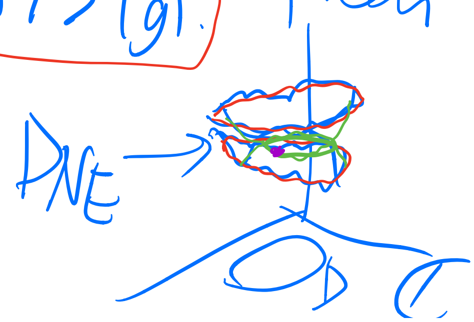
Last time:
- removable singularity (Riemann: f bdd near $z_0 \Rightarrow$ remov.)
- $|f| \to \infty \Leftrightarrow$ pole.
- Casorati-Weierstrass: z_0 essential $\Rightarrow \forall \varepsilon > 0$ $\mathbb{C} = \overline{f(D_\varepsilon(z_0) \setminus \{z_0\})}$

$\hat{\mathbb{C}} \cong S$, f mero on $\hat{\mathbb{C}} \Rightarrow f = \frac{P}{Q}$.

Thm (Argument Principle): f mero on $\overline{R} \subset \Omega$ & f has no zeros or poles on ∂R (= toy contour) $\Rightarrow \frac{1}{2\pi i} \int_{\partial R} \frac{f'}{f} = \#\{\text{zeros}\} - \#\{\text{poles}\}$.

Rouché Thm: S'pose f & $g : \Omega \to \mathbb{C}$ holo & $\boxed{\text{on } C = \partial D, \ |f| > |g|.}$ Then
$\Omega \supset D$

\# zeros of f inside D
$= \#$ zeros of $f+g$ inside D.



Pf: # zeros of f inside $D = \frac{1}{2\pi i}\int_C \frac{f'}{f}$

For $t \in [0,1]$, let $f_t(z) := f(z) + t \cdot g(z)$ jointly cont in $(z,t) \in \Omega \times [0,1]$.

Then $f_t(z)\Big|_C \neq 0$. since $|f(z)| > |g|$ on C.

$\Rightarrow I_t = \frac{1}{2\pi i}\int_C \frac{f_t'}{f_t}\,dz =$ # zeros of f_t inside D. $\in \mathbb{Z}$.

On $C \times [0,1]$ ($=$ cpt), f_t stays uniformly away from zero, $\Rightarrow I_t =$ cont fnct of t.

But I_t is $\mathbb{Z}$-valued $\Rightarrow$ const.

$I_0 =$ # zeros of f in $D = I_1 =$ # zeros of $f+g$ in D ✓.

Over $\mathbb{R}$ f analytic: f cont

$\Downarrow$

$f^{-1}(\text{open}) = \text{open}$.

$f((1,3)) = [2,4]$.

No reason for $f(\text{open}) \stackrel{?}{=} \text{open}$ X

<u>Open Mapping Thm:</u> f hol'c & $f \neq c$

$\Rightarrow f(\text{open}) = \text{open}.$

i.e. $\forall z_0 \in \Omega$, $\forall \delta > 0$ $\exists \varepsilon > 0$ s.t.

$\forall \underline{\underline{w}}$, $|w - \overset{f(z_0)}{w_0}| < \varepsilon \Rightarrow \exists\ \underline{|z - z_0| < \delta}$

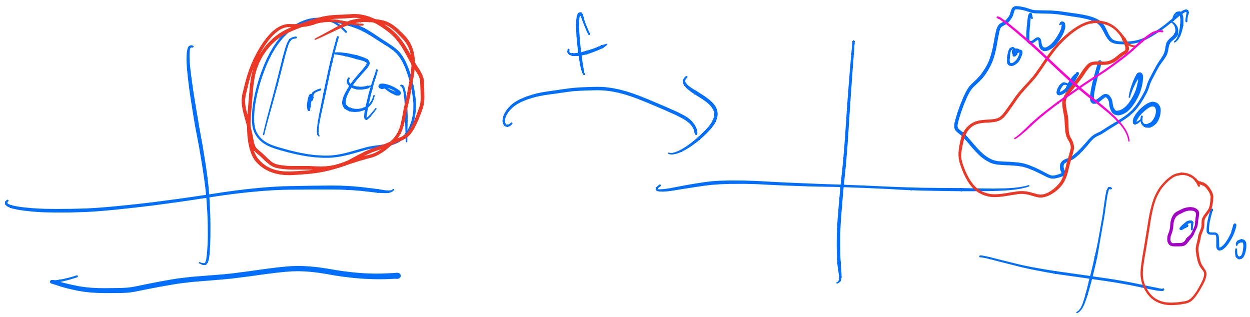
s.t. $f(z) = w$. ie $D_\varepsilon(w_0) \subset \mathrm{Image}(f)$.

