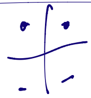
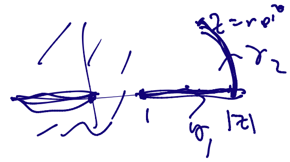
$\frac{d}{dz}\left[(z-z_1)(z-z_2)(z-z_3)(z-z_4)\right]$ $\quad \frac{d}{dz}p(z)\Big|_{z=z_1} = 0 + a_1(z-z_1)+\dots$ $\quad a_1 + O(z-z_1)$ $\quad \frac{p(z)}{z-z_1}\Big|_{z\to z_1} = a_1$





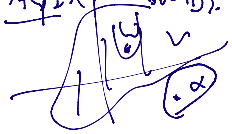
$\frac{d}{dz}(z^4+1) = 4z_1^3$

$z = re^{i\theta}$, γ_2, γ_1, 1, $|z|$

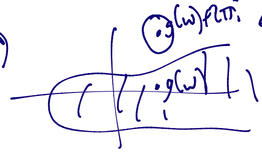
$$\oint_{|z|=r}\frac{dw}{w} \overset{\text{indep}}{=} \underbrace{\int_1^r \frac{dr}{r}}_{\gamma_1} + \underbrace{\int_0^\theta \frac{re^{i\varphi}\, i\, d\varphi}{re^{i\varphi}}}_{\gamma_2,\ w=re^{i\varphi},\ 0<\varphi<\theta} = \log_{\mathbb R} r + i\theta.$$

Last time: Riemann Mapping Thm: $U\subset\mathbb C$ simply connected, proper.

Step 1: (get inside $\mathbb D$).

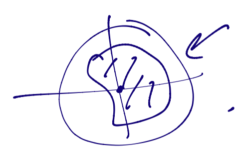


log



$g(w)+2\pi i$ ← get a ball in U^c,

move in ball



U

Step 2: $\mathcal F = \{ f: U\to\mathbb D \mid \text{inj, hol'c}, f(0)=0\}$, Let $1\le S = \sup_{g\in\mathcal F}|g'(0)|<\infty$

Claimed: $\exists\, f\in\mathcal F$ s.t. $|f'(0)| = S$

Step 3: This f is onto. (hence bijective $\Rightarrow$ conformal).

Montel's Thm

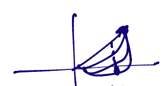
Def: • $\mathcal F$ = family of (hol'c) functions $U\to\mathbb C$, is unif bdd on compacta if: $\forall K\overset{cpt}{\subset}U$, $\exists B: \forall z\in K\ \forall f\in\mathcal F$, $|f(z)|<B$.

• $\mathcal F$ is equicontinuous on compacta if: $\forall K\overset{cpt}{\subset}U$, $\forall\varepsilon>0\ \exists\delta>0: \forall z,w\in K, \forall f\in\mathcal F$, $|z-w|<\delta \Rightarrow |f(z)-f(w)|<\varepsilon$.

• $\mathcal F$ is normal if $\forall$ sequence $\{f_j\}\subset\mathcal F$, $\exists$ subseq $\{f_{n_j}\}$ converges uniformly on compacta. ($\forall K\overset{cpt}{\subset}U$ $\forall\varepsilon>0$, $\exists N: \forall n_j, n_{j'}\ge N, \forall z\in K$, $|f_{n_j}(z)-f_{n_{j'}}(z)|<\varepsilon$)

Thm: $\mathcal F$ is hol'c functions, unif bdd on compacta $\underset{\mathbb C}{\Rightarrow}$ equicontinuous on cpt $\Rightarrow$ normal (Arzela-Ascoli)/$\mathbb R$.

Ex: $f_n(x) = x^n$ on $[0,1]=U$.

