Last time: Poisson summation: $f \in \mathcal{F} \Rightarrow \sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$.

Pf 3: (Baby Trace Formula): Write $G =$ "Lie group" $= \mathbb{R}$.

Write $\Gamma =$ "discrete subgroup of G" $= \mathbb{Z}$. $\mathcal{H} = L^2(G/\Gamma) = L^2(\mathbb{R}/\mathbb{Z}) = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}\cdot e_m$

Laplacian $\Delta = \partial_{xx}$ ("Casimir")

Self-adjoint $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$

(neg) definite.

spectral decomposition.

eigenfunctions of Δ

$e_m(x) = e^{2\pi i m x}$.

$\lambda_m = -4\pi^2 m^2$

Back Matrix World: $\Delta = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$

By change of basis (to $\{e_m\}$), diagonalize

$\Delta \sim \begin{pmatrix} 0 & & & & \\ & -4\pi^2 & & & \\ & & -4\pi^2 & & \\ & & & -4\pi^2\cdot 4 & \\ & & & & -4\pi^2\cdot 4 \\ & & & & & \ddots \end{pmatrix}$

"Construct an automorphic kernel"

(Aside: Cauchy integral rep formula: $(\mathcal{I}\cdot f)(z_0) := \frac{1}{2\pi i}\oint_{\gamma \ni z_0} f(z)\cdot \underbrace{\frac{1}{z-z_0}}_{\text{Kernel } K(z,z_0)} dz = f(z_0)$.

Back to Poisson summation: $f \in \mathcal{F} = \bigcup \mathcal{F}_a$. ~~$\mathcal{F}$~~. Take a "point-pair invariant":

$k : \underset{\mathbb{R}\times\mathbb{R}}{G\times G} \to \mathbb{C}$, $k(x,y) = f(x-y)$. (say f even). $\Rightarrow k(x,y) = k(x+g, y+g)$ $\;\; \boxed{k(g\cdot x, g\cdot y) = k(x,y)}$, $g \in \mathbb{R}$

$(\mathbb{R}, +)$ on G/Γ

Construct automorphic kernel: $K(x,y) := \sum_{n=\gamma\in\Gamma} k(x+n, \cancel{y+n}) = \sum_{n\in\Gamma} k(x, y-n)$, $x, y \in \mathbb{R}/\mathbb{Z}$.

$K : G/\Gamma \times G/\Gamma \to \mathbb{C}$ $\quad (\mathcal{I}_K \cdot g)(x) = \int_{G/\Gamma} K(x,y)\, g(y)\, dy$, $g \in L^2(\mathbb{R}/\mathbb{Z})$

linear operator (∞-dim matrix)

"Trace formula": $\mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{a+d}_{\text{geometric side}} = \underbrace{\lambda_1 + \lambda_2}_{\text{spectral side}}$ ← eigenvalues

Need to spectrally expand $K(x,y) = \sum_{m\in\mathbb{Z}} \langle K(x,\cdot), e_m\rangle e_m(y)$ For fixed x,

$\Delta \circlearrowright$ $K(x,\cdot) \in L^2(\mathbb{R}/\mathbb{Z}) = \bigoplus \mathbb{C} e_m$

$\int_{\mathbb{R}/\mathbb{Z}} K(x,y)\, e_{-m}(y)\, dy = (\mathcal{I} e_m)(x)$

Obs: What happens if I take $\boxed{\Delta \mathcal{I} = \mathcal{I}\Delta}$

$\partial_{xx}[(\mathcal{I}\cdot g)(x)] = \partial_{xx}\left[\int_{\mathbb{R}/\mathbb{Z}} K(x,y)\, g(y)\, dy\right]$

$\lim_{h\to 0} \frac{K(x,y+h) - K(x,y)}{-h}$

$-\partial_y K(x,y)$

$\lim_{h\to 0} \frac{K(x,y-h) - K(x,y)}{h}$

$$= \int_{\mathbb{R}/\mathbb{Z}} \underbrace{\partial_{xx} K(x,y)}_{\partial_{yy}K(x,y)}\, g(y)\, dy \qquad \partial_x K(x,y) = \lim_{h\to 0} \frac{K(x+h,y)-K(x,y)}{h}$$

(Δ selfadjoint)

$$= \int_{\mathbb{R}/\mathbb{Z}} K(x,y)\, \partial_{yy} g(y)\, dy = I(\Delta g)(x).$$

Consequence:

If e_m efunct of Δ, $I(\Delta e_m) = \lambda_m I e_m$ $\Rightarrow$ $I e_m$ is also efunct of Δ with same evalue!

$$\underline{\underline{\Delta}}\, I(e_m)$$

Compute Fourier coefficients of $K(x,\cdot) \rightsquigarrow \langle K(x,\cdot), e_m\rangle = \int_{\mathbb{R}/\mathbb{Z}} K(x,y)\, \overline{e_m(y)}\, dy$, $\quad K = \sum_{n\in\mathbb{Z}} k(x+n,y)$

Fix domain $D \to \mathbb{R}/\mathbb{Z}$, "$(0,1)$"

$$\int_D \sum_{n\in\mathbb{Z}} k(x+n,y)\, \overline{e_m(y)}\, dy = \sum_{n\in\mathbb{Z}} \int_D k(x+n, y+n)\, \overline{e_m(y)}\, dy$$

$y \mapsto y+n$

"unfolding" $$= \sum_{n\in\mathbb{Z}} \int_{D+n} \underbrace{k(x,y)\, \overline{e_m(y+n)}}_{G/\Gamma\ \mathbb{R}/\mathbb{Z}}\, dy = \int_{\mathbb{R}=G} \underbrace{k(x,y)}_{f(x-y)}\, \overline{e_m(y)}\, dy$$

