Last time: Cauchy Integral Rep:

If $f : \underset{D}{\Omega} \to \mathbb{C}$ is hol'c,

$\Rightarrow \forall z \in D$, $C = \partial D$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw$$

Thm: $\forall n$: $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

i.e. hol'c $\Rightarrow C^\infty$.

Cor: (Cauchy inequalities): If $f : \Omega \to \mathbb{C}$, $\overline{D} = \overline{B_R(z_0)} \subset \Omega$, is hol'c then $\forall n$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \cdot \sup_C |f|$$

Pf: $$|f(z_0)| = \left| \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw \right|$$

$$\le \sup_C |f| \cdot \frac{1}{2\pi} \int_C \frac{1}{R}\, dw$$

$$\underbrace{\int_C dw}_{2\pi R}$$

$$\le \sup_C |f|$$

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw\right|$$

$$\le \sup_C |f| \, \frac{n!}{R^{n+1}\, 2\pi} \cdot \cancel{\ell(C)}$$

Thm: hol'c $\Rightarrow$ analytic ($\&\Leftarrow$).

If $f : \overset{\text{open}}{\Omega} \to \mathbb{C}$ hol'c, $\forall z_0 \in \Omega$, $\exists D \subset \Omega$ $(R>0)$, $D = B_R(z_0)$,

s.t. $\forall z \in B_R(z_0)$

$$f(z) = \sum_{n \ge 0} a_n (z-z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Pf: By integral rep know: $\forall z \in B_R(z_0)$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw.$$

want: $w-z \rightsquigarrow w-z_0$. Idea: Solve

$$\underbrace{\frac{1}{w-z}} = \underbrace{\frac{1}{w-z_0}}\big(\underbrace{1+x+x^2+x^3+x^4+\dots}\big).$$

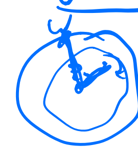
the fcn x:

$$\| \qquad \frac{1}{1-x}. \quad \text{If } |x|<1!$$

$$\frac{1}{w-z} = \frac{1}{w-z_0}\cdot\frac{1}{1-x}, \quad 1-x = \frac{w-z}{w-z_0}.$$

$$x = 1 - \frac{w-z}{w-z_0} = \frac{\cancel{w}-z_0-(\cancel{w}-z)}{w-z_0} = \frac{z-z_0}{w-z_0}.$$

$|x| \le r < 1$ uniformly for $|z-z_0| < r\cdot R$.



$B_R(z_0)$

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = \underbrace{\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}}\left(1+\frac{z-z_0}{w-z_0}+\frac{(z-z_0)^2}{(w-z_0)^2}+\dots\right)dw$$

$\|$ $\qquad\qquad$ $\|$

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + (z-z_0)^2 \frac{f''(z_0)}{2!} + \dots$$

Compare to Hadamard?: $\frac{1}{R} = \overline{\lim}\, |a_n|^{1/n}$:

$$|a_n| = \left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{1}{\cancel{n!}} \cdot \frac{\cancel{n!}}{R^n} \cdot \sup_C |f|$$

$$|a_n|^{1/n} \leq \frac{1}{R} \cdot \underbrace{(\sup |f|)^{1/n}}_{\to 1}$$

$\forall r < R$, the radius of convergence is at least largest R

s.t. $B_R(z_0) \subset \Omega$.

