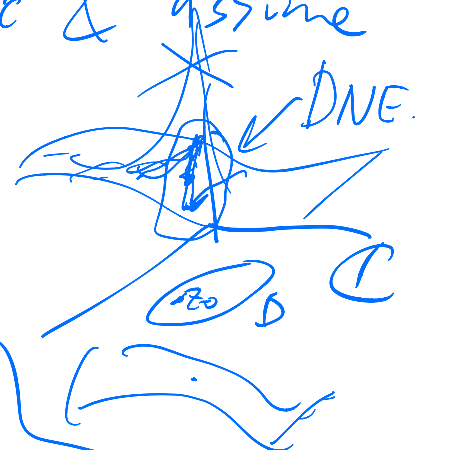
Last time: Residue Thm: $f:\Omega\to\mathbb{C}$, mer'c, hol'c except poles at $z_1, z_2, \dots, z_k \in R$, $R \subset \Omega$.

$$\frac{1}{2\pi i}\int_{\partial R} f(z)\,dz = \sum \mathrm{Res}_{z_j} f.$$

§ Singularities | Riemann Removable Sing. Thm:

Let $f : \mathbb{D}\setminus\{z_0\} \to \mathbb{C}$ hol'c & assume near z_0, f stays bdd. ($\mathbb{D} \subset \Omega$)

$\Rightarrow$ z_0 is removable!!



pf: Look at

$\partial D = C$

$$\frac{1}{2\pi i}\int_{\partial D} \boxed{\frac{f(w)}{w-z}}\,dw =: g(z),$$

has singularities at $w = z$ (denom) & $w = z_0$ (numerator)

$$0 = \int_\gamma \frac{f(w)}{w-z}\,dw \quad \text{Let } \varepsilon \to 0 \Rightarrow \cancel{\int_{\partial B_\varepsilon(z)} \frac{f(w)}{w-z}\,dw = \int_{\partial B_\varepsilon(z_0)} \frac{f(w)}{w-z}\,dw}$$

$$\Rightarrow 0 = \int_C \frac{f(w)}{w-z}\,dw - \underbrace{\int_{\partial B_\varepsilon(z)} \frac{f(w)}{w-z}\,dw}_{\text{(Cauchy)}\ \overset{\shortparallel}{2\pi i}\, f(z)} - \underbrace{\int_{\partial B_\varepsilon(z_0)} \frac{f(w)}{w-z}\,dw}_{|\cdot| \le \underbrace{\sup_{\text{near } z_0}|f|}_{K} \cdot \frac{1}{|z-z_0|-\varepsilon} \cdot \underbrace{\ell(\partial B_\varepsilon)}_{2\pi\varepsilon} \ \to 0 \text{ as } \varepsilon \to 0.}$$

$$\Rightarrow \underbrace{\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw}_{g(z)} = f(z).$$

But $g(z)$ makes perfect sense at z_0! $g(z)$ hol'c at z_0.

$\Rightarrow$ There was no singularity!

Cor: f has a pole at $z_0 \iff |f| \to \infty$ there.

Pf: ($\Rightarrow$) f has a pole $\Rightarrow$ locally, $f(z) = \frac{1}{(z-z_0)^n} \cdot g(z)$, $g \neq 0$ near z_0.

$\Rightarrow |f| \to \infty$.

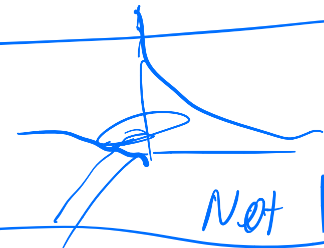
($\Leftarrow$) Assume $|f| \to \infty$. Then $\frac{1}{|f|} \to 0$.

$\Rightarrow$ $1/f$ remains bdd near z_0. $\Rightarrow$

$1/f$ has a removable sing at z_0, $1/f(z_0) = 0$.

$\Rightarrow$ f has a pole.

