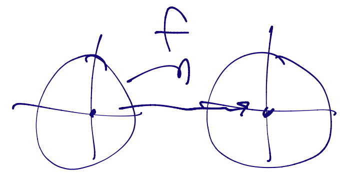
Recall: Schwarz Lemma: $f:\mathbb{D}\to\mathbb{D}$ holo. (not inj, onto), $f(0)=0$.



- $|f(z)|\le|z|$ & $=$ $\Rightarrow$ $f=$ rotation
- $|f'(0)|\le 1$ & $=$ $\Rightarrow$ $f=$ rotation.

$\mathrm{Aut}\,\mathbb{D}=PSU(1,1)$, $\mathrm{Aut}\,\mathbb{H}=PSL_2(\mathbb{R})$

Note: $\mathbb{D}$ is itself defined by quadratic form:

$$\mathbb{D}=\left\{z\in\mathbb{C}\ \middle|\ \underbrace{(\bar z\ \ 1)\begin{pmatrix}-1&0\\0&1\end{pmatrix}}\begin{pmatrix}z\\1\end{pmatrix}>0\right\}.$$

$$(-\bar z\ \ 1)\begin{pmatrix}z\\1\end{pmatrix}=-|z|^2+1>0$$

$$\iff 1>|z|^2 \iff z\in\mathbb{D}.$$

Also, $\mathbb{H}=\left\{z\in\mathbb{C}\ \middle|\ \underbrace{(\bar z\ \ 1)\begin{pmatrix}0&i\\-i&0\end{pmatrix}}\begin{pmatrix}z\\1\end{pmatrix}>0\right\}$

$\{\mathrm{Im}\, z>0$

$\iff -i(z-\bar z)=2\,\mathrm{Im}\, z$

$$0<(-i\ \ +i\bar z)\begin{pmatrix}z\\1\end{pmatrix}=-iz+i\bar z$$

$$=-i(z-\bar z)=-i(2iy)$$

$$=2y>0$$

Riemann Mapping Thm: Every $U,V\subset\mathbb{C}$ proper, conn, ($U\neq\emptyset,\mathbb{C}$) simply connected sets $\Rightarrow$ U & V are conformal!!!!

Follows from: Every such U conformal to $V=\mathbb{D}$. $U\to\mathbb{D}\to V$.

Step 1: Get U inside $\mathbb{D}$. Given arbitrary $U \neq \mathbb{C}$.

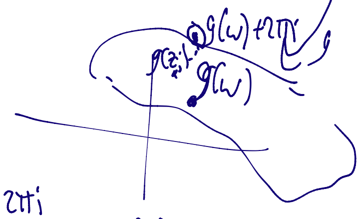
$\exists\, \alpha \in \mathbb{C} \setminus U$. Then $f(z) = z - \alpha$
$\neq 0$ on U.

$\Rightarrow \exists\; g(z) = \log f(z) = \int_{w \to z} \frac{f'}{f} + C$, s.t. $e^{g(z)} = f(z)$.

U simply conn $\Rightarrow$ well-defined

Claim: $\nexists\, z \in U$ s.t.

$$\underline{g(z) = g(w) + 2\pi i} \notin g(U).$$



If such exists, then $\underset{z-\alpha}{\underset{\parallel}{e^{g(z)}}} = e^{g(w)+2\pi i} = e^{g(w)} = f(w) = w - \alpha$

$\Rightarrow z = w. \Rightarrow g(z) = g(w)$

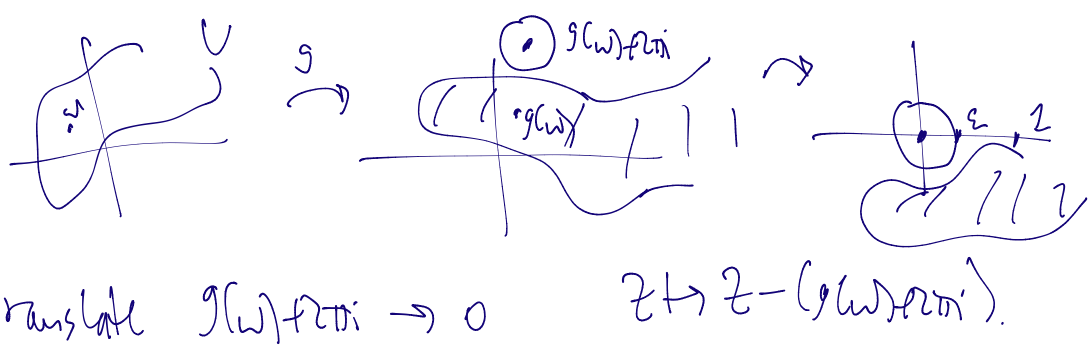
Claim: $D_\varepsilon(g(w)+2\pi i) \cap g(U) = \emptyset$.