Pf: Let z_0 be given, set $w_0 = f(z_0)$,

Look at $\overset{g(z)}{f(z) - w_0}$. has a zero

at z_0 & $\exists\ \delta_0$ s.t. $\forall_{0 < |z - z_0| < \delta_0}$,

$f(z) - w_0 \neq 0$. Let $_{0 <} \delta < \delta_0$.



Look at $C_\delta = \partial D_\delta(z_0)$.

cpt.

$$g(C_\delta) \neq 0 \quad \forall z \in C_\delta,$$

$$\Rightarrow \exists\, \varepsilon > 0 \text{ s.t. } |g(z)| \geq \varepsilon.$$

Let $w \in D_\varepsilon(w_0)$ be arbitrary. (fix).

$$\text{Let } h(z) = \underbrace{f(z) - w_0}_{F(z)} + \underbrace{w_0 - w}_{\substack{G(z) \\ |\cdot| < \varepsilon}}$$

$$\text{"} \quad f(z) - w.$$

on C_δ, $|F| \geq \varepsilon$ $\geq$

$\Rightarrow |F| > |G|$ on C_δ.

Rouché $\Rightarrow$ # zeros of F $\cancel{\boxed{1}}$ inside $D_\delta(z_0)$

z_0 need not be simple

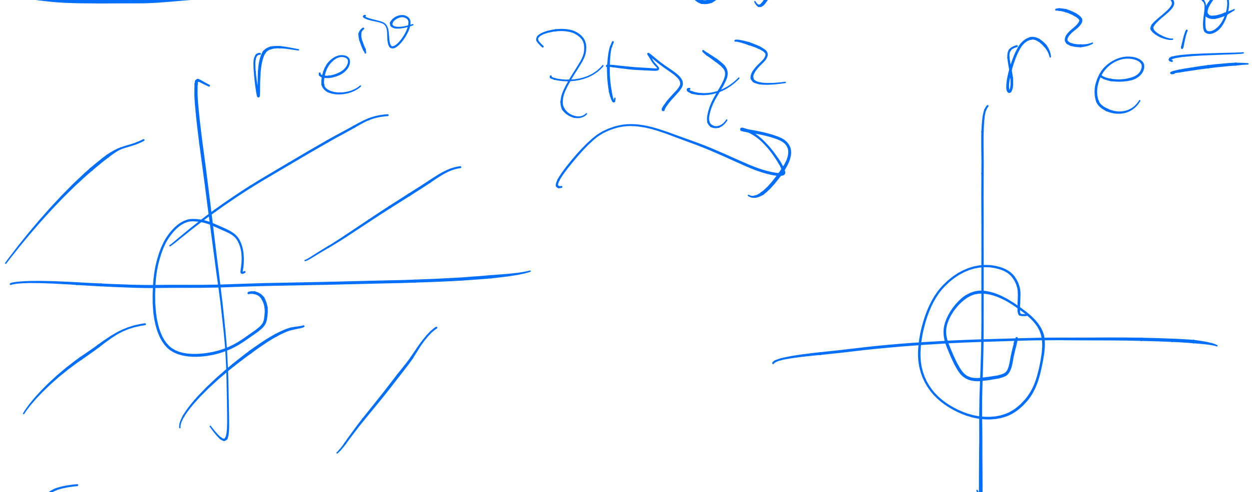
$=$ # zeros of $F+G$ inside $D_\delta(z_0)$

$\Rightarrow F+G = f(z) - w$ has a zero!

$\Rightarrow \exists z$ s.t. $f(z) = w$.

~~Rmk: Pf shows that f is locally 1-1.~~

Non-Ex: $f(z) = z^2$.

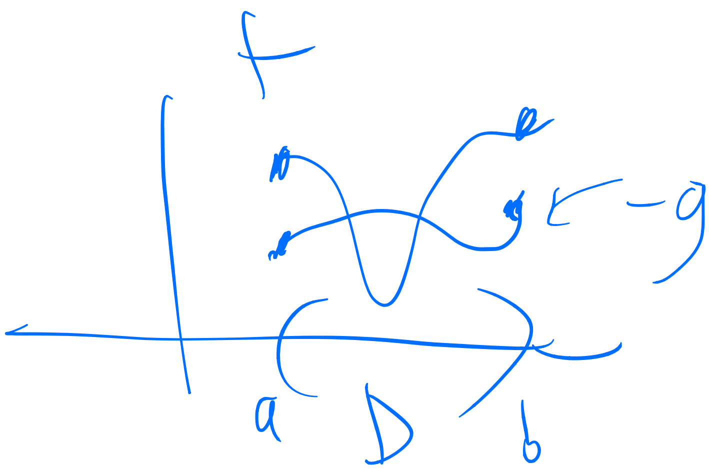


Rmk: Proof shows that if z_0 is a simple zero of $g(z) := f(z) - w_0$.

then locally f is 1-1. (near z_0)

Where does pf fail $/\mathbb{R}$?

It fails on Rouché!!



$f+g = f-(-g)$.

$\partial D = \{a,b\}$

$|f|>|g|$ on a, b.

