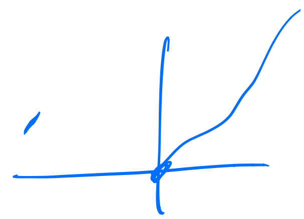
Last time: • $\forall\, \Omega \subset \mathbb{C}\setminus\{0\}$, simp connected, $\exists \log_\Omega$, branch cut.



• $f: \Omega \to \mathbb{C}$, $f \neq 0$. simp conn, $\exists\, g = \log f$. $\int \frac{f'}{f} + w_0$.

• Mean Val Prop: $\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = f(z_0)$

$$\frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0+re^{i\theta})e^{-in\theta}\,d\theta = \begin{cases} f^{(n)}(z_0) & n \geq 0 \\ 0 & \text{oth.}\end{cases}$$

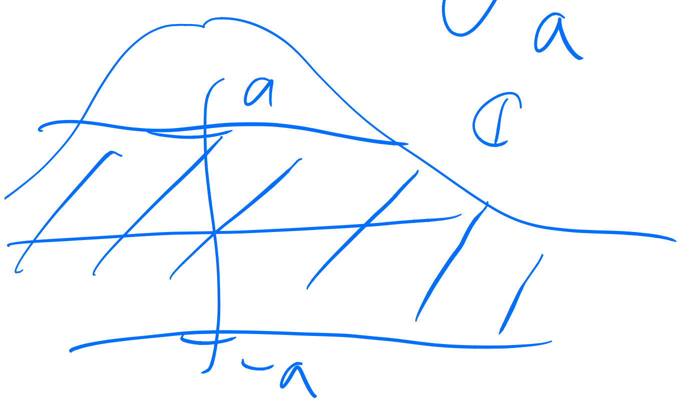
Fourier analysis on S^1 ⇝ Fourier analysis on $\mathbb{R}$.

Def: $f \in L^1 \Rightarrow \hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx$.

Q: To what extent do we have F. Inversion, i.e.

$$\int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \stackrel{?}{=} f(x)\,?$$

"Good enough" class of functions for our purposes: $\mathcal{F}_a^{(a>0)} := \{ f$ hol on $\{|\operatorname{Im} z| < a\}$, s.t. $\exists C>0$: $\forall |y|<a$, $\forall x \in \mathbb{R}$, $|f(x+iy)| \leq \frac{C}{1+x^2} \}$



Do we have $\mathcal{F}_a \neq \emptyset$?

What about $f(z) = e^{-2\pi z^2}$

Which $\mathcal{F}_a$ (if any) is $\uparrow$ in?

Then $\left|e^{-2\pi(x+iy)^2}\right| = e^{-2\pi(x^2-y^2)}$

$$= e^{-2\pi x^2} \cdot \underbrace{e^{+2\pi y^2}}_{< e^{2\pi a^2} = C}.$$

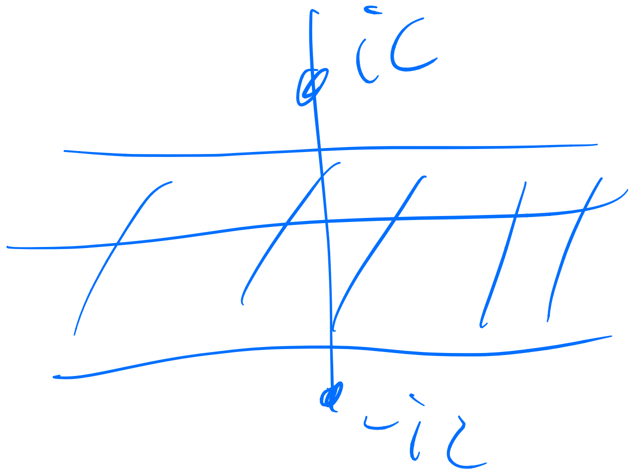
If $|y|<a$,

$$f(x+iy) \leq C \cdot e^{-2\pi x^2}.$$

So Gaussian $\in \bigcap_{a \geq 0} \mathcal{F}_a$.

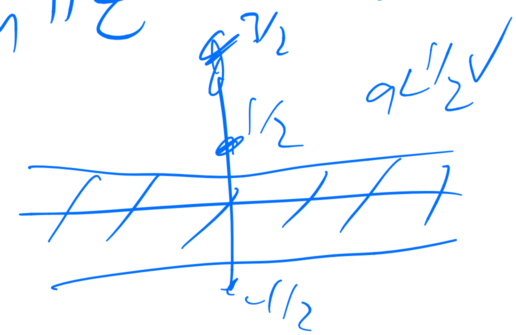
Def: $\mathcal{F} = \bigcup_a \mathcal{F}_a$.