Cor: If f is entire ($\Omega = \mathbb{C}$),

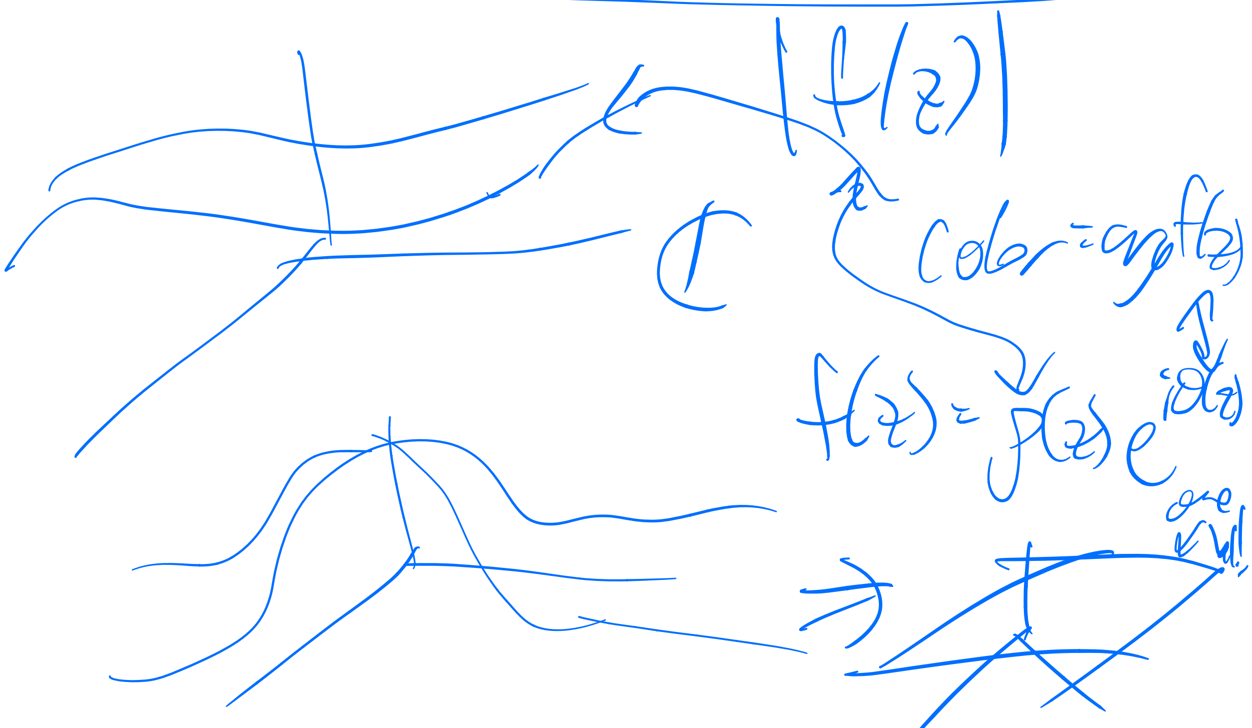
$\Rightarrow$ f has convergent power series about 0 $\forall z \in \mathbb{C}$,

$$f(z) = \sum a_n z^n.$$

Cor: (Liouville) If f is entire & bdd on $\mathbb{C} \Rightarrow f = c$.

Pf: $|f'(z_0)| \le \frac{1!}{\cancel{2\pi}} \cdot \underbrace{\sup|f|}_{C} \cdot \frac{\cancel{R}\,\cancel{2\pi}}{R^2}$,

True for all $R \Rightarrow$ $< K$, $f' \equiv 0$.



Fundamental Thm of Alg:

$\ell\ f \neq C$,

If $f \in \mathbb{C}[x] \Rightarrow$

$\exists z \in \mathbb{C}$ s.t. $f(z) = 0$.

($\Rightarrow$ If f has deg $n \Rightarrow f$ has n roots in $\mathbb{C}$ (w/ multiplicity)).

Pf: Assume $f(z) = a_0 + a_1 z + \dots + a_n z^n$, $a_n \neq 0$

& $\underbrace{f \neq 0}$ on all of $\mathbb{C}$.

Then $\frac{1}{f}$ also entire.

$$\frac{1}{f} = \frac{1}{a_n z^n \left(1 + \underbrace{b_1/z + b_2/z^2 + \dots + b_n/z^n}\right)}.$$

If $|z| > 10 \max |b_k|$, > 0,

$$\frac{1}{|f|} \le \frac{K}{|z|^n} \text{ as } |z| \to \infty,$$

$\to 0$. $\Rightarrow \frac{1}{|f|} \le K$ $\forall z \in \mathbb{C}$,

$\Rightarrow f = \frac{1}{c}$, $f \equiv c$.

Once we have one root,

$$f(z) = (z - \alpha) \cdot g(z)$$

(poly long division) & g has

degree $n-1$. Induct down to $g \equiv c$. (degree $=0$).

