}, {-1, 2}}]
         ,
         Graphics[Point[{2, Pi^2 / 6}]]
         ,
         Graphics[Point[{-1, -1 / 12}]]
        ]
```

Out[1]=



```
In[2]:= Plot[Zeta[s], {s, -10, 10}, PlotRange → {{-10, 1}, {-.1, .1}}]
```

Out[2]=

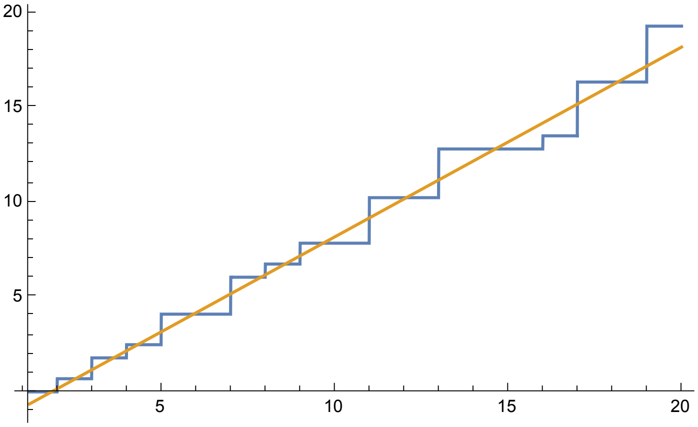


```
In[3]:= Zeta'[0] / Zeta[0]
```

```
Out[3]= Log[2 π]
```

```
In[4]:= Plot[{Sum[MangoldtLambda[n], {n, 1, x}]
           ,
          x - Log[2 Pi]
         }
         , {x, 1.2, 20}]
```

Out[4]=

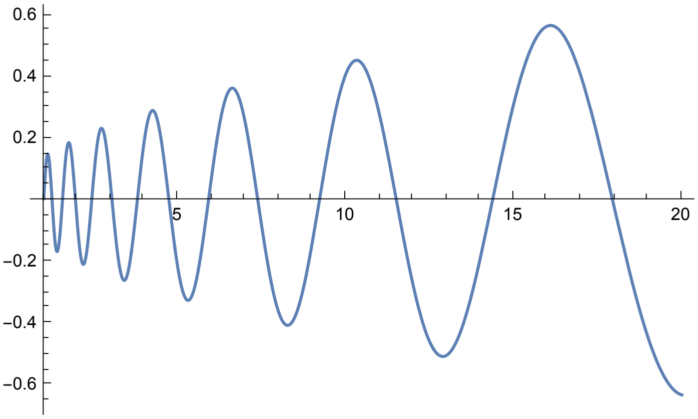


