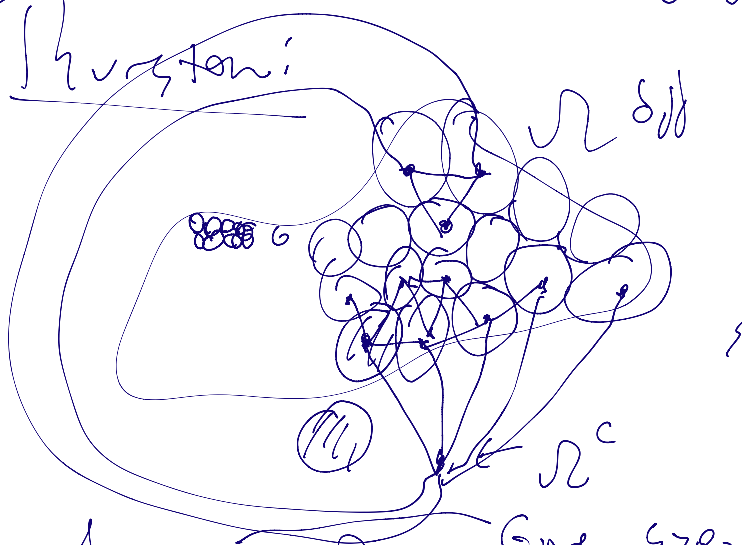
Explicit Riemann mapping Thm:

(Version by Koebe). $\mathcal{F} := \{ f : U \to \mathbb{D} \mid f(0)=0, \text{ holo, inj} \}$,

Found $f \in \mathcal{F}$ with $\sup_{g \in \mathcal{F}} |g'(0)| = |f'(0)|$.

Thurston:



Fix $\varepsilon > 0$, pack Ω with D_ε, s.t. no D_ε can be added & no $D_\varepsilon \subset \Omega^c$

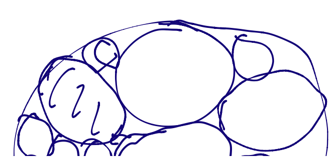
Step 1: Pack Ω.

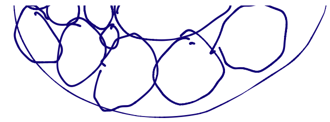
Step 2: Unravel Combinatorics of the packing

Take "nerve" of packing. i.e. make a planar, triangulated graph of disk tangencies.

Step 3: Geometrize (Koebe-Andreev-Thurston Thm)

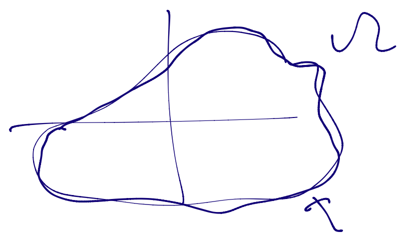
topological/combinatorial $\longrightarrow$



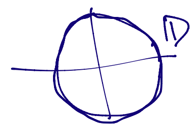
graph $\longrightarrow$ 

Step 4: Send $\varepsilon \to 0$. (Limit as $\varepsilon \to 0$ is a Riemann map $\Omega \to \mathbb{D}$ Rodin–Sullivan).

§ "Dirichlet Problem" $\Omega \subset \mathbb{R}^2$



Want to solve
$$\begin{cases} \Delta u = 0 \text{ . (} u \text{ harmonic)} \\ (\partial_{xx} + \partial_{yy}) \\ u|_{\partial\Omega} = f \leftarrow \text{given} \end{cases}$$
&



In $\mathbb{D}$, Known ("easy"): $u(z) = u(re^{i\theta}) =$ Convolve with Poisson Kernel!

$P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$

$$= \int_{\partial\mathbb{D}} P_r(\varphi) f(\theta - \varphi)\, d\varphi.$$

Facts: Solves Dirichlet problem

How to solve elsewhere? Riemann Map!

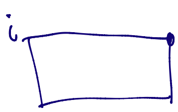
If $\Delta u = 0 \Rightarrow u = \mathrm{Re}(g)$, g = hol.

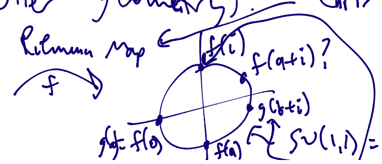
Solve $\begin{cases} \Delta \tilde{u} = 0 \\ \tilde{u}|_{\partial\mathbb{D}} = \underbrace{f \circ F}_{\tilde{f}} \end{cases}$ on $\mathbb{D}$ $\Rightarrow \tilde{u} = \mathrm{Re}(\tilde{g})$, $\tilde{g}$ hol. on $\mathbb{D}$.