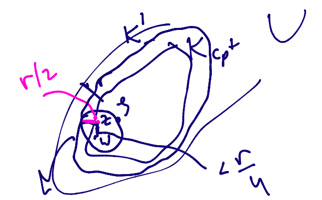


x_0

Unif Bdd ✓ Not equicont, $|f_n(1)-f_n(x_0)|\to 1$ as $n\to\infty$. ($f_n(x_0)\to 0$)

Pf: Assume unif bdd on all cpta, Want: equicont on cpta. Given $K\overset{cpt}{\subset}U$.

Given $\varepsilon > 0$. $\exists\, r>0$ s.t. $\operatorname{dist}(K, U^c) = r > 0$. Given $f \in \mathcal{F}$.



Let $z, w \in K$, $|z-w| < \frac{r}{4}$

$$|f(z) - f(w)| \le \left|\frac{1}{2\pi i}\int_{\partial D_{\frac{r}{2}}(z)} \underset{\in B}{f(\xi)}\underbrace{\left[\frac{1}{\xi - z} - \frac{1}{\xi - w}\right]}_{\frac{|\xi-w-(\xi-z)|}{|(\xi-z)(\xi-w)|}}\, d\xi\right|$$

$$|\xi - z| \ge \frac{r}{2}, \quad |\xi - w| \ge \frac{r}{4}.$$

Note: $\underline{\underline{K'}} = \overline{\bigcup_{z\in K} D_{\frac{r}{2}}(z)} \subset U$ still cpt.

$$\le \underbrace{\frac{B|z-w|}{2\pi \,\frac{r}{2}\cdot\frac{r}{4}}\cdot 2\pi\cdot\frac{r}{2}}$$

So $\exists\, B < \infty$ s.t. $\forall\, \xi \in K'$, $|f(\xi)| < B$.

To make $\uparrow < \varepsilon$, choose $\delta =$ Exer

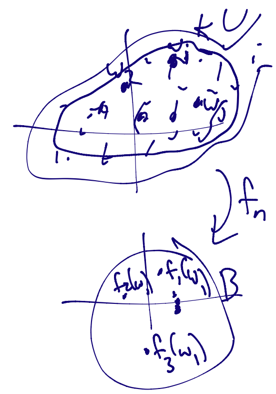
Now: Assume $\mathcal{F}$ is unif bdd on cpts & equicont on cpts. Want: normal.

Given $\{f_i\} \subset \mathcal{F}$. $U \subset \mathbb{C}$. $\mathbb{Q}(i) = a + bi$, $a, b \in \mathbb{Q}$ are dense countable.

Let $\{w_i\} \subset U$ dense countable.

Look at $f_j(w_1)$ on $K = \{w_1\}$. Unif bdd $\Rightarrow \exists$ subseq

$\{f_{n,1}\} \subset \{f_i\}$ s.t. $f_{n,1}(w_1)$ converges. (only at w_1).



Look at $f_{n,1}(w_2)$ $\leftarrow$ unif bdd $\Rightarrow \exists$ subseq

$\{f_{n,2}\} \subset \{f_{n,1}\}$ $f_{n,2}(w_2)$ converges (& w_1).

$\vdots$

$\forall j$, $\{f_{n,j}\}$ on which have convergence at $w_1, \ldots, w_j$.

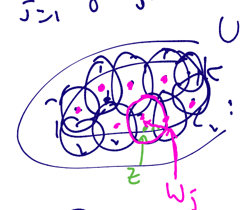
Let $g_n := f_{n,n} \in \mathcal{F}$ Then g_n converges at w_j, $\forall\, j = 1, 2, \ldots$

Fix some $K \overset{\text{cpt}}{\subset} U$ Claim: g_n converges unif on K. Let $\varepsilon > 0$ be given,

Let $\delta > 0$ come from equicontinuity (s.t. $\forall\, z, w \in K$, $\overset{\forall g_n \in \mathcal{F}}{|z-w| < \delta} \Rightarrow |g_n(z) - g_n(w)| < \varepsilon$).
$\delta = \delta(\varepsilon, K)$.