```
In[5]:= ZetaZero[1] // N
        ZetaZero[2] // N
        ZetaZero[3] // N

Out[5]= 0.5 + 14.1347 i

Out[6]= 0.5 + 21.022 i

Out[7]= 0.5 + 25.0109 i
```

```
In[8]:= kthContr[k_, x_] := x ^ ZetaZero[k] / ZetaZero[k] +
          x ^ (1 - ZetaZero[k]) / (1 - ZetaZero[k])
```

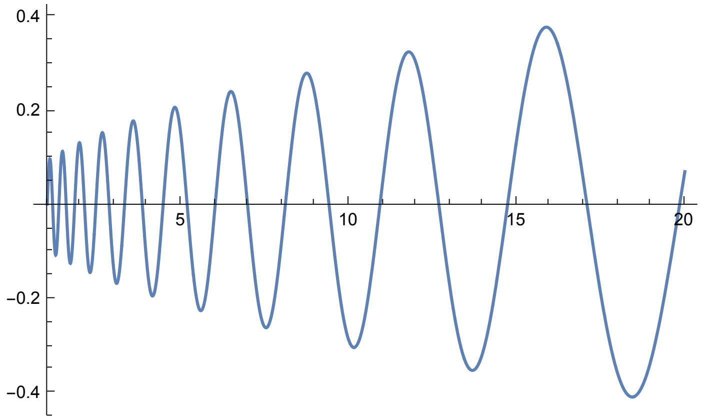
```
In[9]:= Plot[kthContr[1, x], {x, 1, 20}]
```

Out[9]=



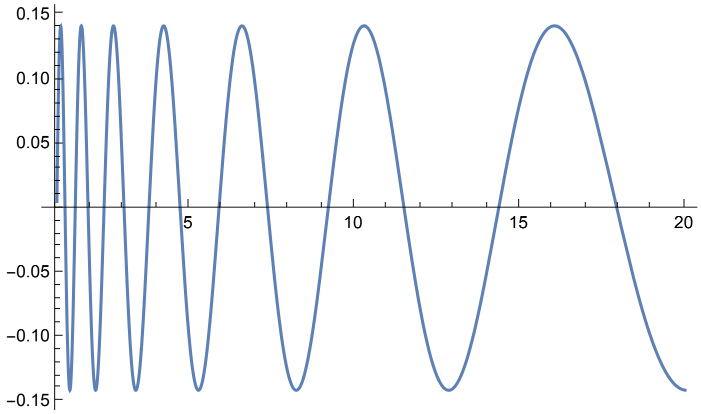
In[10]:=
```
Plot[kthContr[2, x], {x, 1, 20}]
```

Out[10]=



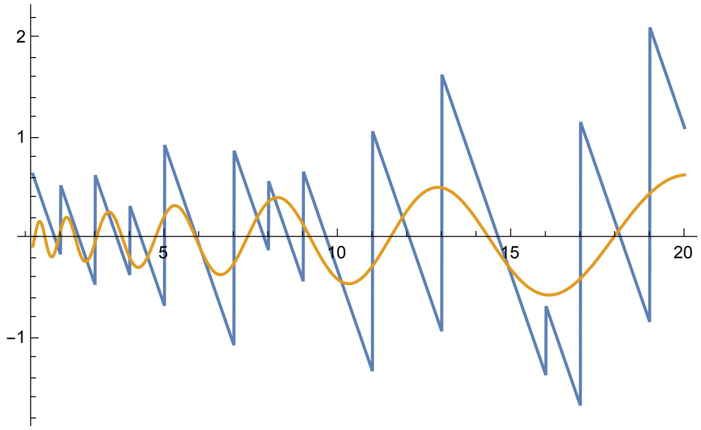
In[11]:=
```
Plot[kthContr[1, x] / x^.5, {x, 1, 20}]
```

Out[11]=



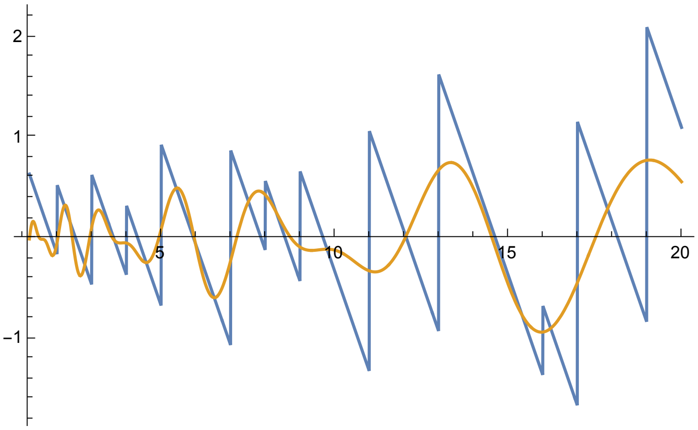
In[12]:=
```
Plot[{Sum[MangoldtLambda[n], {n, 1, x}]
    -
    (x - Log[2 Pi])
   ,
   -kthContr[1, x]
  }
 , {x, 1.2, 20}]
```

Out[12]=



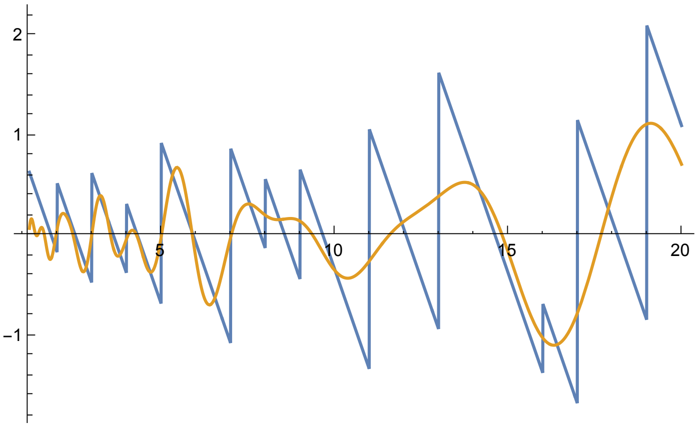
```
In[13]:= Plot[{Sum[MangoldtLambda[n], {n, 1, x}]
            -
            (x - Log[2 Pi])
           ,
           -kthContr[1, x]
            - kthContr[2, x]
          }
          , {x, 1.2, 20}]
```

Out[13]=



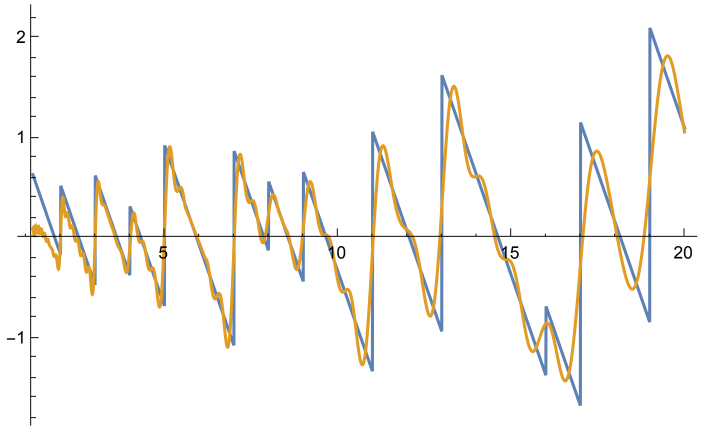
```
In[14]:= Plot[{Sum[MangoldtLambda[n], {n, 1, x}]
            -
            (x - Log[2 Pi])
           ,
           -kthContr[1, x]
            - kthContr[2, x]
            - kthContr[3, x]
          }
          , {x, 1.2, 20}]
```

Out[14]=



In[15]:=
```
Plot[{Sum[MangoldtLambda[n], {n, 1, x}]
    -
    (x - Log[2 Pi])
   ,
   -Sum[kthContr[k, x], {k, 1, 30}]
  }
 , {x, 1.2, 20}]
```

Out[15]=



In[16]:=
```
Plot[{Sum[MoebiusMu[n], {n, 1, x}], Sqrt[x], -Sqrt[x]}, {x, 1, 10 000}]
```

Out[16]=