If not, $\exists\, z_j \in U$ s.t. $g(z_j) \to g(w) + 2\pi i$

$\Rightarrow \underset{z_j-\alpha}{\underset{\parallel}{e^{g(z_j)}}} \to e^{g(w)+2\pi i} = w - \alpha \Rightarrow z_j \to w$

$\Rightarrow g(z_j) \to g(w) \neq g(w)+2\pi i$.

Note: $g : U \to g(U)$ is a conformal map. (hol & has inverse).

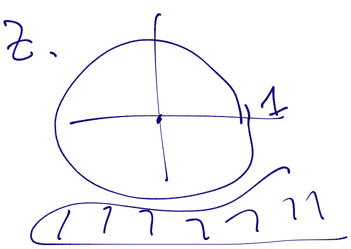


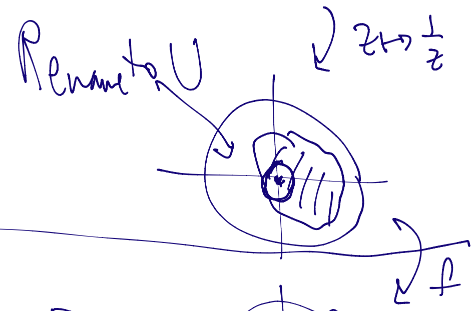
Translate $g(w)+2\pi i \to 0$ $\quad z \mapsto z - (g(w)+2\pi i)$.

Rescale $D_\varepsilon(0) \to D_1(0)$, $z \mapsto \frac{1}{\varepsilon} z$.

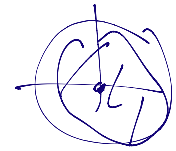
Apply $z \mapsto \frac{1}{z}$ $z \neq 0$. (hol'c),

& assume (translation) $0 \in U$.





Step 2: Assume $0 \in U \subset \mathbb{D}$.



Look at $\mathcal{F} := \left\{ f : U \to \mathbb{D} \;\middle|\; \text{hol'c, inj}, \; f(0) = 0 \right\}$.

$\neq \emptyset$, $z \mapsto z$.

$\boxed{\text{Magic:}}$ Let $1 \le S = \sup_{f \in \mathcal{F}} |f'(0)| \overset{\le \frac{1}{\varepsilon}}{<} \infty$?.

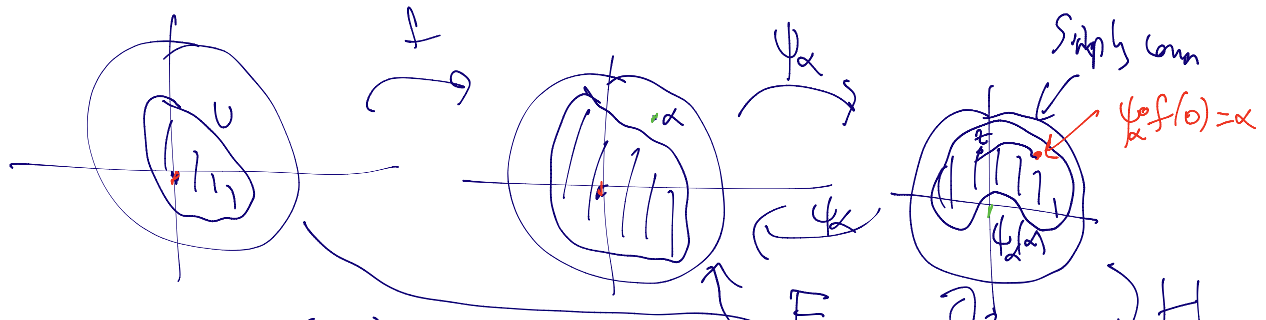
Recall Cauchy rep $|f'(0)| \le \left| \frac{1}{2\pi i} \int_{\partial D_\varepsilon(0) \subset U} \frac{f(z)}{(z-0)^2} \, dz \right| \le \frac{1}{2\pi} \frac{1}{\varepsilon^2} 2\pi\varepsilon$