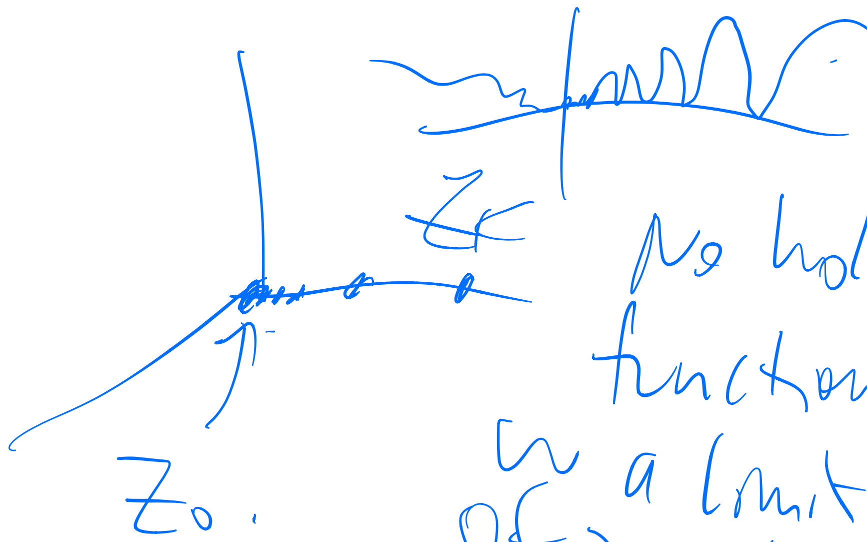
Thm: Analytic Continuation:

If $f : \Omega \to \mathbb{C}$ (analytic) ~~holo~~.

& $\exists$ $z_k \to z \in \Omega$ ($\cup$ distinct)

s.t. $f(z_k) = 0$. $\Rightarrow f \equiv 0$.

Ex:



No holo function w/ a limit pt of zeros!!!

(true for $\mathbb{R}$-analytic, all we use is convergent power series)

Pf: Say $z_k \to z_0 \in \Omega$, ← connected

f is analytic at z_0 so

$\exists\, D \ni z_0$ s.t. $\forall z \in D$,

$f(z) = \sum a_n (z-z_0)^n$,

$\boxed{\text{If } f \not\equiv 0}$, then $\exists$ least N s.t. $a_N \neq 0$. (Note $N \geq 1$, since $f(z_0) = 0$ by continuity).

Then $\forall z$ close enough to z_0, $f(z) = a_N (z-z_0)^N (1 + g(z))$,

$\underline{1} = \underline{g(z)} \to 0$ as $z \to z_0$.

So for z near z_0,

$|1+g(z)| \geq 1/2$. Evaluate $f(z)$ at $z = z_k$, k large enough, that z_k is close enough to z_0,

$$\underbrace{f(z_k)}_{=0} = \underbrace{a_N}_{\neq 0} \underbrace{(z_k - z_0)^N}_{\neq 0} \underbrace{(1+g(z))}_{|\;| \geq 1/2}$$

Contradiction. $\Rightarrow f \equiv 0$.
(in neigh of z_0).

Let $U = \{f=0\}^{\circ} \neq \emptyset$.

Claim U also closed (rel Ω).

i.e. $U = \Omega \cap$ (Closed set of $\mathbb{C}$).

If $\{z_K\} \to z_0$, $z_K \in U$ then

$z_0 \in U$?? (by what was proved!). Cont $\Rightarrow f(z_0)$.

If $V = \Omega \setminus U$, $\Rightarrow V$ is open,

$\Omega = U \sqcup V$ both open. (?)

Connected $\Rightarrow$ <u>One of U & V is empty!</u>

$\Rightarrow V = \text{empty} \Rightarrow U = \Omega \Rightarrow f \equiv 0$ on Ω.