Where does pf of Rouché fail?

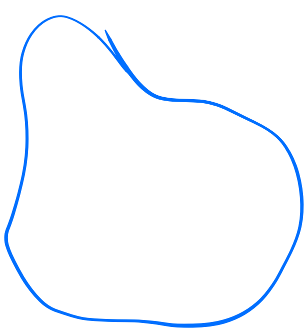
$$\int_{\partial D} \frac{f'}{f} = \frac{f'}{f}(b) - \frac{f'}{f}(a) \notin \mathbb{Z}.$$

f_t is cont int & ~~$\mathbb{Z}$-valued~~.

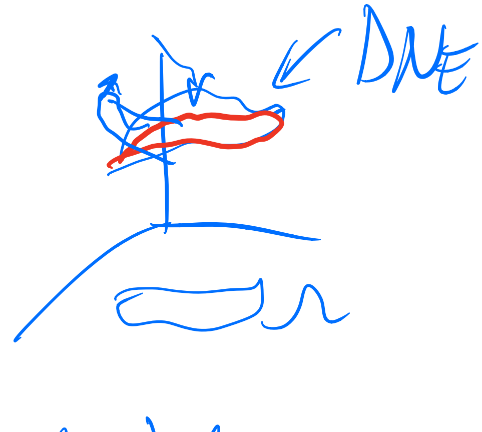
No Arg Principle

Thm (Max Modulus Principle): f hol'c & ~~non-constant~~ on Ω & bdd, & assume f cont on $\overline{\Omega}$

Then $\underbrace{\sup |f|}_{\text{on } \Omega} \leq \sup_{z \in \overline{\Omega} \setminus \Omega} |f|.$



$\partial\Omega$



Rmk: Essential Heat

Ω is bdd:

<u>Ex</u>: $f(z) = e^{z^2}$: if $z = e^{\pm \frac{i\pi}{4}} \cdot r$

then $z^2 = \pm i \cdot r$

$|e^{z^2}| = 1$

blows up e^{x^2}.

(<u>Later</u>: Phragmen-Lindelöf Principle: As long as you don't blow up too fast, behavior on ∂ $\rightsquigarrow$ behavior inside).

Could max value happen

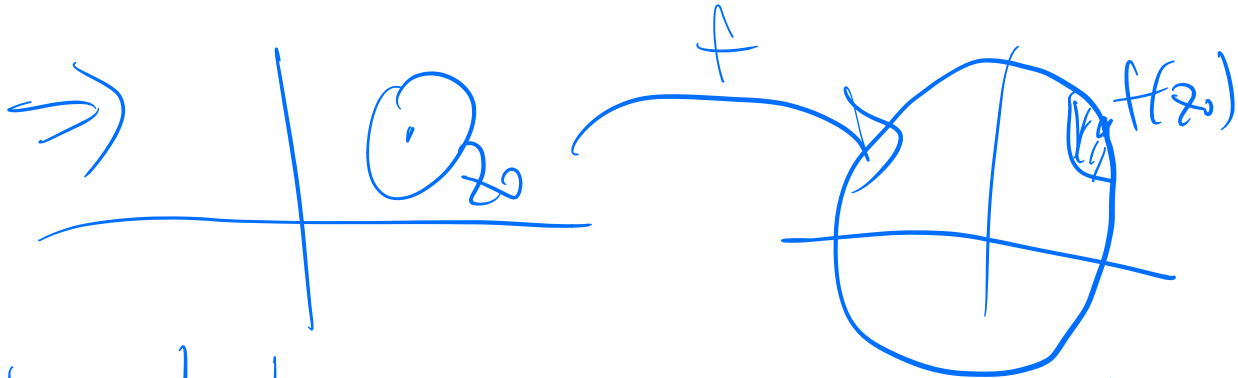
both on inside & on ∂?

Then smaller region around interior maximum

Also has max on its boundary

Pf: Assume $\exists z_0 \in \Omega$

s.t. $\forall z \in \Omega$, $|f(z)| \leq |f(z_0)|$,



Violates open mapping Thm $\Rightarrow f=c$.

§ Homotopy: Two curves

$\gamma_0 : [0,1] \to \Omega$ with $\gamma_0(0)=\gamma_1(0)$

$\gamma_1 : [0,1] \to$ and $\gamma_0(1)=\gamma_1(1)$

homotopic

if $\exists$ $\underbrace{\gamma : [\overset{s}{0,1}] \times [\overset{t}{0,1}] \to \Omega}_{\text{homotopy}}$

$\gamma(s,t) = \gamma_s(0) = \alpha,\ \gamma_s(1) = \beta$, both cont in

$\gamma_0(t) = \gamma(0,t)$ & $\gamma_1(t) = \gamma(1,t)$.

Thm: f hol on Ω

& γ_0 & γ_1 homotopic curves in Ω

$\Rightarrow \int_{\gamma_0} f = \int_{\gamma_1} f$.