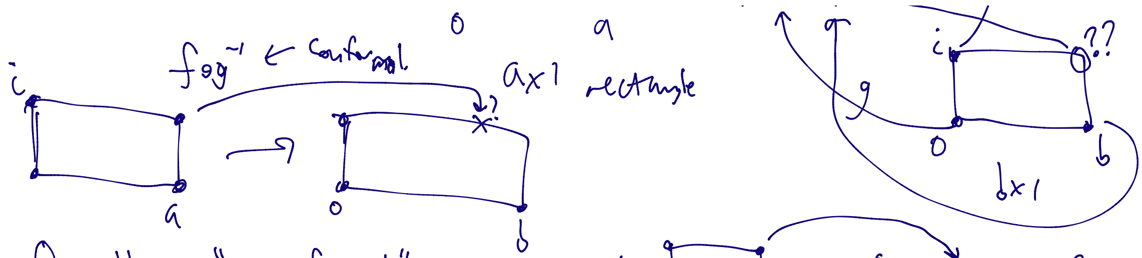
Then $g = \underbrace{\tilde{g} \circ F^{-1}}_{\text{hol.}} : U \to \mathbb{C}$ hol., $\mathrm{Re}(g) = u$ harmonic. $u|_{\partial U} = f$

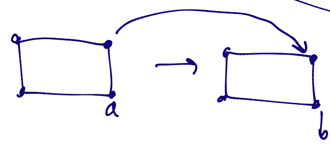
§ Quasi-conformality (Teichmüller geometry). Christoffel?

Basic Version:





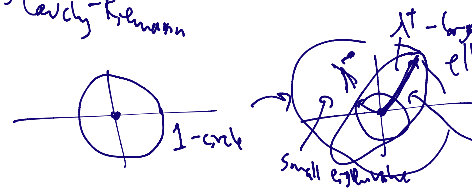


Q: How "unconformal" ?? must a map:  be?

Recall if $f:\Omega\to\mathbb{C}$ is $\mathbb{R}^2$-differentiable, ⇒ near z_0, $f(z)=f(z_0)+L\cdot(z-z_0)+o(z-z_0)$.

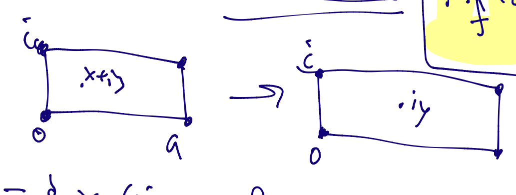
$f=u+iv$. $\quad \frac{\partial}{\partial z}=\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\frac{1}{2}$, $\quad \partial_{\bar z}=\frac{1}{2}\left(\partial_x-\frac{1}{i}\partial_y\right)$. $\quad \begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}$ $\quad \begin{cases}u_x=v_y,\\ u_y=-v_x.\end{cases}$

f hol ⇔ $\partial_{\bar z}f=0$ ⇔ Cauchy-Riemann

L = linear map 

Exercise: outradius $=|f_z|+|f_{\bar z}|$
inradius $=|f_z|-|f_{\bar z}|$.

"Unconformality" measured by dilatation $\boxed{K_f(z_0):=\frac{|f_z(z_0)|+|f_{\bar z}(z_0)|}{|f_z(z_0)|-|f_{\bar z}(z_0)|}}$.

Eg: Find map  $(b>a)$. 

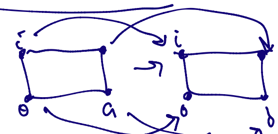
$f(x+iy)=\frac{b}{a}\cdot x+iy$ affine. $\quad K_f(z_0)\overset{?}{=}\frac{f_z+f_{\bar z}}{f_z-f_{\bar z}}=\frac{\frac{b}{a}}{1}=\frac{b}{a}\geq 1.$

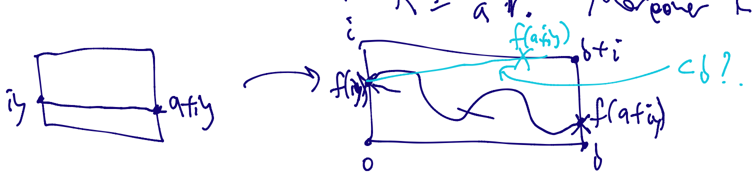
indep of z_0

$f_z=\frac{1}{2}\left(f_x+\frac{1}{i}f_y\right)=\frac{1}{2}\left(\frac{b}{a}+\frac{1}{i}i\right)=\frac{1}{2}\left(\frac{b}{a}+1\right).$