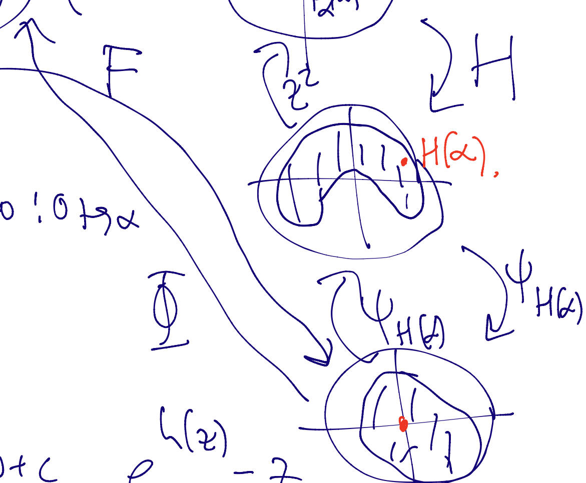
$|\cdot| \le 1$.

Claim: $\exists f \in \mathcal{F}$ s.t. $|f'(0)| = S = \sup_{g \in \mathcal{F}} |g'(0)|$.

Step 3: Such an f is onto (hence U conf to $\mathbb{D}$).

Assume $\alpha \in \mathbb{D} \setminus f(U)$ (Claim: $|f'(0)|$ not maximal).





Blashke factors $(\alpha \in \mathbb{D})$: $\psi_\alpha = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$; $\alpha \mapsto 0 : 0 \mapsto \alpha$

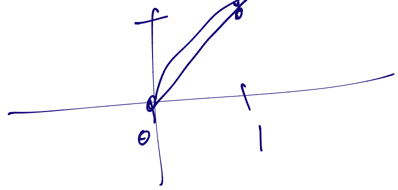
Then $0 \notin \psi_\alpha \circ f(U)$ so

$$\exists\ h(z) = \log z = \int_{\alpha \to z} \frac{1}{w}\, dw + c, \quad e^{h(z)} = z.$$

$$\exists\ H(z) = e^{\frac{1}{2} h(z)} \text{ "=" } \sqrt{z}.$$

(Idea: increases things, which shouldn't be possible if f is maximal).

Why this? $0 < r < 1$, $\sqrt{r} > r$.



$$F := \psi_{H(\alpha)} \circ H \circ \psi_\alpha \circ f : U \to \mathbb{D}, \text{ hol, inj, } F(0) = 0.$$

So $F \in \mathcal{F}$. $f = \underbrace{\psi_\alpha \circ (z \mapsto z^2) \circ \psi_{H(\alpha)}}_{\Phi} \circ F = \Phi \circ F$

$\Phi(z) = \psi_\alpha \circ (z \mapsto z^2) \circ \psi_{H(\alpha)} : \mathbb{D} \to \mathbb{D}$, hol'c, Not injective!

$\Phi(0) = \psi_\alpha\Big(\underbrace{\big(\psi_{H(\alpha)}(0)\big)^2}_{H(\alpha)^2 = \alpha}\Big) = 0.$

$\Rightarrow$ (Schwarz) $|\Phi'(0)| \leq 1$, ~~(If $=1 \Rightarrow \Phi$ rotation)~~,

