Recall: Thm: $u, v: \mathbb{R}^2 \to \mathbb{R}$ are $\mathbb{R}^2$-diff'ble & Cauchy Riemann $\iff$ $f = u + iv$ is hol'c.

Thm (Hadamard): Let $f(z) = \sum_{n \geq 0} a_n z^n$. Then $\exists R \left(= \frac{1}{L}, \; L = \overline{\lim} |a_n|^{1/n} \right) \in [0, \infty]$ s.t.

(i) $\forall |z| < R$, series conv abs &

(ii) $\forall |z| > R$, series diverges.

& moreover convergence is uniform $\forall |z| < r < R$.

(Recall: $|z| < r < R = \frac{1}{L}$, make ε small that
$L|z| < r_1 < 1$,
$(L + \varepsilon)|z| < r_1 < 1$.

$\sum_{n \geq N} |a_n||z|^n \leq \sum ((L+\varepsilon)|z|)^n = \sum_{n \geq N} r_1^n < \infty$.

uniform

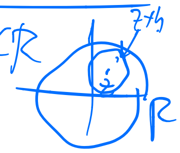
Thm: Analytic $\Rightarrow$ hol'c.

i.e. $f(z) = \sum a_n z^n$, $R > 0$.

$\Rightarrow f'(z)$ exists & $f'(z) = g(z) = \sum n a_n z^{n-1}$

& g has same R ✓ $(n^{1/n} \to 1)$.

$|z|, |z+h| < r < R$



pf: $\left| \dfrac{f(z+h) - f(z)}{h} - g(z) \right| \leq$

$$\boxed{S_N(z) := \sum_{n \leq N} a_n z^n.}$$

$$\leq \left| \frac{f(z+h)-f(z)}{h} - \frac{S_N(z+h) - S_N(z)}{h} \right|$$

$$+ \left| \frac{S_N(z+h) - S_N(z)}{h} - S_N'(z) \right|$$

$$+ \left| S_N'(z) - g(z) \right|$$

- $\forall \varepsilon > 0 \ \exists N_1$ s.t. $\forall |z| < r$, $\forall N > N_1$, $|S_N'(z) - g(z)| < \varepsilon$.
- $\forall N$, $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall |h| < \delta$,

(δ small enough that $|z+h| < r < R$

$$\left| \frac{S_N(z+h) - S_N(z)}{h} - S_N'(z) \right| < \varepsilon.$$

$$\to = \left| \frac{1}{h} \sum_{n>N} a_n \left[(z+h)^n - z^n \right] \right|$$

$$= \left| \frac{1}{\cancel{h}} \sum_{n>N} a_n \cancel{h} \left((z+h)^{n-1} + (z+h)^{n-2} z + \cdots + z^{n-1} \right) \right|$$

(using $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$

Recall $|z+h|, |z| < r < R$

$$\leq \sum_{n>N} |a_n| n r^{n-1} < \varepsilon.$$

By abs conv of g on $|z| < R$,

$\forall \varepsilon > 0 \ \exists N_2 \ \forall N > N_2$,)

Choose arbitrary $\varepsilon > 0$,

Let $\underline{N} > \max(N_1, N_2)$

$\exists \delta > 0$ s.t.

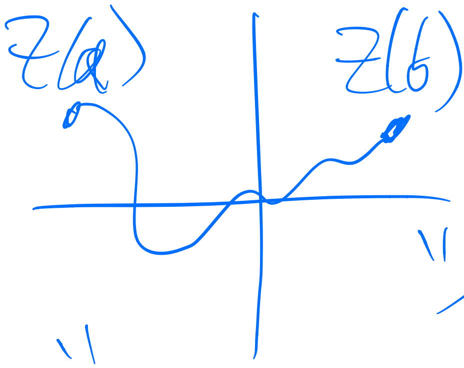
$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| < 3\varepsilon.$$

Cor: f analytic $\Rightarrow$

$f \in C^\infty_{\mathbb{C}}$, each $f^{(n)}$ has same R & is given by termwise differentiation.

§ Integration over curves

Parametric curve: $z : [a,b] \to \mathbb{C}$, "C^1"



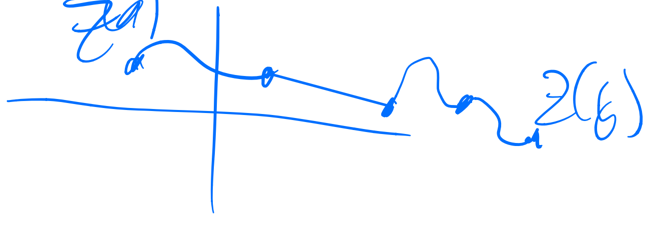
$\mathbb{C}$

cont'ly diff'ble.
(at endpoints $z'(a^+)$
$z'(b^-)$)

"smooth",

"piecewise smooth": $z : [a,b] \to \mathbb{C}$ cont

& cont'ly diff'ble on $a = a_0 < a_1 < \cdots < a_n = b$.



