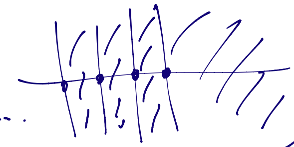
Last time: Mellin transform / inversion: $f : \mathbb{R}_{>0} \to \mathbb{C}$,

$$\tilde{f}(s) := \int_0^\infty f(t)\, t^s \frac{dt}{t}, \qquad \frac{1}{2\pi i}\int_{(2)} \tilde{f}(s)\, t^{-s}\, ds = f(t)$$

$(2) \leftarrow$ $2-i\infty \uparrow 2+i\infty$

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \frac{dt}{t}, \qquad \Gamma(s+1) = s\Gamma(s)$$

mer'c cont with simple poles at $s = 0, -1, -2, -3, \dots$



$\mathrm{Re}\, s > 0$.

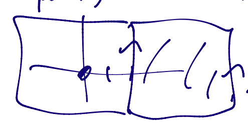
$\mathrm{Re}(s-1) > -1$

E.g.: $f(u) = \mathbb{1}_{\{0<u<1\}}$, $\quad \tilde{f}(s) = \int_0^{1} \cancel{\mathbb{1}_{(0,1)}}\, u^s \frac{du}{u} = \frac{u^s}{s}\Big|_0^1 = \frac{1}{s}$

Pole at $s=0$.

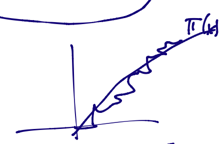
inversion? $\frac{1}{2\pi i}\int_{(2)} \frac{1}{s} u^{-s} ds \stackrel{\text{"Perron's Formula"}}{=} \begin{cases} 0, & u>1 \leftarrow \text{pull right} \\ 1, & u<1 \end{cases}$

$= f(t) = \mathbb{1}_{(0,1)}$.



(Legendre-Gauss Conj): $\pi(x) = \#\{p<x\} \stackrel{?}{\sim} Li(x) = \int_2^x \frac{dt}{\log t}$

$\sim \frac{x}{\log x}$



Recall: Poisson Summation formula: f on $\mathbb{R}$ "nice"

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

If $t>0$, & $f_t(x) = f(xt)$, then

Exercise 1: $\boxed{\hat{f}_t(\xi) = \frac{1}{t}\hat{f}\left(\frac{\xi}{t}\right)}$

$$\sum_n f(nt) = \frac{1}{t}\sum_m \hat{f}\left(\frac{m}{t}\right).$$

Assume f is even $\Rightarrow$ $\boxed{\hat{f} = \text{even} \quad \text{Exercise 2:}}$

$$= f(0) + 2\underbrace{\sum_{n\geq 1} f(nt)}_{\vartheta_f(t) = \text{"theta function"}} = \frac{1}{t}\hat{f}(0) + \frac{2}{t}\sum_{m\geq 1}\hat{f}\left(\frac{m}{t}\right).$$

E.g.: $f(x) = e^{-\pi x^2}$ Gaussian, $\hat{f}(\xi) = e^{-\pi\xi^2}$.

$\vartheta_f(t) = \sum_{n\geq 1} e^{-\pi n^2 t^2}$ at $t \to \infty$, $\boxed{\text{Exercise 3: } |\vartheta_f(t)| \leq C\cdot e^{-t} \text{ as } t\to\infty}$

As $t \to 0$? $\boxed{\vartheta_f(t) = \underbrace{\frac{1}{t}\sum_{m\geq 1}\hat{f}\left(\frac{m}{t}\right)}_{\to 0} + \underbrace{\frac{1}{2t}\hat{f}(0)}_{\sim \frac{1}{t}} - \underbrace{\frac{f(0)}{2}}_{C}}$

$|\vartheta_f(t)| \leq \frac{C}{t}$ as $t \to 0$.

Key idea in Riemann's 1859 memoir (Tate's thesis).

Compute Mellin transform of theta function!!!

$$\widetilde{\vartheta_f}(s) = \int_0^\infty \left(\sum_{n\geq 1} f(nt)\right) t^s \frac{dt}{t}. \quad (\underline{\operatorname{Re} s > 1})$$

$\int_0 |t^{s-2}| < \infty \Leftrightarrow \operatorname{Re} s - 2 > -1$

interchange

$$= \sum_{n\geq 1}\int_0^\infty f(nt)\,\underset{\text{character}}{t^s}\,\frac{dt}{t}.$$

$y = nt \leftarrow$ Haar measure $dy = n\cdot dt$, $\boxed{\frac{dy}{y} = \frac{dt}{t}}$.