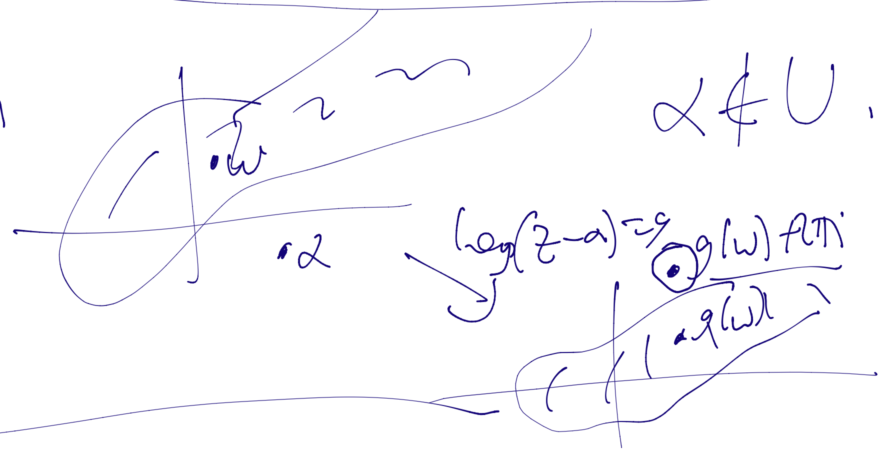
$\Rightarrow |\Phi'(0)| < 1.$

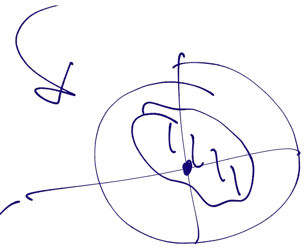
$$|f'(0)| = |\Phi'(\underbrace{F(0)}_{0})| \cdot |F'(0)| < |F'(0)|.$$

But f has largest $|f'(0)|$ in $\mathcal{F}$ ↯ !

Again

$\alpha \notin U$.