E.g.: $z: [0,1] \to \mathbb{C}$, $z(t) = e^{2\pi i t}$

$\tilde{z}: [0, 2\pi] \to \mathbb{C}$, $\tilde{z}(t) = e^{it}$.

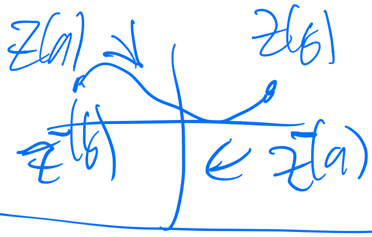
Def: $z: [a,b] \to \mathbb{C}$ equivalent to

$\tilde{z}: [c,d] \to \mathbb{C}$ iff $\exists\, t: [c,d] \to [a,b]$ cont, bijection, diff'ble

with $\tilde{z}(\underbrace{s}_{[c,d]}) = z(t(s))$, $t' > 0$ (equivalence relation)

Note: z^- (opposite curve): $[a,b] \to \mathbb{C}$:

$$z^-(t) = z(a+b-t).$$



Different curves

Def: A smooth curve $\gamma = [z]$.

= parametriz curve $/ \sim$.

If $f : \Omega \to \mathbb{C}$ is cont,

& $\gamma \subset \Omega$ is a smooth curve

<u>Def:</u> $\displaystyle\int_\gamma f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$ ~~$\nleftarrow$~~

$[z] = \gamma$

<u>Lemma:</u> Def is independent of parametrization,

i.e. If $\tilde{z}(s) = z(t(s))$.

$t \to t(s)$.

$$\Rightarrow = \int_c^d f(z(t(s)))\, z'(t(s))\, t'(s)\, ds$$

$$\tilde{z}'(s)\,ds = z'(t(s)) \cdot t'(s)\, ds.$$

$\frac{ds}{dt}$.

$$= \int_c^d f(\tilde{z}(s))\, \tilde{z}'(s)\, ds \checkmark$$

Def: $\text{length}(\gamma) = \ell(\gamma) = \int_a^b |z'(t)|\,dt.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\psi}{z}$

Lemma: "$f:\Omega\to\mathbb{C}$ cont, $\gamma\subset\Omega$",

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_\gamma |f| \cdot \ell(\gamma).$$

pf:

$$\left(\left|\int_a^b f(z(t))\,z'(t)\,dt\right|\right.$$

$$\le \int_a^b \underbrace{|f(z(t))|}\,|z'(t)|\,dt.$$

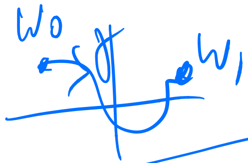
Def: If $f:\Omega \xrightarrow{\text{open}} \mathbb{C}$ &

$F:\Omega\to\mathbb{C}$ s.t. $F' = f$

$\leftarrow$ hole

$\Rightarrow$ F is called a "<u>primitive</u>" of

Thm: If $f:\Omega\to\mathbb{C}$ (cont) has primitive F & $\gamma\subset\Omega$, then $\int_\gamma f(z)\,dz = F(w_1)-F(w_0)$.

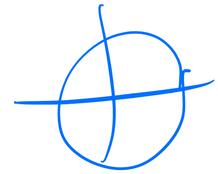


Pf: $\;= \int_a^b \underbrace{\overset{F'}{f}(z(t))\, z'(t)}_{\frac{d}{dt}F(z(t))}\,dt = F(z(b)) - F(z(a))$.

Cor: If $f:\Omega\to\mathbb{C}$ (cont.) & has a primitive & $\gamma\subset\Omega$ is closed $w_0=w_1$,

$$\Rightarrow \oint_\gamma f(z)\,dz = F(w_1)-F(w_0) = 0.$$

E.g.: $f(z)=\frac{1}{z}$, $\gamma =$



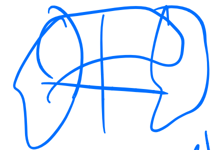
$z:[0,1]\to\mathbb{C}$
$: t \mapsto e^{2\pi i t}$.

$$\int_\gamma f(z)\,dz = \int_0^1 f(e^{2\pi i t})\, 2\pi i\, e^{2\pi i t}\,dt$$

$z'(t) = 2\pi i\, e^{2\pi i t}$

$\cancel{e^{-2\pi i t}} = 2\pi i \neq 0.$

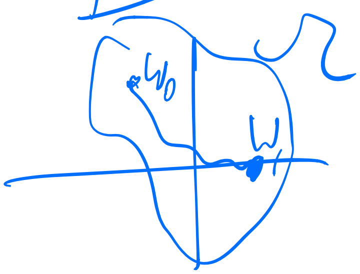
$\Rightarrow f(z)$ does not have a prim. in Ω

For any $\Omega \supset S^1$. 