$x^{-1}y$

Generally: $k(x,y) = f(x^{-1}y)$, $k(gx,gy) = f(x^{-1}g^{-1}gy)$

$$= \int_{\mathbb{R}} f(x-y)\, e^{-2\pi i m y}\, dy \qquad e^{2\pi i m(y+x)}$$

$z = x-y$, $dz = -dy$, $y = z-x$

If $\Gamma = \sqrt{2}\mathbb{Z}$, $e_m(x) = e^{\frac{2\pi i m x}{\sqrt{2}}}$, $\Delta e_m = \left(\frac{-4\pi^2 m^2}{2}\right) e_m$

$$= \underbrace{\int_{\mathbb{R}} f(z)\, e^{-2\pi i m(z)}\, dz}_{\hat{f}(m)} \cdot \underbrace{e^{2\pi i m x}}_{e_m(x)}.$$

$\{e_m\}_{m\in\mathbb{Z}}$ discrete, $e_m \in L^2$, basis for $L^2(\mathbb{R}/\mathbb{Z})$

$\{e_\xi\}_{\xi\in\mathbb{R}}$, $e_\xi \notin L^2(\mathbb{R})$, basis for $L^2(\mathbb{R})$ — Fourier inversion formula

Continuous spectrum $f(x) = \int_{\mathbb{R}} \langle f, e_\xi\rangle\, \overline{e_\xi}\, d\xi$

evalues: $\lambda_\xi = -4\pi^2\xi^2$ continuous.. $(\Delta - sI)^{-1}$ resolvent kernel.

$$I_k \overline{e_m} = \hat{f}(m)\, e_m$$

"Selberg / Harish-Chandra transform"

$\operatorname{Spec}\Delta = \{ s \in \mathbb{C} \mid$ where ? has no inverse $\}$.

$$K(x,y) = \sum_{\underline{\underline{m\in\mathbb{Z}}}} \underbrace{\hat{f}(m)\, e_m(x)}_{\langle K(x,\cdot), e_m\rangle} \cdot e_m(y)$$

Spectral expansion of automorphic kernel K.

$$\sum f(n) = \sum \hat{f}(m) \qquad \sum_n \left(f(n) + f\left(\tfrac{n}{2}\right)\right) = \sum_m \left(\hat{f}(m) + 2\hat{f}(2m)\right)$$

$$\sum f\left(\tfrac{n}{2}\right) = \sum 2\hat{f}(2m)$$

should be abs?

$$(I_k g)(x) = \int_{\mathbb{R}/\mathbb{Z}} k(x,y) g(y)\, dy \qquad k(x,y) = \sum_{m\in\mathbb{Z}} \hat{f}(m) \overline{e_m(x)} e_m(y)$$

$$\boxed{\begin{aligned} &k(x,y) \\ &= f(x-y) \end{aligned}}$$

Take $\operatorname{tr} I := \underbrace{\int_{\mathbb{R}/\mathbb{Z}} k(x,x)\,dx}_{\text{geometric side}} =$ Spectral side.

$$= \int_{D} \sum_{n\in\mathbb{Z}} k(x+n,x)\, dx = \sum_{n\in\mathbb{Z}} \int_D \underbrace{k(x+n,x)}_{f(x+n-x)} dx = \sum_{n\in\mathbb{Z}} f(n)\cdot 1$$

$$= \int_D \underbrace{\sum_{m\in\mathbb{Z}} \hat{f}(m) \overline{e_m(x)} e_m(x)}_{k(x,x)} dx = \sum_{m\in\mathbb{Z}} \hat{f}(m)\cdot 1$$

$$k(x,y) = f(x^{-1}y)$$

$$\int_D k(\gamma x, x) = f(x^{-1}\gamma^{-1} x)$$

If G/Γ ~~cpt~~, $L^2(G/\Gamma) = \bigoplus e_m \Rightarrow \sum_{[\gamma]\in\Gamma} f(\gamma) = \sum_{\lambda_m \in \operatorname{Spec}\Delta} \hat{f}(\lambda_m)$.

SL/HC:

What to do in "general": $I e_m =$ ___ $\cdot e_m$ on G "Know" efuncts of Δ

has same eval of Δ

$\langle k(x,\cdot), e_\xi\rangle$

___ $\cdot e_\xi(x)$

e_ξ (not on G/Γ)

$$g(x) = f\left(\tfrac{2}{x}\right)$$

$$\sum_{n\in\mathbb{Z}} f\left(\tfrac{2}{n}\right) = \sum_{n\in\mathbb{Z}} g(n) = \sum_{m\in\mathbb{Z}}$$

$$\hat{g}(\xi) = \int g(x) e^{2\pi i \xi x} dx = \int f\left(\tfrac{2}{x}\right) e^{-2\pi i \xi x} dx$$

$x^2 - 5y^2 = -4$

$x^2 - 5y^2 = 4$

$\frac{1+\sqrt{5}}{2}$, 1

Madhava

Dirichlet class # formula $\quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \cdots = \frac{\pi}{4}$. Leibniz

$$L(1,\chi_5) = \underbrace{1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{11} - \frac{1}{12} - \frac{1}{13} + \frac{1}{14} \cdots}_{} = \frac{h(\mathbb{Q}(\sqrt{5}))}{\sqrt{5}\cdot \log\varphi}.$$

$\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 1 \end{matrix}$

$\frac{1}{\sqrt{5}\log\varphi}$ this