Recall: $e^{1/z}$ near $z_0 = 0$, $\cancel{\neq 0}$

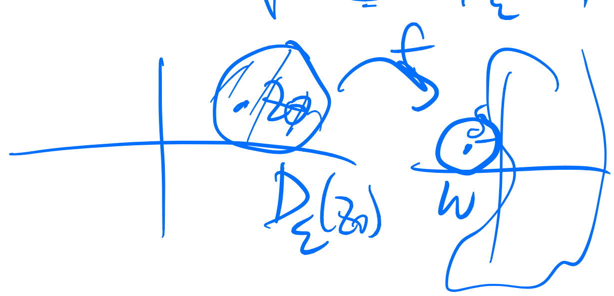


Thm (Casorati-Weierstrass): Assume f has essential sing at z_0. Then

$\forall \varepsilon > 0$, $\quad f(D_\varepsilon(z_0) \setminus \{z_0\}) = \mathbb{C}$.

Pf: Assume $\exists w \in \mathbb{C}$ s.t. $\exists \delta > 0$ $\forall z \in D_\varepsilon(z_0) \setminus \{z_0\}$

$|f(z) - w| > \delta$



Let $g(z) := \dfrac{1}{f(z)-w}$. Near z_0,

$$|g(z)| \le \frac{1}{|f(z)-w|} < \frac{1}{\delta} \Rightarrow z_0 \text{ is removable}$$

$$g(z_0) = \begin{cases} 0 \\ \neq 0 \end{cases}$$

$\Rightarrow f$ has removable sing at z_0

$g(z_0) = \dfrac{1}{f(z_0)-w}$

$f(z_0) = w + \dfrac{1}{\underbrace{g(z_0)}_{\neq 0}}$.

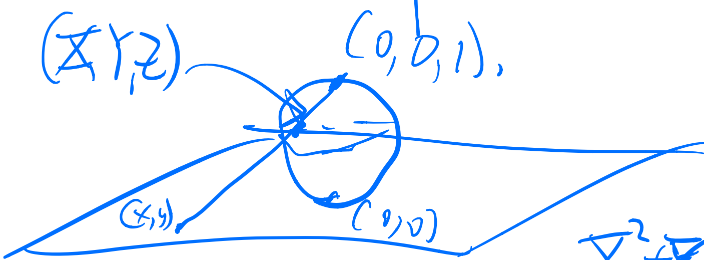
$\dfrac{1}{|g(z)|} \to \infty$ as $z \to z_0$, $\dfrac{1}{g} = f - w$.

$\Rightarrow |f| \to \infty \Rightarrow f$ has a pole at z_0

$\{$ Mero functions on $\hat{\mathbb{C}}$

(extended complex plane) $\to \mathbb{C} \cup \{\infty\}$.

Riemann sphere $\mathbb{S} = S^2$))



$\subset \mathbb{R}^3$.

$\mathbb{C} = (\mathbb{R}^2, 0)$

$X^2+Y^2+Z^2=1$.

$\boxed{\mathbb{C} \to \mathbb{S}}$

$\mathbb{S} \to \mathbb{C}$: (X,Y,Z) Line pt $\leftrightarrow (0,0,1)$.

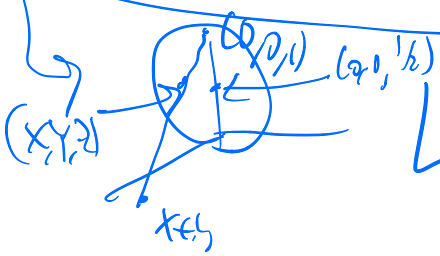
$$L_t: t(X,Y,Z) + (1-t)(0,0,1).$$

Value of t s.t. Z part $=0$?