Then $\bigcup_{j=1}^{\infty} D_\delta(w_j) \supset K$. $\Rightarrow \exists$ finite sub cover $K \subset \bigcup_{j=1}^{J} D_\delta(w_j)$.

g_n converge at w_j, i.e. $\forall \varepsilon > 0$, $\exists N_j$ s.t. $n, m \ge N_j$,
then $|g_n(w_j) - g_m(w_j)| < \varepsilon$. Let $N = \max_{1 \le j \le J}(N_j) < \infty$.



So: If $n, m \ge N$. Then: $\forall z \in K \Rightarrow \exists w_j$ s.t. $z \in D_\delta(w_j)$.

$$|g_n(z) - g_m(z)| \le \underbrace{|g_n(z) - g_n(w_j)|}_{< \varepsilon \text{ by equicont.}}$$

This suffices for our application but see "Exhaustion"

$<\varepsilon$ since $g_n(w_j)$ converges at w_j. $+|g_n(w_j)-g_m(w_j)|$ $+|g_m(w_j)-g_m(z)|$ $<3\varepsilon$.

Note: let $K_\ell := \{z\in U, |z|\le \ell, \operatorname{dist}(z,U^c)\ge \frac{1}{\ell}\}$ is compact

Then $U=\bigcup_{\ell=1}^{\infty} K_\ell$, & any $K \overset{cpt}{\subset} U$ is contained in some K_ℓ.

$g_n^{(1)}$ converges unif on K_1, $g_n^{(2)}$ conv unif on K_2, ..., $h_n = g_n^{(n)}$ conv unif on all K_ℓ's.

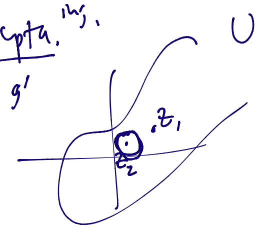
Prop: If U connected, $\{f_n\}$ of inj, holo functions on U s.t. f_n converge unif on compacta. Then $f=\lim f_n = \begin{cases}\text{inj} & \text{or}\\ \text{Const}.\end{cases}$

Pf: If f is not injective, $\exists z_1,z_2\in U$: $f(z_1)=f(z_2)$. Look at

$g_n(z) := f_n(z)-f_n(z_1)$ $\leftarrow$ Converge unif on cpta. inj.

$g_n \to g = f(z)-f(z_1)$. $\neq 0$ except at $z=z_1$. $g_n'\to g'$

f non const $\Rightarrow$ if z_2 is a zero of g then it is isolated. So $g\neq 0$ on $\partial D_\varepsilon(z_2)$.



Look at:

$$\frac{1}{2\pi i}\int_{\partial D_\varepsilon(z_2)} \frac{g_n'(w)}{g_n(w)}\,dw = 0$$

By unif conv on $K=\partial D_\varepsilon(z_2)$, $n\to\infty$, $\frac{1}{g_n}\neq 0$ on $\partial D_\varepsilon(z_2)$

$$\Downarrow$$

$$\frac{1}{2\pi i}\int_{\partial D_\varepsilon(z_2)} \frac{g'(w)}{g(w)}\,dw = 1.$$ $\times$

Back to: $\mathcal{F}=\{f: U\to \mathbb{D} \mid \text{inj, holo}, f(0)=0\}$, $1\le S = \sup_{g\in\mathcal{F}} |g'(0)| <\infty$

Let $\{f_n\}$ with $|f_n'(0)|\nearrow S$. By Montel, $\{f_n\}$ has sub seq, normal (conv unif on cpta). By Prop, $\lim f_{n_j} = f \leftarrow$ holo, non const ($|f'(0)|\ge 1$) $\Rightarrow$ inj. $f(0)=0$,

Know: (continuity) $|f(z)|\le 1$ By Max Modulus $|f(z)|<1$. $\forall z\in U$.

$\Rightarrow f\in\mathcal{F}$.