Cor: If $f: \Omega \to \mathbb{C}$ is hol (open, connected $\Rightarrow$ path connected!)

& $f' \equiv 0$ on Ω. Then $f \equiv c$.

pf: Fix any $w_0 \in \Omega$, $\forall w_1 \in \Omega$ $\exists$

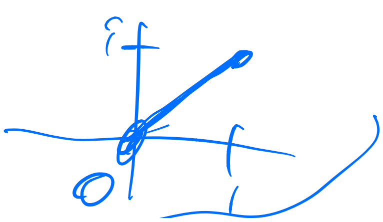


$\gamma: w_0 \to w_1$

$$0 = \int_\gamma \underbrace{f'(z)}_{=0} dz = f(w_1) - f(w_0).$$

$\Rightarrow f(w_1) = f(w_0)$ $\forall w_1 \in \Omega$. $\Rightarrow f \equiv c$.

E.g.: $f(z) = 1$, $\gamma: 0 \to 1+i$.

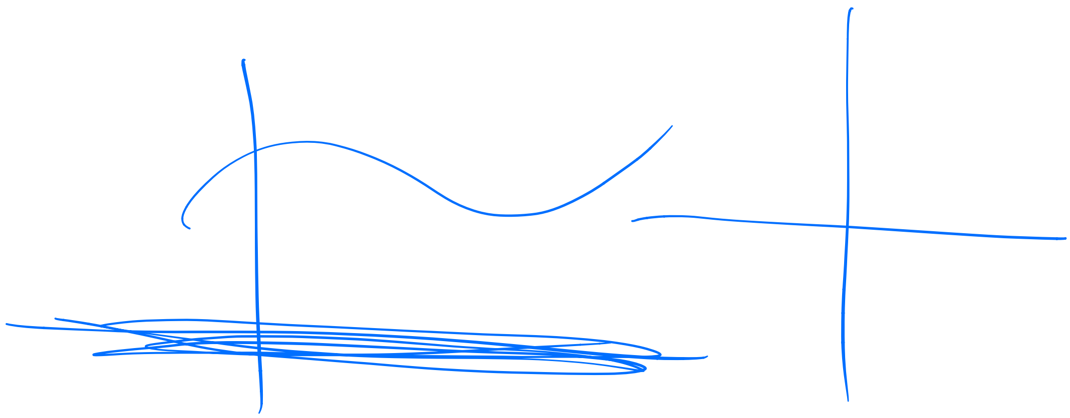


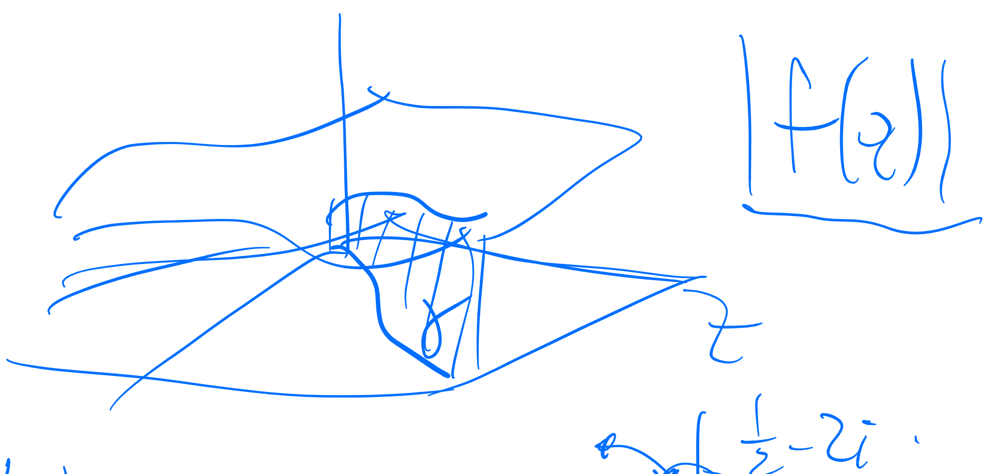
$$\int_\gamma f(z) dz = \int_0^{1+i} 1 \, dz = z\Big|_0^{1+i} = 1+i - 0 = 1+i$$

$F(z)=z$, $F'=1$, $\cancel{f}$, Fine.

$z(t) = t+it$, $0 \le t \le 1$,

$= \int_0^1 1 \cdot (1+i)\,dt = 1+i$.





$f(z) \cdot z'(t)$