$tZ + (1-t)\cdot 1 = 0$, $t(Z-1) = -1$, $\boxed{t = \frac{1}{1-Z}}$

$$\mathbb{C} \ni x+iy \leftrightarrow (x,y) \leftrightarrow (x,y,0) = \left(\underbrace{\frac{X}{1-Z}}_{x}, \underbrace{\frac{Y}{1-Z}}_{y}, \cancel{\frac{Z}{1-Z}+1-\frac{1}{1-Z}}\right)$$

0.



$$L_t: t(x,y,0) + (1-t)(0,0,1).$$

$$=(tx, ty, 1-t) \text{ on } S^2.$$

$$t^2x^2+t^2y^2+\left((1-t)-\tfrac{1}{2}\right)^2 = 1/4$$

$$t^2x^2 + t^2y^2 + \underbrace{\tfrac{1}{4} - \tfrac{1}{2}\cdot 2t + t^2} = 1/4.$$

$$\underset{\substack{t=0 \\ \text{North pole}}}{t}\Big(\underbrace{-1 + t(x^2+y^2+1)}_{\neq 0}\Big) = 0.$$

$$\Rightarrow \quad t = \frac{1}{x^2+y^2+1} \qquad \text{but } \neq 1.$$

$$\text{pt } (X, Y, Z) = \left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1}\right)$$

North pole in $S \longleftrightarrow \infty$ in $\hat{\mathbb{C}}$.

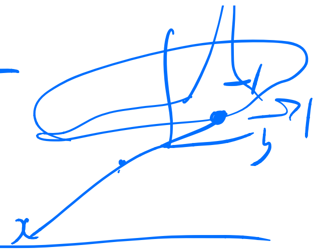
What is a zero or pole at ∞ mean?

<u>Def^n</u> f has a $\left\{\begin{matrix}\text{removable} \\ \text{pole} \\ \text{ess}\end{matrix}\right\}$ at ∞

if $F(z) = f(1/z)$ has a $\left\{\begin{matrix}\text{removable} \\ \text{pole} \\ \text{ess}\end{matrix}\right\}$ at 0.

(Equiv, $|f| \to \infty$ as $z \to \infty$)

Ex: $f(z) = \frac{z}{z+1} = \frac{1}{1+1/z}$



Thm: f is meri on $\hat{\mathbb{C}}$
Classification $\Rightarrow f = \frac{P}{Q}$, $P, Q \in \mathbb{C}[z]$
= rational.

Pf: f meri at $\infty \Rightarrow$ has fin many poles.

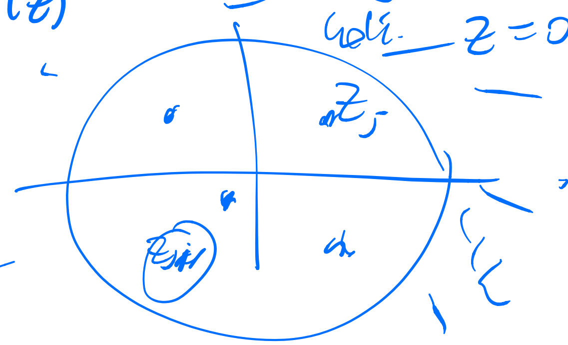
$\Updownarrow$

$F(z) = f\left(\frac{1}{z}\right)$ meri at 0, ($\underset{\text{pole}}{\text{remov}}$) $\overset{\text{if pole}}{\Longrightarrow} \underbrace{\frac{1}{f(1/z)}}_{1/z \to 0} \to 0$ as $z \to \infty$.

in neigh of $\frac{1}{z} \in B_\epsilon(0)$, fin many poles, (else $\frac{1}{f}$ has ∞ly many 0s)

For ϵ small enough, $\underbrace{f\left(\frac{1}{z}\right)}_{F(z)}$ $\overset{\text{holo}}{\text{regular}}$ except at $\cancel{\frac{1}{z} = 0}$ holo. $z = 0$.

$\Rightarrow \forall |z| > \frac{1}{\epsilon}$, f is holo.