$f_{\bar z}=\frac{1}{2}\left(\underbrace{\frac{b}{a}-1}_{>0}\right)$ $\quad \boxed{\text{If } f \text{ is hol, } f_{\bar z}=0 \Rightarrow K_f=1.}$

Def: $f:U\to\mathbb{C}$ is K-quasiconformal if $\forall z\in U$, $K_f(z)\leq K$.

Thm (Grötzsch): Let f:  $(b>a)$ be K-q.c. & f: corners ↦ corners. Then $K\geq\frac{b}{a}$. Moreover $K=\frac{b}{a}\Rightarrow f$ is affine.

Pf: 

$f_x=f_z+f_{\bar z}$.

For fixed $y\in(0,1)$.

$$b \le |f(a+iy)-f(iy)| = \left|\int_0^a f_x(x+iy)\,dx\right| \le \int_0^a |f_x(x+iy)|\,dx \le \int_0^a \left(|f_z|+|f_{\bar z}|\right) dx.$$ Integrate in y.

$$b \le \int_0^1\int_0^a \left(|f_z|+|f_{\bar z}|\right) dx\,dy.$$

$$\left[\frac{(|f_z|+|f_{\bar z}|)^{1/2}}{(|f_z|-|f_{\bar z}|)^{1/2}}\,(|f_z|+|f_{\bar z}|)^{1/2}\cdot(|f_z|-|f_{\bar z}|)^{1/2}\right] = K_f(x+iy)^{1/2}\,\underbrace{\left(|f_z|^2-|f_{\bar z}|^2\right)^{1/2}}_{(Jac\, f)^{1/2}}.$$

Cauchy-Schwarz

$$b^2 \le \underbrace{\int_0^1\int_0^a K_f(x+iy)\,dx\,dy}_{\le K}\cdot\underbrace{\int_0^1\int_0^a |Jac\, f|\,dx\,dy}_{= b}$$

assumed

$\boxed{\text{Exercise: } |f_z|^2-|f_{\bar z}|^2 = Jac(f)}$

$$\int_0^1\int_0^b 1\,dx\,dy = \int_0^1\int_0^a |Jac_f|\,dx'\,dy'$$
$x,y \mapsto f^{-1}(x+iy) = (x',y')$.

$(x',y') \xrightarrow{f} (x,y)$

$$\Rightarrow b^2 \le K\cdot a\cdot b.$$

$\Rightarrow K \ge \frac{b}{a}$. If $K = \frac{b}{a}$ then all "$\le$" are "$=$", $\boxed{f_x = u_x,\ f_y = i v_y.}$

$$L = \begin{pmatrix} u_x & \cancel{v_x} \\ \cancel{u_y} & v_y \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix} \quad \forall z.$$

$$b = \int_0^a f_x\,dx\,dy = \int_0^a |f_x|\,dx\,dy = \int_0^a \left(|f_z|+|f_{\bar z}|\right) dx\,dy$$

$f_x = u_x + i v_x \Rightarrow v_x = 0.$

$f_x = f_z + f_{\bar z}.$

$\boxed{f_z = u_x + v_y,\quad f_{\bar z} = u_x - v_y}$

$\boxed{f_z = \tfrac12\left(f_x + \tfrac1i f_y\right),\quad f_{\bar z} = \tfrac12\left(f_x - \tfrac1i f_y\right).}$ $\Rightarrow u_y = 0.$

When does CS lose nothing? $\left(\int f\cdot g\right)^2 = \int f^2\cdot\int g^2.$
$\Rightarrow f, g$ constants.

$\Rightarrow K_f = K$ & $J_f = J$ indep of $z \in$ rectangle.

$$K_f = \frac{|f_z|+|f_{\bar z}|}{|f_z|-|f_{\bar z}|}, \qquad J = |f_z|^2-|f_{\bar z}|^2 = \left(|f_z|+|f_{\bar z}|\right)\left(|f_z|-|f_{\bar z}|\right).$$

$\Rightarrow f$ is affine.

product & quotient = const

$\Rightarrow$ $f_z + f_{\bar z}$ = const, $f_z - f_{\bar z}$ = const; $u_x = c$, $v_y = c'$.