Ex: $f(z) = \frac{1}{\pi} \frac{c}{c^2+z^2} \in \mathcal{F}_a$ $(c>0)$ $\underline{a<c}$



Ex: $f(z) = \dfrac{1}{\cosh \pi z} \in \mathcal{F}_a$

for $a = ???$



$$\cosh \pi z = \frac{e^{\pi z} + e^{-\pi z}}{2} = 0.$$

$$e^{\pi z} = -e^{-\pi z}$$

$$e^{2\pi z} = -1.$$

$$2\pi z = \pi i(2n+1),$$

$$z = i\frac{(2n+1)}{2}.$$

__Exercise:__ $f \in \mathcal{F}_a \Rightarrow \cancel{\forall b<a}, f^{(n)} \in \mathcal{F}_b$

"Typically" say $f, f^{(n)} \in L^1$.

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx$$

$$f \in L^1 \Rightarrow \left| \hat{f}(\xi) \right| \le C.$$

$f' \in L^1$

$$\left| \hat{f}(\xi) \right| = \left| \int_{\mathbb{R}} \underline{f(x)} \, \underbrace{e^{-2\pi i x \xi}} \, dx \right|$$

$$= \left| - \int_{\mathbb{R}} f'(x) \frac{e^{-2\pi i x \xi}}{-2\pi i \underline{\xi}} \, dx \right|$$

$$\le \frac{1}{|\xi|} \cdot C.$$

$f^{(n)} \in L^1 \Rightarrow |\hat{f}(\xi)| \leq \frac{C_n}{|1+|\xi||^n}$

So: If $f \in C^\infty$ & $f^{(n)} \in L^1$,

Then $\hat{f}(\xi)$ has super polynomial decay!

let $a_n = \boxed{n^d}$ $- - e^{d \log n}$, $b_n = e^n$.

$\exists ?$ c_n s.t. $\frac{a_n}{c_n} \to 0$ &

$\frac{b_n}{c_n} \to \infty$?. Ex: $c_n = e^{\sqrt{n}}$.

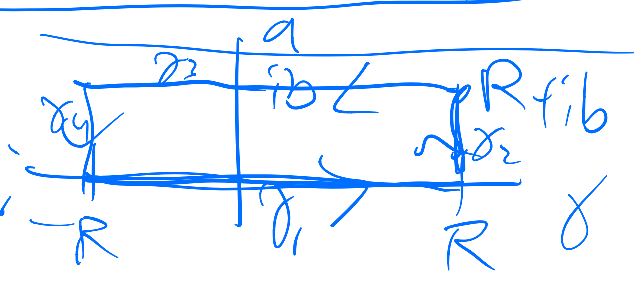
$c_n = e^{(\log n)^2}$.

So there is a big difference between superpolynomial decay & exp decay!

If f can be "deformed" into $\mathbb{C}$, just a little bit, $f \in \mathcal{F}_a$ for some a, Then we get exp decay!

Thm: $f \in \mathcal{F}_a \Rightarrow \forall b < a, \exists C: |\hat{f}(\xi)| < C e^{-b|\xi|}$.

Pf: Look at:



$g(z) := f(z)\,e^{-2\pi i z\xi}$ is hol'c in γ, so $\int_{\gamma_1}+\cdots+\int_{\gamma_4} = \int_\gamma g = 0$.

$\hat{f}(\xi) \leftarrow \gamma_1$

$$\left|\int_{\gamma_2}\right| = \left|\int_0^b f(R+iy)\,e^{-2\pi i\xi(R+iy)}\, i\,dy\right|$$

$\gamma_2 \leftarrow z = R+iy,\ 0<y<b,\ z'(y)=i$

$$\leq \int_0^b \frac{C}{1+R^2}\,e^{2\pi|\xi|y}\,dy \leq \frac{e^{2\pi|\xi|b}\cdot b}{1+R^2} \longrightarrow 0 \text{ as } R\to\infty$$

(ξ fixed, b fixed)

Same $\left|\int_{\gamma_4}\right| \to 0$.

$$\left|\int_{\gamma_3}\right| = \left|-\int_{\gamma_3^-}\right| = \left|-\int_{-R}^{R} f(x+ib)\,e^{-2\pi i\xi(x+ib)}\cdot 1\cdot dx\right|$$

$\gamma_3^- \leftarrow z = x+ib,\ -R<x<R$

$$\le \int_{-R}^{R} \frac{C}{1+x^2}\, e^{2\pi \xi b}\, dx \le e^{2\pi \xi b} \cdot C.$$

(not same!)

indep of R.

$$0=\int_{\gamma} \to \left|\hat{f}(\xi)\right| = \left|\lim \int_{\gamma_3}\right| \le C e^{2\pi \xi b}.$$

If $\xi<0$, is exactly exp decay!