$$= \sum_{n\geq 1} \frac{1}{n^s}\left[\int_0^\infty f(y)\,(y)^s\,\frac{dy}{y}\right] = \tilde{f}(s)\cdot\zeta(s).$$

$$\left(\frac{y}{n}\right)^s = y^s\cdot\frac{1}{n^s}$$

$$\tilde{f}(s)\cdot\zeta(s) = \int_0^\infty \vartheta_f(t)\,t^s\,\frac{dt}{t} = \int_1^\infty + \int_0^1$$

$$= \int_1^\infty \vartheta_f(t)\,t^s\,\frac{dt}{t} + \int_0^1 \left[\frac{1}{t}\vartheta_{\hat{f}}\left(\frac{1}{t}\right) + \frac{1}{2t}\hat{f}(0) - \frac{f(0)}{2}\right] t^s\,\frac{dt}{t}.$$

$$= \int_1^\infty \vartheta_f(t)\,t^s\,\frac{dt}{t} + \int_0^1 \frac{1}{t}\vartheta_{\hat{f}}\left(\tfrac{1}{t}\right)t^s\,\frac{dt}{t} + \frac{\hat{f}(0)}{2}\,\frac{t^{s-1}}{s-1}\Big|_0^1 - \frac{f(0)}{2}\,\frac{t^s}{s}\Big|_0^1.$$

$$\int_1^\infty y\cdot\vartheta_{\hat{f}}(y)\,y^{-s}\,\frac{dy}{y}$$

$$\begin{cases} y = \frac{1}{t}, & dy = \frac{-1}{t^2}dt. \\ \frac{dy}{y} = \frac{-\frac{1}{t^2}dt}{1/t} = -\frac{dt}{t}. \end{cases}$$

$$= \underbrace{\int_1^\infty \left[\vartheta_f(t)\,t^s + \vartheta_{\hat{f}}(t)\,t^{1-s}\right]\frac{dt}{t}}_{\text{Entire!!! all } s\in\mathbb{C}.} - \underbrace{\frac{\hat{f}(0)}{2(1-s)}}_{\substack{\text{simple pole}\\ \text{at } s=1}} - \underbrace{\frac{f(0)}{2s}}_{\substack{\text{pole at}\\ s=0.}} = \tilde{f}(s)\cdot\zeta(s).$$

mero on $\underline{\underline{\text{all}}}$ of $\mathbb{C}$

gives analytic cont of ζ.

If $f(x) = e^{-\pi x^2}$, then $\tilde{f}(s) = \int_0^\infty e^{-\pi x^2}x^s\,\frac{dx}{x}$.

$$\boxed{\frac{dt}{t} = \frac{2\pi x\,dx}{\pi x^2} = 2\frac{dx}{x}}$$

$$= \int_0^\infty e^{-t}\left(\frac{t}{\pi}\right)^{s/2}\frac{1}{2}\frac{dt}{t} = \boxed{\frac{1}{2}\cdot\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)}$$

$t = \pi x^2$, $dt = 2\pi x\,dx$.

$x = \left(\frac{t}{\pi}\right)^{1/2}$

$$\underbrace{\tfrac{1}{2}\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s)}_{=\frac{1}{2}\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)} = \underbrace{\int_1^\infty \vartheta_{e^{-\pi x^2}}(t)\left[t^s + t^{1-s}\right]\frac{dt}{t} - \frac{1}{2(1-s)} - \frac{1}{2s}}$$

symmetric in s and $1-s$!!!!

Gives new definition of ζ (Riemann Zeta!).

1950 Tate: "Completed" Zeta function

$$\xi(s) = \prod_{p\le\infty} \zeta_p(s), \quad \zeta_\infty(s) = \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right), \quad \zeta_p(s) = \frac{1}{1-p^{-s}}.$$

same function!

adelic Poisson sum.

p-adic Mellin transf.

Where are the primes??.. ($\mathrm{Re}\, s > 1$).