If $F(z)$ has pole at 0,

$f\left(\frac{1}{z}\right) = \underbrace{\tilde{f}_\infty(z)}_{\text{principal part}} + \underbrace{\tilde{g}_\infty(z)}_{\text{holo near } 0}$

$\frac{a_{-N}}{z^N} + \cdots + \frac{a_{-1}}{z}$. For $|z| < 1/2$, finitely many poles at $z_1, \dots, z_k$, say. Near z_i:

$f(z) = \underbrace{f_i(z)}_{\text{principal part}} + g_i(z)$

Let $f_\infty(z) = \tilde{f}_\infty(1/z)$

$g_\infty(z) = \tilde{g}_\infty(1/z)$.

Look at $G(z) := \underbrace{f - f_\infty - \sum_1^k f_i}$. ($f -$ all principal parts)

Then G is hol'c, entire, & bdd. $\Rightarrow G = C$.

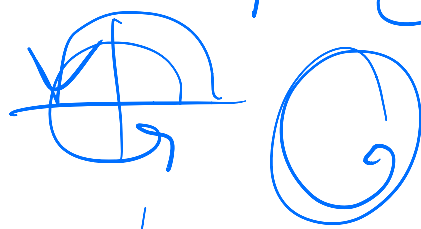
$\Rightarrow f = f_\infty + \sum_1^k f_i + C = \frac{P}{Q}$.

Cor: mero functions on $\hat{\mathbb{C}}$ are det'd by their zeros & poles (w/ mult). up to constant

$\rightsquigarrow$ § Complex Logarithm.

$\log f$ $\qquad [-\pi, \pi)$ $\qquad f = |f| e^{i \arg f}$

$" = " \log|f| + i \arg f.$



def up to 2π.

$$\log(f \cdot g) \neq \underbrace{\log f + \log g}\ \underbrace{+ 2\pi i}_{\text{error}}$$

"Better" attempt:

$$"\frac{d}{dz} \log f" = \frac{f'}{f} \leftarrow \text{"log-deriv"}.$$

$f \mapsto \dfrac{f'}{f}$ is "nice".

Claim: $\dfrac{(f \cdot g)'}{f \cdot g} = \dfrac{f'}{f} + \dfrac{g'}{g}$.

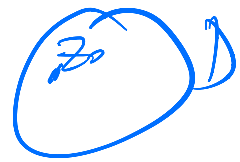
$$" = " \ \frac{f' \cdot g + f \cdot g'}{f \cdot g} \quad //$$

Thm (Argument Principle):

- if $f(z_0)=0$, $D \ni z_0$, (holo & $\neq 0$ else)

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'}{f}\,dz = N$$

= order of zero.



Pf: Near z_0, $f(z) = (z-z_0)^N \cdot g(z)$, $g \neq 0$.

$$\Rightarrow \frac{f'}{f} = \underbrace{\frac{\boxed{N}\cancel{(z-z_0)^{N-1}}}{(z-z_0)^N}}_{\text{Principal part}} + \frac{g'}{g}$$

$\mathrm{Res}_{z_0} \frac{f'}{f} = a_{-1}$ Principal part

- if f has pole at z_0,

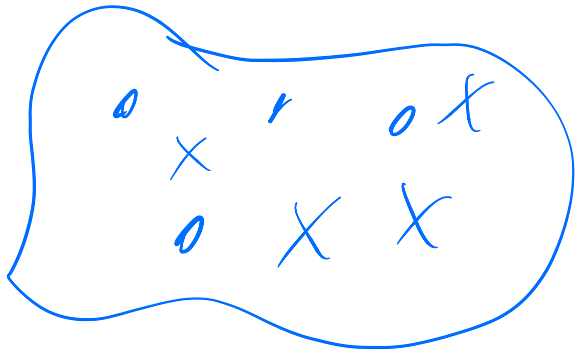
$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'}{f} = -N$$

Pf: Near z_0, $f = (z-z_0)^{-N} \cdot g(z)$
$\neq 0$.

So $\frac{f'}{f} = \frac{-N}{z-z_0} + \frac{g'}{g}$

Arg Principle: If f mero on $R \subset \mathbb{C}$,

$\frac{1}{2\pi i} \int \frac{f'}{f} = \#\text{zeros}$
OR $\quad - \#\text{poles}$
(w/ mult).



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