If $\xi>0$,

$\ldots \le C e^{-2\pi \xi b}.$

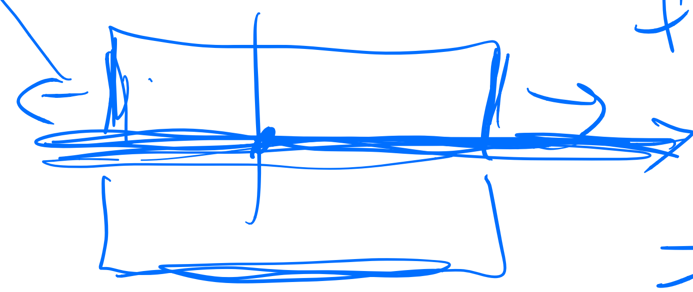
So: $\left|\hat{f}(\xi)\right| \le e^{-2\pi b \cdot |\xi|}.$

From being able to pull contours, we gain exp decay (not just super-poly).
& how much we gain $\Leftrightarrow$ how far pull!

Thm: $f \in \mathcal{F} \Rightarrow \forall x \in \mathbb{R}$

$$\int_0^\infty + \int_{-\infty}^0 = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi = f(x).$$

Pf:



$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx$$

$$= \int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i \xi(u-ib)}\, du$$

Then

$$\int_0^\infty \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi$$

$\xi > 0.$

$$= \int_0^\infty \int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i \xi u}\, e^{-2\pi \xi b}\, du\, d\xi$$

$$\le \frac{C}{1+u^2} \qquad |e^{-2\pi \xi b}| \qquad e^{2\pi i x\xi} \le 1$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \_\_ \ \_\_ \ \_ \ . \ d\xi \, du.$$

$$= \int_{-\infty}^{\infty} f(u-ib) \left[\underbrace{\int_0^{\infty} e^{-2\pi\xi(b+i(u-x))}}_{\text{''}} \, d\xi \right] du$$

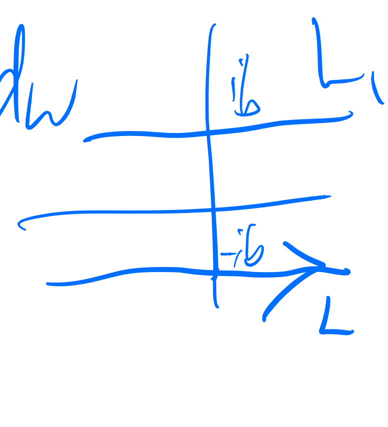
$$\lim_{R\to\infty} \int_0^R \underbrace{\_ \ \_ \ \_}_{} \, d\xi = (\ \ )\Big|_0^R \to \frac{0-1}{-2\pi(b+i(u-x))}$$

$$\frac{e^{-2\pi\xi(b+i(u-x))}}{-2\pi(b+i(u-x))}$$

primitive!

$$\int_0^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} \, d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u-ib)}{u-ib-x} \, du$$

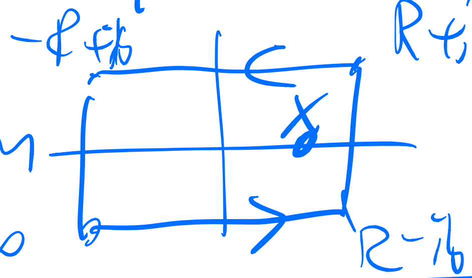
$$= \frac{1}{2\pi i} \int_L \frac{f(w)}{w-x} \, dw$$



But $\displaystyle\int_{-\infty}^{0} d\xi = -\frac{1}{2\pi i}\int_{L_1} \frac{f(w)}{w-x}\, dw$ ← Cauchy integral rep!

(Exercise)

So $\displaystyle\int_{-\infty}^{\infty} d\xi = \lim_{R\to\infty}$



$\to f(x)$

Inversion by Physics:

$$\int_{\mathbb{R}} \underbrace{\hat{f}(\xi)} e^{2\pi i x\xi}\, d\xi$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)\, e^{-2\pi i u\xi}\, du\, e^{2\pi i x\xi}\, d\xi$$

$$= \int_{\mathbb{R}} f(u) \underbrace{\int_{\mathbb{R}} e^{-2\pi i(u-x)\xi}\, d\xi}\, du$$

$\delta''_{x=u}.$

$= f(x).$

Another way: integrate by parts forwards & back:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f''(u)\,\frac{e^{-2\pi i u\xi}}{(-2\pi i\xi)^2}\,du\,e^{2\pi i x\xi}\,d\xi$$

$$= \int_{\mathbb{R}} f''(u)\int_{\mathbb{R}} \frac{e^{-2\pi i\xi(u-x)}}{(-2\pi i\xi)^2}\,d\xi\,du.$$

Double integrate by parts back $\to f(x).$