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad |\cdot| < 1. \qquad -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} \dots$$

$$\frac{d}{ds}\left[\log\zeta(s) = \sum_p \underbrace{-\log\left(1 - \frac{1}{p^s}\right)} = \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots\right)\right]$$

$$\frac{1}{p^s} = e^{-s\log p} \qquad \frac{d}{ds} e^{-s\log p} \cdot (-\log p)$$

$$\frac{\zeta'}{\zeta}(s) = \sum_p \left(\frac{-\log p}{p^s} - \frac{2\log p}{2p^{2s}} - \frac{3\log p}{3p^{3s}} - \dots\right).$$

$$= -\sum_{n\ge1} \frac{\Lambda(n)}{n^s} \longleftarrow \text{"von Mangoldt function".}$$

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, \ k\ge1. \\ 0, & \text{else} \end{cases}$$

$$\boxed{\mathrm{Re}\, s > 1.}$$

Obs: Look at "inverse Mellin convolution":

$$\boxed{\left(\hat f \cdot \hat g\right)^\vee = f * g.}$$

entire, nice decay etc.

$$\frac{1}{2\pi i}\int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right) \tilde\varphi(s)\, x^s\, ds$$

$$\tilde\varphi(s) = \int_0^\infty \varphi(u)\, u^s \frac{du}{u}$$

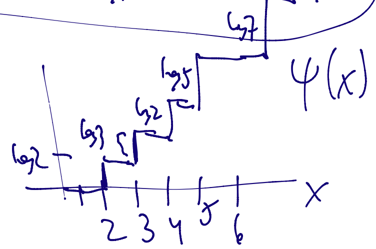
$$\varphi(u) = \frac{1}{2\pi i}\int_{(2)} \tilde\varphi(s)\, u^{-s}\, ds$$

$$= \frac{1}{2\pi i}\int_{(2)} \sum_{n\ge 1} \Lambda(n)\, \tilde\varphi(s) \left(\frac{x}{n}\right)^s ds$$

$$= \sum_{n\ge1} \Lambda(n)\, \underbrace{\varphi\left(\frac{n}{x}\right)}_{\mathbb{1}_{\frac{n}{x}<1} \Leftrightarrow \mathbb{1}_{n<x}}$$

Eg.: $\varphi(x) = \mathbb{1}_{\{x<1\}}$, $\tilde\varphi(s) = \frac{1}{s}$.

$$= \underbrace{\sum_{n<x} \Lambda(n)}\ =$$



$\psi(x)$ = Chebyshev function

$$= \sum_{p^e<x} \log p$$

Fact:

$$\psi(x) \sim x$$

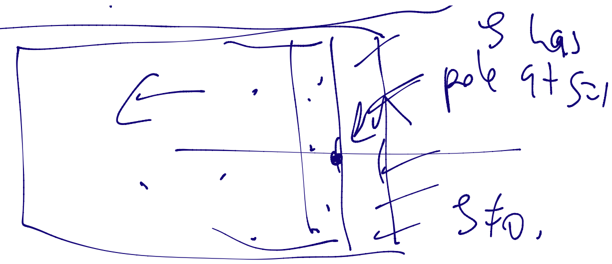
$$\Updownarrow$$

$$\pi(x) \sim \mathrm{Li}(x) \sim \frac{x}{\log x}.$$

Exercise 4: $\psi(x) = \sum_{p<x} \log p + O\left(x^{1/2+\varepsilon}\right)$

$$\left(\sum_{n\ge1} \Lambda(n)\, \varphi\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{(2)} \left(-\frac{\zeta'}{\zeta}(s)\right) \tilde\varphi(s)\, x^s\, ds.\right)$$

Encounter poles at poles of $\frac{\zeta'}{\zeta} \Leftarrow$ pole $(\Rightarrow)$



$\zeta =$ pole or $= 0$

We only care about poles of $\frac{\zeta'}{\zeta}$ (which happen to be zeros of ζ).

$\boxed{\text{Recall } \operatorname{Re} s > 1 \Rightarrow \zeta \neq 0.}$

$\to$ Pull contours: Recall Argument Principle $\operatorname{Res} \frac{\zeta'}{\zeta} = \begin{cases} 1 & \zeta = 0 \\ -1 & \zeta = \infty \end{cases}$

$$= \cancel{\frac{1}{2\pi i} \int_{(-1000)} \left(\underbrace{-\frac{\zeta'}{\zeta}(s)} \right) \tilde{\varphi}(s) X^s \, ds} + \sum_{\substack{\rho: \\ \zeta(\rho) = 0}} (-1) \tilde{\varphi}(\rho) X^{\rho} + \underset{(s=1)}{1 \cdot \tilde{\varphi}(1) \cdot X^1}$$

$(-1000) \to \infty$

$\to$ "Poisson summation on the primes":

$$\sum_{n \geq 1} \Lambda(n) \varphi\left(\frac{n}{X}\right) = \tilde{\varphi}(1) \cdot X - \sum_{\rho} \tilde{\varphi}(\rho) X^{\rho}.$$

sum on primes

dual sum, sum on zeros

Want: $\varphi(x) = \mathbb{1}_{0 < x < 1}$ then $\psi(x) \sim X$

Remain to show: $\operatorname{Re} \rho < 1$.