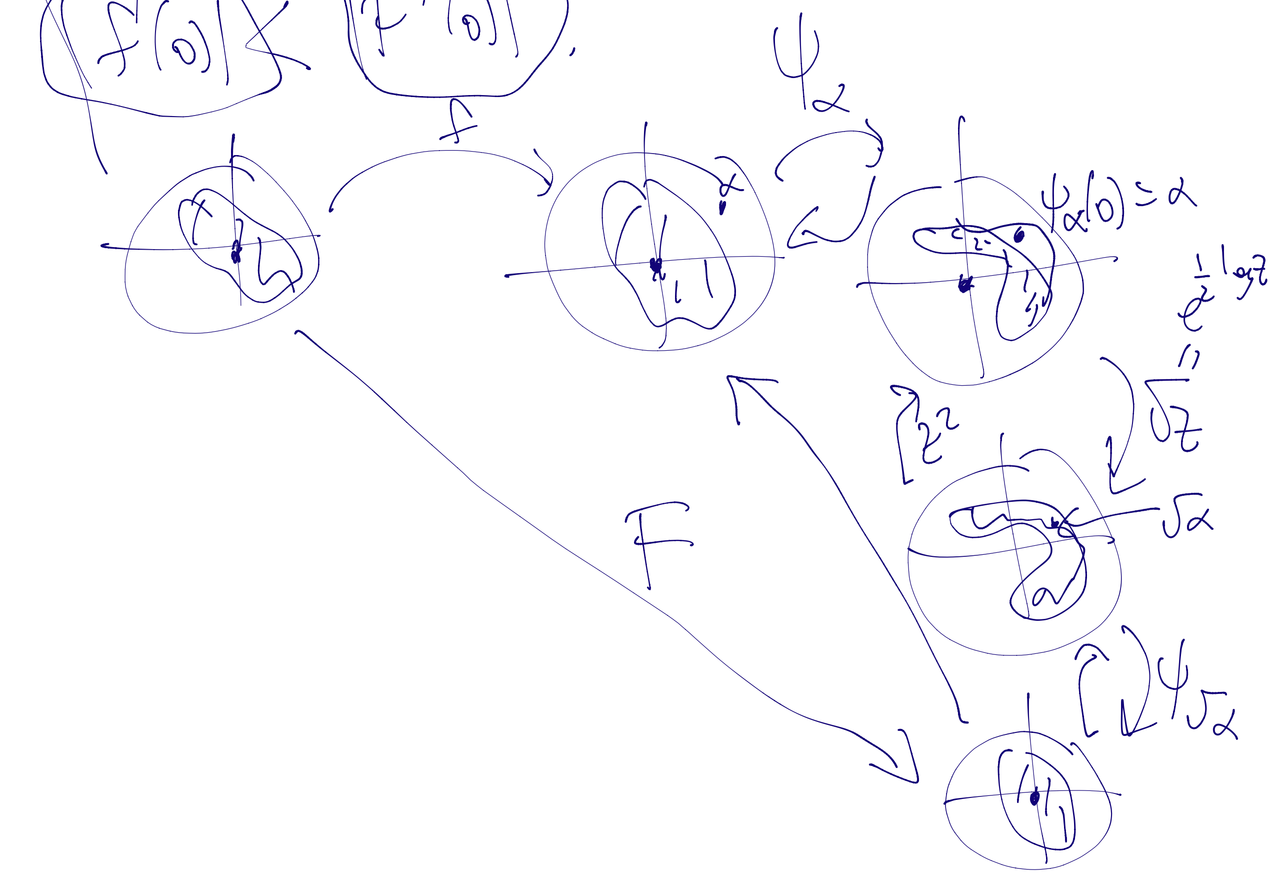


$\mathcal{F} = \{ f : U \to \mathbb{D} \mid \text{holo, inj}, \ f(0) = 0 \}$.

Suppose: $\exists f$ s.t. $|f'(0)| = \sup_{g \in \mathcal{F}} |g'(0)|$.

Claim: f is onto.

Contrapositive: if not onto, then $\exists F \in \mathcal{F}$, $|f'(0)| < |F'(0)|$.



Why does such an f exist in $\mathcal{F}$?

$\mathcal{F} = \{ f : U \to \mathbb{D},\ \text{hol'c},\ \text{inj},\ f(0) = 0 \}$.

If $s = \sup_{g \in \mathcal{F}} |g'(0)|$, then

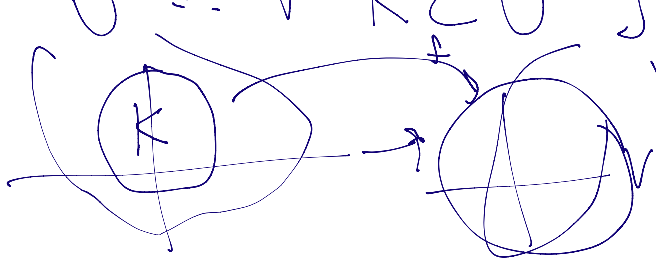
$\exists\ f_1, f_2, \ldots \subset \mathcal{F},\ |f_j'(0)| \to s$.

Want: $\exists$ subseq $f_j \to f \in \mathcal{F}$???

Thm (Montel): Let $\mathcal{F}$ family of hol'c functions $: U \to \mathbb{C}$ which is uniformly bdd on compacta.

i.e. $\forall\ K \overset{\text{cpt}}{\subset} U\ \exists B > 0 : \forall f \in \mathcal{F}, \forall z \in K,$

$|f(z)| < B$



Then (i) $\mathcal{F}$ is equicontinuous on compacta

i.e. $\forall K \subset U$, $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon, K) > 0$

$\forall f \in \mathcal{F}$, $\forall z, w \in K$,

$$|z-w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon.$$

(ii) $\mathcal{F}$ is normal (For any seq $f_1, f_2, \ldots \in \mathcal{F}$, $\exists$ subseq converging unif on compacta (not nec in $\mathcal{F}$!)).

$f_{n_j} \to f$ unif on compacta: $\forall K \overset{\text{cpt}}{\subset} U$, $\forall \varepsilon > 0$ $\exists N = N(\varepsilon, K)$:

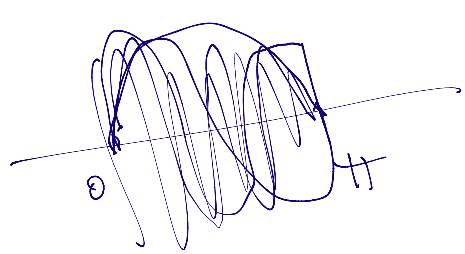
$\forall w \in K$

$$n_j \geq N \Rightarrow |f_{n_j}(w) - f(w)| < \varepsilon.$$

Rmk 1: (i) is truly / $\mathbb{C}$

E.g.: $f_n = \sin nx$ on $[0, \pi]$.

Unif bdd ✓.



Not equi-cont.

So equicont crucially uses hol'c:

Rmk?: But Arzela-Ascoli /$\mathbb{R}$,
unif bdd & equicont $\Rightarrow$ normal.