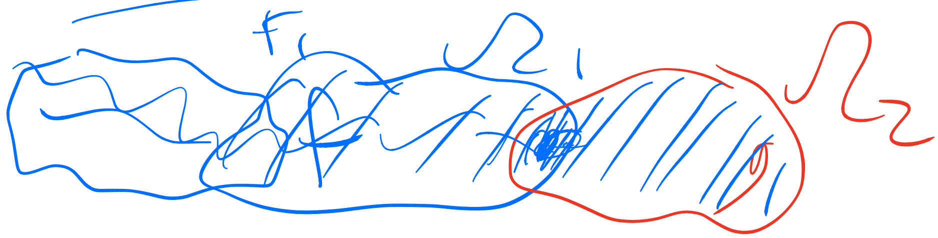
<u>Cor</u>: (Analytic Continuation):

If $f : \Omega_1 \to \mathbb{C}$ & $g : \Omega_2 \to \mathbb{C}$

& $\Omega_1 \cap \Omega_2 \neq \emptyset$ & $f|_{\Omega_1 \cap \Omega_2} = g|_{\Omega_1 \cap \Omega_2}$

$\exists! \, F : \Omega_1 \cup \Omega_2 \to \mathbb{C}$ holc,

$F|_{\Omega_1} \equiv f \quad \& \quad F|_{\Omega_2} \equiv g$



Pf: Let $F = \begin{cases} f & \text{on } \Omega_1 \\ g & \text{on } \Omega_2 \setminus \Omega_1 \end{cases}$ (holo),

Let F_1 be any other function which agrees with f on Ω_1.

Then $F_1 - F \equiv 0$ on Ω_1.

$F_1 - F \equiv 0$ wherever they are defined.

Ex: $1 + 2 + 4 + 8 + 16 + \dots = -1$,

$$1+x+x^2+\cdots = \frac{1}{1-x}$$

$x=2$

$$f(x) := 1+x+x^2+\cdots$$

$$g(x) := \frac{1}{1-x} \qquad \Omega_2 = \mathbb{C}\setminus\{1\}$$

$f \equiv g$ on $|z|<1$.

g is the analytic cont of f

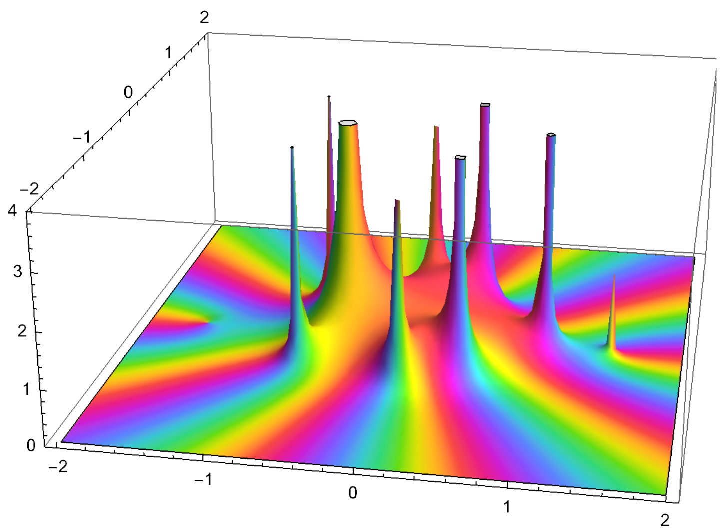
$\cancel{f(z)} = g(z),$

In[29]:=
```
Solve[3 z^10 - 7 z^8 + 3 z^3 - z^2 + z + 1 == 0, z]
```

Out[29]=
```
{{z → 1}, {z → -1.57…}, {z → -0.466…}, {z → 1.48…},
 {z → -0.724… - 0.578… i}, {z → -0.724… + 0.578… i}, {z → 0.0513… - 0.806… i},
 {z → 0.0513… + 0.806… i}, {z → 0.451… - 0.589… i}, {z → 0.451… + 0.589… i}}
```

In[23]:=
```
f[z] := 3 z^10 - 7 z^8 + 3 z^3 - z^2 + z + 1;
ComplexPlot3D[1 / f[z], {z, -2 - 2 I, 2 + 2 I}, PlotRange → {All, All, {0, 4}}]
```

Out[24]=




```
ComplexPlot3D[f[z], {z, -2 - 2 I, 2 + 2 I}]
```

Out[26]=